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On Cumulative Jump Models for Random Deterioration Processes

ABSTRACT. There exists variety of phenomena of cumulative type which are adequately described by random sums of random variables, or - when dependence on time is essential - by cumulative random processes of the form: $X(t) = X_1 + X_2 + \cdots + X_{N(t)}$ where $X_i, i = 1, 2, \ldots, N(t)$ are random variables characterizing elementary increments (or jumps) of the process in quastion and $N(t)$ is a counting process.

In the paper we expound the existing results and new problems associated with such cumulative processes under various assumptions on random variables X_i and process $N(t)$. As a possible application, the modelling of fatigue accumulation is briefly discussed.

1. Introduction. In the midst of a rich research activity of D. Szynal essential contributions are connected with randomly indexed sequences of random variables (cf. [9], [10], [23]). The papers cited concentrate on the asymptotic behavior of randomly indexed sequences and sums of random variables, in particular, on the central limit theorem and the rate of convergence. The problems studied by Szynal and his co-workers have, to a great extent, close connections with the applied problems in risk theory and reliability analysis where a total effect (for example, the total claim amount, or the total damage experienced by a machine element) is represented as a random sum of random partial contributions (cf. [1], [16]). There exist a variety of real phenomena of cumulative type, which can be adequately described by random sums of random variables, or - when the

dependence on time is essential - by cumulative random processes of the form $X(t,\gamma) = \sum_{i=1}^{N(t)} X_i(\gamma)$, where $X_i(\gamma)$ are random variables characterizing elementary increments (or, jumps) of the process in question, and $N(t)$ is a counting stochastic process characterizing a number of the increments in time interval $[0,t]$; γ denotes an element of the space of elementary events Γ on which a probability structure is defined.

The key quantity of reliability theory is the life-time of a device or system and the main problem arising is that of finding the probability distribution of the lifetime. In the statistical approach, this distribution is inferred from the empirical data on the lifetimes of a number of identical copies of the device, mechanical/structural element etc. However, a more satisfactory approach to the characterization of the lifetime of engineering device (or, system) is to consider a deterioration process including damage and responsible for the ultimate failure. The lifetime is then defined to be the hitting time of the critical threshold value by the deterioration process . As far as mechanical components are concerned, the most common deterioration (or degradation) processes are wear, creep, fatigue, plastic deformation and others. Analysis of deterioration models provides information on the development of the damage in time and, consequently, it can be used for the reliability updating and reliability control of the system.

In this paper we wish to expound the basic models for random deterioration phenomena, with special emphasis on those associated with random sums of random variables including their "continuous" extensions in the form of integrals with respect to random measures.

2. Cumulative jump models. Lets us denote by $D(t,\gamma)$ a stochastic process and interpret it as the amount of deterioration in the engineering element at time t. Real deterioration phenomena are mostly cumulative in nature. Process $D(t,\gamma)$ can be regarded as a non-decreasing and discontinuous random process consisting of a random number of jumps, each with random magnitude. Therefore, its representation is as follows

$$D(t,\gamma) = D_0 + X(t,\gamma), \qquad X(t,\gamma) = \sum_{i=1}^{N(t)} Y_i(\gamma), \tag{2.1}$$

where D_0 denotes the initial deterioration of sufficient amount to growth; it can be regarded as deterministic, or random. Random variables $Y_i(\gamma)$ characterize the magnitudes of partial (elementary) increments of degradation and $N(t)$ is an integer-valued stochastic process (a counting stochastic process) characterizing the number of increments in the interval $[0,t]$.

The model-process $D(t,\gamma)$ constitutes a class of cumulative models in which particular members are specified by the hypotheses posed on the counting process $N(t)$ and on the random variables $Y_i(\gamma)$, $i = 1,2,\ldots$. A function of prime interest is the probability distribution of the degradation model-process $D(t,\gamma)$ at an arbitrary time t, that is

$$F_D(x;t) = P\{D(t,\gamma) \leq x\} \tag{2.2}$$

or equivalently, the probability density function $f_D(x;t)$. Let T be a positive random variable that characterizes a random time at which the deterioration process $D(t,\gamma)$ reaches a fixed critical value ξ. Of course,

$$P\{T > x\} = P\{D(t,\gamma) < \xi\}\,. \tag{2.3}$$

This means that the lifetime distribution and the distribution of the deterioration process are directly related to each other, namely

$$F_T(t) = 1 - F_D(x;t)|_{x=\xi}\,. \tag{2.4}$$

The determination of explicit probability distribution $F_D(x;t)$ of the model-process (2.1) is, in general, a hard problem. To make the analysis effective one has to introduce simplifying hypotheses concerning both the process $N(t)$ and random variable $Y_i(\gamma)$.

2.1. $N(t)$ – Poisson process, $Y_i(\gamma)$ – i.i.d. random variables. The case when $N(t)$ is the Poisson process with intensity $\lambda_0 > 0$ and $Y_i(\gamma)$ are i.i.d random variables is the simplest one (compound Poisson process). But even in this case, the explicit, exact probability distribution $F_D(x;t)$ is not available in general. It can be determined only in some particular cases; for example if the distribution of random variables $Y_i(\gamma)$ is the exponential one, that is $f_Y(y) = a\exp(-ay)$, $y > 0, a > 0$. Then, as well known

$$f_X(x;t) = \sqrt{\frac{a\lambda_0 t}{x}}\,\exp(-\lambda_0 t - ax) I_1\left(2\sqrt{a\lambda_0 t x}\right)\,, \tag{2.5}$$

where $I_1(\cdot)$ is a modified Bessel function of the first order given for $p > -1$ by

$$I_p(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+p+1)}\,(z/2)^{2k+p}\,. \tag{2.6}$$

To obtain formula (2.5) it is assumed that random variables $Y_i(\gamma)$ are independent of process $N(t)$.

In general case, when we only assume that common probability density $f_Y(y)$ (or, characteristic function $\varphi_Y(\theta)$) of random variables $Y_i(\gamma)$ is known, the characteristic function of $X(t,\gamma)$ has a general form

$$\varphi_X(\theta) = \exp\{\lambda_0 t[\varphi_Y(\theta) - 1]\}\,. \tag{2.7}$$

Unfortunately, there is no general rule yielding the explicit and exact inverse transform and providing the corresponding probability density. All what we can evaluate from (2.7) are the moments of $X(t,\gamma)$; for example

$$E[X(t,\gamma)] = \lambda_0 t E[Y]\,, \quad \operatorname{var} X(t,\gamma) = \lambda_0 t E[Y^1 0]\,. \tag{2.8}$$

In order to characterize a distribution of $X(t,\gamma)$ the appropriate approximations are necessary. Of course, the best known result is when the distribution of $X(t,\gamma)$ at the fixed t is asymptotically normal with mean and variance given by (2.8) for $\lambda_0 \to \infty$. For fixed λ_0 two most popular approximations are the Edgeworth and saddlepoint approximations (cf. [3], [6], [11]).

The Edgeworth approximation uses the first few central moments of the distribution and therefore gives a good approximation in the center of the distribution only where $x - \lambda_0 t m_Y$ is of order $(\lambda_0 t Y^2)^{1/2}$; the relative error tends to infinity in the tail of the distribution. Contrary to this the saddlepoint approximation gives a small relative error for $x = \lambda_0 t m_Y$; this approximation is valid when $t \to \infty$, if the Laplace transform $g(s) = E[\exp(-sY)]$ of $f_Y(y)$ exists in a neighborhood of zero, the equation $g'(s) = m_Y$ has a solution and the characteristic function of Y belongs to L_q for some $q \geq 1$.

The question which has risen much attention in the recent years is: What happens if x is very large in $P\{X(t,\gamma) > x\}$ but t is not large. It turns out (cf. [8], [9]) that for a large class of densities of $Y = Y_i$ the relative error of the saddlepoint approximation tends to zero for $x \to \infty$ irrespective of the value of t. The classes of densities are the so called Gamma-like, Beta-like and log-concave and belong to L_q, for some $p > 1$. The saddlepoint approximation has the form

$$P\{X(t,\gamma) > x\} \approx \frac{G(s)\exp(-sx)}{s\sigma(s)}\, B_0(s\sigma(s))\,, \tag{2.9}$$

where

$$\begin{aligned} &G(s) = E[\exp(sX(t))] = \exp\{\lambda_0 t[g(s) - 1]\}\,, \quad g(s) = E[\exp(sY)] \\ &\sigma^2(s) = \frac{d}{ds^2}\ln G(s)\,, \quad B_0(z) = z\exp(z^2/2)[1 - \Phi(z)] \end{aligned} \tag{2.10}$$

with $\Phi(\cdot)$ the standard normal distribution function, and where the saddlepoint $s > 0$ is determined by

$$\frac{d}{ds}\ln G(s) = x\,. \tag{2.11}$$

The above approximation is often called the Esscher approximation due to his works in 1932 and 1963.

2.2. $N(t)$ – pure birth process, $Y_i(\gamma)$ – i.i.d random variables. In modelling of deterioration processes (e.g. in modelling of fatigue crack growth - cf. [17]) it is essential that the intensity of a jump-counting process is state dependent. Pure birth process, for which the probability of transition from the state k to the state $k+1$ in the interval $[t, t+\Delta t]$ is proportional to the state k, has the features which make it interesting not only in population growth. In this case the intensity of $N(t)$ is: $\lambda_k = \lambda^0 k, k = 1, 2, \ldots, \lambda^0 > 0$. In contrast to Poisson process whose mean and variance are linear functions of time, in this case:

$$E[N(t)] = \exp(\lambda^0 t), \quad \operatorname{var} N(t) = \exp(\lambda^0 t)[\exp(\lambda^0 t) - 1].$$

For arbitrary probability distribution of random variables $Y_i(\gamma) = Y(\gamma)$ with the corresponding characteristic function $\varphi_Y(\theta)$ we have the general formula

$$\varphi_x(\theta) = E[\exp(i\theta X(t))] = \frac{\varphi_Y(\theta)\exp(-\lambda_0 t)}{1 - \varphi_Y(\theta)[1 - \exp(-\lambda_0 t)]} \tag{2.12}$$

yielding the general expressions for moments; for example

$$\begin{aligned} &E[X(t,\gamma)] = \exp(\lambda_0 t)E[Y] \\ &\operatorname{var} X(t,\gamma) = \exp(\lambda_0 t)\operatorname{var} Y + E[Y^1 0]\exp(\lambda_0 t)\{\exp(\lambda_0 t) - 1]\big|. \end{aligned} \tag{2.13}$$

In order to obtain the probability density of $X(t,\gamma)$ one has to evaluate the inverse Fourier transform (2.12). If random variables $Y_i(\gamma) = Y(\gamma)$ have exponential distribution with parameter α, then (cf. [18])

$$f_X(x;t) = \alpha\exp(-\lambda_0 t)\exp[-\alpha x\exp(-\lambda_0 t)], \quad x > 0,\ \alpha > 0,\ t > 0. \tag{2.14}$$

Therefore, the distribution of $X(t,\gamma)$ for each t is an exponential one with "parameter" $\alpha\exp(-\lambda_0 t)$.

The distribution of random variable T, the time at which the deterioration process $D(t,\gamma)$ reaches the critical value ξ, is easily obtained by using formula (2.3). The result is (D_0 is assumed to be deterministic constant)

$$F_T(t) = \exp[-\alpha(\xi - D_0)\exp(-\lambda^0 t)]. \tag{2.15}$$

Differentiation with respect to t yields the probability density of the life time

$$f_T(t) = b\lambda^0\exp[-\lambda^0 t - b\exp(-\lambda^0 t)], \quad b = (\xi - D_0), \tag{2.16}$$

which is of the form of an extreme-type (or, Gumbel) distribution. Analysis of experimental data on the deterioration caused by the growth of fatigue crack reported in [8] has indicated that the extreme type distribution of the form (2.16) can constitute a satisfactory model for the first passage time of this process. More details can be found in [18], where also an attempt is made to estimate the model parameters (e.g. λ^0, α) using the empirical information about fatigue deterioration. Formulae (2.13-2.16) have been obtained under the assumption that the distribution of $Y_i(\gamma)$ is exponential. In general case, one may look for some asymptotic results. However, according to the author's knowledge the saddlepoint approximations for cumulative models (2.1) with underlying birth process seem to be lacking.

2.3. $N(t)$ – Poisson process, $Y_i(\gamma)$ – correlated random variables. The assumption on independence of random variable $Y_i(\gamma)$ process (2.1) is an essential weakness of the model. It is clear that in real phenomena, successive increments in deterioration are not independent. A reason for this is that an accumulation of deterioration (damage) usually results in a loss of resistance of further damage, so the magnitudes of successive increments should be treated as dependent random variables.

Summation of dependent random variables creates very serious difficulties. Perhaps, one may say that this is one of most challenging problems of applied probability theory. Even calculation of a few of first moments of the sum constitutes a non-trivial computational problem.

Recently, the authors of [13] provided the recursive method-algorithm (with respect to n) for calculating $E[S_n^k]$ where

$$S_n(\gamma) = Y_1(\gamma) + Y_2(\gamma) + \cdots + Y_n(\gamma). \tag{2.17}$$

The basic formula is as follows

$$E[S_n^k] = E[(S_{n-1} + Y_n)^k] = E[S_{n-1}^k] + E[Y_n^k] + \sum_{i=1}^{k-1} \binom{k}{i} E[S_{n-1}^i Y_n^{k-1}]. \tag{2.18}$$

This means that each recursion requires that $k-1$ expectations

$$E[S_{n-1}Y_n^{k-1}], \ldots, E[S_{n-1}^{k-1}Y_n] \tag{2.19}$$

are available. Assuming that the sequence $Y_1(\gamma), \ldots, Y_n(\gamma)$ is stationary, one can obtain recursion formulae for expectations (2.19)

$$E[S_{n-1}^i Y_n^{k-1}] = E[(Y_1 + \overline{S}_{n-1})^i Y_n^{k-i}] = E[S_{n-2}^i Y_{n-1}^{k-1}] + E[Y_1^i Y_n^{k-i}] + \sum_{j=1}^{i-1} \binom{i}{j} E[Y_1^j (\overline{S}_{n-1})^{i-j} Y_n^{k-i}], \tag{2.20}$$

where $\overline{S}_{n-1} = Y_2 + \cdots + Y_{n-1}$, and $E[(\overline{S}_{n-1})^i Y_n^{k-i}]$ has been replaced by $E[(\overline{S}_{n-2})^i Y_{n-1}^{k-i}]$. Therefore, formula (2.20) becomes a recursion formula in n. The last sum of expectations in (2.20) can be regarded as simple if the power $i - j$ is not greater than 2. Since $i - j \leq k - 2$ we conclude that the recursive formulae above are computationally manageable (for the computerized symbol manipulation for $k \leq 4$). In [13] the above formulae have been used to a sequence of polynomials of degree ν of standard Gaussian variables (i.e. $Y_i = P_\nu(Z_i), i \in \mathbb{N}$), and $Z_1, Z_2, \ldots, Z_n, \ldots$ is a stationary sequence of standardized Gaussian variables with given covariances $c(r) = \operatorname{cov}(Z_1, Z_{1+r})$. Also the results obtained have been used to judge the validity of the central limit theorem for dependent random variables (cf. [22], [24]) in specific applicational situations; especially, to judge the speed of convergence of the distribution of S_n/σ_n^2 to the standard normal distribution. Of course, such results are of prime interest in the estimation of the probability distribution of the integrals of non-Gaussian processes.

Lets us focus our attention again on our main process-model (2.1) with correlated random variables. From practical point of view, it is important to construct a joint probability distribution (density) of dependent random variables $Y_1(\gamma), \ldots, Y_n(\gamma)$ in terms of known one-dimensional (marginal) distributions and covariances (or, correlation coefficients). A construction adopted in [16] is known as the Morgenstern model (cf. [14]) according to which the joint probability density is presented by the formula

$$f(y_1, \ldots, y_n) = \prod_i f_{Y_i}(y_i)\Big\{1 + \sum_{i<j} a_{ij}[1 - 2F_{Y_i}(y_i)]\,[1 - 2F_{Y_j}(y_j)]$$

$$+ \sum_{i<j<k} a_{ijk}[1 - 2F_{Y_i}(y_i)]\,[1 - 2F_{Y_j}(y_j)]\,[1 - 2F_{Y_k}(y_k)] + \ldots\Big\}, \tag{2.21}$$

where $F_{Y_i}(y_i)$ are the marginal cumulative distributions,

$$a_{12\ldots k} = \frac{\rho_{12\ldots k}}{(-2)^k Q_1 Q_2 \ldots Q_k} \tag{2.22}$$

and $\rho_{12\ldots k}$ is the k-dimensional, normalized joint central moment, i.e.

$$\rho_{12\ldots k} = E\left[\left(\frac{Y_1 - m_1}{\sigma_1}\right)\left(\frac{Y_2 - m_2}{\sigma_2}\right)\ldots\left(\frac{Y_i - m_i}{\sigma_i}\right)\right], \tag{2.23}$$

where m_i and $\sigma_i (i = 1, 2, \ldots, k)$ are the mean and standard deviation of $Y_i(\gamma)$, $E[\cdot]$ denotes the mean value, and

$$Q_i = \int \left(\frac{Y_i - m_i}{\sigma_i}\right) f_{Y_i}(y_i) F_{Y_i}(y_i) dy_i\,. \tag{2.24}$$

Of course, since the joint density (2.21) has to be non-negative, the Morgenstern model is valid under the restrictions on $a_{12\ldots k}$. These restrictions are expounded in [12].

The moment of order k of process $X(t,\gamma)$ in (2.1) where $Y_i(\gamma)$ are dependent is

$$E[X^k(t,\gamma)] = \sum_{n=0}^{\infty} E[X^k(t,\gamma) \mid N(t) = n]\, P_n(t) = \sum_{n=0}^{\infty} E_n^k P_n(t)\,, \tag{2.25}$$

where E_n^k – the conditional moment of order k is

$$E_n^k = E\left[\sum_{i=0}^{n} Y_i(\gamma)\right]^k = \sum_{h_1+\cdots+h_n=k} \frac{k!}{h_1!\ldots h_n!} E[Y_1^{h_1}\ldots Y_n^{h_n}]\,, \tag{2.26}$$

and the summation is extended over all possible combinations of n elements giving the sum equal to k. To evaluate the joint moment in (2.26) we make use of the Morgenstern representation (2.21) when only binary correlations are accounted for the joint moments occurring in (2.26). In this case [19]

$$E[Y_1^{h_1}\ldots Y_n^{h_n}] = \prod_{i=1}^{n} E[Y_i^{h_i}] + \sum_{i<j} a_{ij} \prod_{\substack{r\neq i\\ r\neq j}} E[Y_i^{h_i}] M_{h_i} M_{h_j}\,, \tag{2.27}$$

where

$$M_{h_r} = \int y_r^{h_r} f_{Y_r}(y_r)[1 - 2F_{Y_r}(y_r)]dy_r\,. \tag{2.28}$$

Formulae (2.25) - (2.28) give a general representation of the moment of order k of the process $X(t,\gamma)$ under the approximation (2.21) and with account for the binary correlation only.

Let us assume that:

(1) $N(t)$ is the Poisson process with intensity λ_0,

(2) random variables $Y_i(\gamma)$ have exponential distribution with parameters a_i, that is $F_{Y_i}(y) = a_i \exp(-a_i y), y > 0$, and $a_i = a/i^v, a > 0, v$ - natural number (which means that $E(Y_i) = i^v/a$ increases with the number of jump i; the parameter v can be calibrated to empirical data),

(3) the binary correlation coefficient between $Y_i(\gamma)$ and $Y_j(\gamma)$ is as follows

$$a_{ij} = B\left[1 - \frac{|j-i|}{\max(i,j)}\right]\,,\quad i,j = 1,\ldots,n,\ i \neq j\,, \tag{2.29}$$

where B is a positive constant selected in such a way that the joint probability density (2.21) is non-negative. Under the above hypotheses, general formulae given above yield [19]

$$E[X^k(t,\gamma)] = \sum_{l=1}^{k(v+1)} c_l(B) R_l(\lambda_0 t)\,, \tag{2.30}$$

where $c_l(B)$ are the appropriate algebraic expressions including B, whereas $R_l(z)$ are the Stirling polynomials of degree l.

For example, when $v = 2$ we have the following results for the first moments of $X(t,\gamma)$

$$E[X(t,\gamma)] = \frac{\lambda_0 t}{6a}\left[2(\lambda_0 t)^2 + 9\lambda_0 t + 6\right] \tag{2.31}$$

$$\begin{aligned} E[X^2(t,\gamma)] = \frac{\lambda_0 t}{720a^2}\Big[&5(\lambda_0 t)^5(16+3B) + 18(\lambda_0 t)^4(88+13B) \\ &+ 345(\lambda_0 t)^3(28+3B) + 60(\lambda_0 t)^2(348+23B) \\ &+ 360\lambda_0 t(38+B) + 1440\Big]\,. \end{aligned} \tag{2.32}$$

It is worth noting that the effect of parameter B on the mean square of $X(t,\gamma)$ is explicit and - as calculations show - significant, especially for larger values of time.

Formulae (2.25)-(2.28), or - in particular situation formula (2.30) - express the moments for arbitrary order k of the jump cumulative process $X(t,\gamma)$ in terms of the probability distribution of elementary increments $Y_i(\gamma)$, intensity of jumps and the correlation coefficients between $Y_i(\gamma)$. Except for the asymptotic results which might be obtained via the central limit theorem it does not seem to be possible to obtain analytical formula for the probability distribution of the process $X(t,\gamma)$ at an arbitrary time t. However, having the moments, one can construct an approximate probability distribution using the information contained in the given moments. A possible, effective approach is the use of the maximum entropy principle (cf. [20]), as it has been done in [19].

2.4 Generalization. The models discussed above represent the cumulative deterioration level $X(t,\gamma)$ of the "system" at time t as a result of summation of the partial (elementary) increments (or jumps) which occur in random times according to the process $N(t)$. Of course, the cumulation of deterioration may take place in a more complicated way. For example,

the partial deterioration may depend both on its magnitude and on the deterioration level already existing in the system. Such a more general formulation can be accomplished by use of the counting integral with respect to a random Poisson measure (cf. [15]). We mean here the representation of the deterioration level $X(t,\gamma)$ in the form

$$X(t,\gamma) = \int_{t_0}^{t}\int_{0}^{\infty} h(H_{s-},y)N(ds,dy) \tag{2.33}$$

which means that the mechanism of a system deterioration is as follows. The instants of time and magnitudes of partial deterioration form a collection of random points (τ_i, Y_i) in $\mathbb{R}_+ \times \mathbb{R}_+$. The number $N(A)$ of such pairs falling into a Borel set A has the Poisson distribution; if the deterioration jump (increment) occurs at time s with magnitude y when the existing deterioration level just before s is X_{s-} then the partial damage to the system is $h(X_{s-},y)$. The overall deterioration due to events (jumps) which occurred in time interval $[t_0,t]$ with non-negative magnitudes is given by (2.33). It is clear that integral (2.33) is in fact a sum. Indeed, if the points of the Poisson random measure are enumerated so that

$$N(A) = \sum_i I_A(\tau_i, y_i)\,, \tag{2.34}$$

where $I_A(\cdot)$ denotes the indicator function of the Borel set A then

$$\int_{t_0}^{t}\int_{0}^{\infty} h(X_{s-},y)N(ds,dy) = \sum_{\tau_i \le t} h(X_{\tau_i-},y_i) = \sum_{i=1}^{N_t} h(X_{\tau_i-},y_i)\,, \tag{2.35}$$

where

$$N_t = \int_{t_0}^{t}\int_{0}^{\infty} N(ds,dy) \tag{2.36}$$

characterizes a random number of jumps occurring in the interval $[t_0,t]$ with magnitudes $y \in \mathbb{R}_+$. The counting integral (2.33) or (2.35) as interpreted above provides an adequate model for a wide class of jump cumulative phenomena; in fact, any homogenous process with independent increments and no continuous component can be represented in such a form. Of course, model-process (3.33) constitutes a stochastic integral equation with respect to a Poisson measure. The attempts to characterize the reliability problem along this line are presented in papers [2], [5].

3. Fatique–induced deterioration. It is widely accepted that fatigue fracture in real engineering materials takes place via formation and growth

of cracks. However, according to many experimental investigations fatigue crack grows intermittently. The damaging stress experienced by the material at different times are random and the instances of occurring of such stresses are random as well. This is supposed to be mainly due to the fact that the fatigue process is primary generated by such "factors" as peaks of the stress process, its rises and falls, etc., which are random and discrete in time. It is therefore justified to regard the crack growth process as a discontinuous random process consisting of a random number of jumps, each with random magnitude. Such an approach to modelling of fatigue crack growth has been presented in the papers [16], [18], [19]. It should be underlined that the cumulative models of the form (2.1) have features that make them applicable in modelling various complicated growth process; for example - in modelling of crack growth with retardation due to occasional overloads. In this situation (cf. [17]) a decrease in growth rate is observed which follows a high overloads. To capture mathematically such a phenomena a compound pure birth process with specially defined intensity has been adopted in paper [4]. The idea is that instead of the intensity $\lambda_k = \lambda^0 k$ of the birth process $N(t)$, the following intensity is proposed

$$\lambda_k = \lambda_{OL}(t)\lambda_k \tag{3.1}$$

where λ_{OL} is constructed to characterize as adequately as possible the amount of retardation due to overloadings. The intensity function $\lambda_k(t)$ introduced in [23] is as follows

(3.2)

$$\lambda_k(t) = \lambda_k(t; t_1, \ldots, t_M)$$
$$= \lambda^0 k \prod_{i=1}^{M} [1 - \mu(t, \zeta)(t - t_i - \theta_i)^{\beta_i}] \exp\{-a_i(t - t_i - \theta_i)\} H(t - t_i - \theta_i)$$

where $t_1, \ldots, t_M$ are the instants of time in which overstresses occurred, $\mu(t, \zeta_i)$ is a retardation magnitude function depending on time and a collection of relevant variables (e.g. overloading ratio, stress biaxiality, etc.) denoted by ζ, the parameter θ is introduced to reflect a delay of the start of the retardation, a_i characterizes a decay in the retarded growth after i-th overload, and $H(\cdot)$ is the Heaviside unit step function. The physical features of the phenomenon enter into the model through the parameters ζ, a, β and θ. The interpretation of these parameters in terms of experimental data are discussed in [23]. In order to model a curvilinear random fatigue crack growth the vectorial jump cumulative processes were constructed and analyzed in papers [21], [25].

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