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Selfdecomposability: an Exception or a Rule?

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ABSTRACT. Selfdecomposability may be defined as a decomposability property of the probability measure or as a limiting distribution for some sequences of independent but not necessarily identically distributed random variables. We present many classes of selfdecomposable distributions that include t-Student, F-distribution, log-normal, generalized hyperbolic distributions among others. We utilize random integral representations of selfdecomposable distributions to prove their absolute continuity or existence of some moments.

Introduction. Class L of *selfdecomposable probability distributions* (known also as Lévy class L probability measures) appears in probability theory as the solution to the central limit problem. It is exactly the class of limit distributions of normalized partial sums of independent (not necessarialy identically distributed) random variables, (rv). Class L is a proper subset of the set ID, of all infinitely divisible distributions, and it contains stable probability measures, i.e., limits as mentioned above but for *identically distributed* rv's. Stable laws attracted very substantial interest: existence of densities, domains of attraction and the associated theory of regular varying functions, stable processes, stable type Banach spaces, stable random measures and integrals, etc. On the other hand, Feller (1966) devotes about two pages to class L distributions although many of his examples, of infinitely divisible distributions, are indeed in class L. Except for the unimodality property of selfdecomposable measures, it seems that class L was in a "shadow" of stable laws.

In last two decades or so, selfdecomposable distributions had appeared in many statistical and physical models. Among others, in connection to Ising model of ferromagnetism, Sherrington-Kirkpatrick spin glass model, in mathematical finance, in autoregressive sequences and queueing modelling, as well in the number theory. In fact, in many of those models stable distributions are excluded or give worse estimates than class L non-stable distributions.

In the paper we present many families (not necessarily disjoint) of class L distributions and indicate procedures or properties that allow to give non-selfdecomposable laws or to show why such distributions cannot be in L. Selfdecomposable distributions have random integral representations (integrals with respect to Lévy processes). We show how these representations can be used to prove absolute continuity, or existence of moments of selfdecomposable measures.

1. Basic characterizations of selfdecomposability. We say that a random variable (rv) X is *selfdecomposable* or X belongs to *Lévy class* L, if

$$\forall (t>0)\,\exists (X_t \sim X) \quad X \stackrel{d}{=} X_t + e^{-t}X, \tag{1.1}$$

where: $\stackrel{d}{=}$ means equality in distribution, $\sim$ means that rv are independent. Iterating (1.1) we obtain

$$X \stackrel{d}{=} X_t + e^{-t}(X_s + e^{-s}X) = X_t + e^{-t}X_s + e^{-(t+s)}X \stackrel{d}{=} X_{t+s} + e^{-(t+s)}X$$

for all $t, s > 0$. Hence $X_{t+s} \stackrel{d}{=} X_t + e^{-t}X_s$, in other words the family $\{X_t : t \geq 0\}$ of the "remainders" or "cofactors" in (1.1) satisfies a cocyle equation. More important, the conditions on $X_t's$ allow to construct a *Lévy process* Y such that

$$X \stackrel{d}{=} \int_{(0,t]} e^{-s}dY(s) + e^{-t}X, \quad \text{for } t \geq 0, \tag{1,2}$$

cf. [13, Chapter 3] . Recall that by a Lévy process we mean a process $Y(t), t > 0$, with stationary independent increments, $Y(0) = 0$ a.s. and cadlag paths, i.e., $Y(\cdot, \omega)$ are in Skorohod space $D[0, \infty)$.

From (1.2) we get

$$(1.3)\qquad X \in L \text{ iff } X \stackrel{d}{=} \int_{(0,\infty)} e^{-t} dY(t) \text{ and } \mathbf{E}[\log(1+|Y(1)|)] < \infty,$$

cf. [13]. The process Y in (1.3) is uniquely determined up to the distribution and is called the *background driving Lévy process* of X; for short: Y is BDLP for X. Note that the logarithmic moment condition is necessary and sufficient for the existence of the improper integral in (1.3). Cadlag paths of Y allow us to define the random integrals

$$\int_{(a,b]} h(t)dY(t,\omega) := h(t)Y(t,\omega)\Big|_{t=a}^{t=b} - \int_{(a,b]} Y(t,\omega)dh(t)$$

provided h is of bounded variation on $(a,b]$, where $0 < a \le b < \infty$.

From (1.3) we have a characterization of L in terms of the Fourier transform. Namely, for $\varphi(t) = \mathbf{E}[e^{itX}]$ and $\psi(t) = \mathbf{E}[e^{itY(1)}]$ we have

$$(1.4)\qquad \begin{aligned} \varphi \in L \quad \text{iff} \quad \log\varphi(t) &= \int_0^1 \log\psi(st)s^{-1}ds \\ &= \int_0^t \log\psi(r)r^{-1}dr, \quad t \in \mathbb{R}. \end{aligned}$$

The logarithms are well defined as X and $Y(1)$ have infinitely divisible distributions. Furthermore, if $X \stackrel{d}{=} [a,\sigma^2,M]$ and $Y(1) \stackrel{d}{=} [b,s^2,N]$, where the tripletes: real number, non-negative number and (Lévy) spectral measure are those from the Lévy characterization of infinitely divisible measures in terms of Fourier transform; cf. for instance [13, p. 33]. Thus (1.4) equivalently gives

$$(1.5)\qquad \begin{aligned} [a,\sigma^2,M] \in L \quad \text{iff} \quad M(A) = \int_0^\infty N(e^t A)dt, \\ \text{for all Borel sets } A \subseteq \mathbb{R}^* := \mathbb{R} \setminus \{0\}, \end{aligned}$$

where the Lévy spectral measure N integrates $\log(1+|x|)$ outside every neighbourhood of zero. Finally, for $X \stackrel{d}{=} [a,\sigma^2,M]$, (1.1) reads

$$(1.6)\qquad \begin{aligned} [a,\sigma^2,M] \in L \quad \text{iff} \quad M(A) \ge M(c^{-1}A) \text{ for all } 0 < c < 1 \\ \text{and all Borel } A \subseteq \mathbb{R}^*, \end{aligned}$$

i.e., there are no restrictions on shifts a and variance σ^2. We also have that

$$(1.7)\quad [a,\sigma^2,M]\in L \quad\text{iff}\quad M(A)=\int_A h(x)dx \ \text{ and the function } \ x\mapsto xh(x) \ \text{ is non-increasing on } (-\infty,0) \text{ and } (0,\infty).$$

Cf. [13, p. 94].

Finally, we should observe the following property of the whole class L:

(1.8) *L is a convolution (addition of independent sunmands) semigroup, closed under weak convergence, and affine change of scale (types).*

Last but not least, class L is a limiting class of distributions in the following scheme:

(1.9) *$X\in L$ iff there exist a sequence of independent rv's $\xi_1,\xi_2,\ldots$, a sequence of positive numbers $a_1,a_2,\ldots$ and a sequence of real numbers $x_1,x_2,\ldots$ such that*

$$a_n(\xi_1+\ldots+\xi_n)+x_n \xrightarrow{d} X \quad as \quad n\to\infty,$$

and the triangular array $\{a_n\xi_j : 1\le j\le n\}$ is uniformly infinitesimal, i. e., $\max_{1\le j\le n} P[|\xi_j|\ge \varepsilon a_n^{-1}\}\to 0$, for each $\varepsilon>0$.

From (1.9) one may conclude that class L is properly contained in the class ID, of all infinitely divisible laws, while the later can be identified with the family of all Lévy processes (defined above).

Remark 1.1. All the above descriptions of class L rv's or distributions or Fourier transforms hold true for Banach space valued rv's. However, in the present note we deal only with real valued rv's.

Proposition 1.1. *Let $M(dx)=h(x)dx, h\ge 0$ on $\mathbb{R}^*:=(-\infty,0)\subset(0,\infty)$. Then $\mu:=[a,\sigma^2,M]\in L$ iff $A\mapsto\int_A(-x)dh(x)-M(A)$, A is Borel subset of $\mathbb{R}^*$, is the Lévy (spectral) measure N of the BDLP corresponding to μ.*

Proof. For a Lévy (spectral) measure M (on $\mathbb{R}^*$) we define its corresponding Lévy (spectral) function L_M as follows: $L_M(x):=-M([x,\infty))$, for $x>0$, and $L_M(a):=M((-\infty,x])$, for $x<0$. Then one has

$$\int_{\mathbb{R}^*} g(x)M(dx)=\int_{\mathbb{R}^*} g(x)dL_M(x),$$

for any measurable (integrable) g. Furthermore, (1.5) with (1.7) imply that, if $[b,s^2,N]$ is the BDLP Y for $[a,\sigma^2,M]\in L$ then $L_N(x)=-xh(x)$ for

$x \neq 0$. Hence, for a positive and measurable g, one gets

$$\int_{\mathbb{R}^*} g(x)dL_N(x) = -\int_{\mathbb{R}^*} g(x)h(x)dx - \int_{\mathbb{R}^*} g(x)xdh(x)$$
$$= -\int_{\mathbb{R}^*} g(x)M(dx) + \int_{\mathbb{R}^*} g(x)(-x)dh(x).$$

In particular, $N(A) = \int_A (-x)dh(x) - M(A)$, which completes the proof.

Corollary 1.1. *If $M(dx) = h(x)dx, h'(x)$ exists and $\mu = [a, \sigma^2, M] \in L$ then $N(dx) = -(xh(x))'dx$ is the Lévy (spectral) measure of the* BDLP *corresponding to μ.*

Corollary 1.2. *If $M(dx) = h(x)dx$ and $[a, \sigma^2,] \in L$ then $A \mapsto \int_A(-x)dh(x)$, A is a Borel subset of $\mathbb{R}^*$, is a Lévy (spectral) measure.*

The results above can be used to identify BDLP, of class L distributions, using corresponding Lévy (spectral) measures/functions.

2. Examples and properties of class L distributions. Of course, stable laws (limits of sequences in (1.9) but with i.i.d rv's ξ_j) are in L. Their BDLP processes are stable Lévy processes (cf. [13, Section 4.4]. Other families of class L distribution include:

(a) *Gamma distibutions $\gamma_{\alpha,\lambda}$ are in L.*

These are *rv's* with the probability densities

$$f_{\alpha,\lambda}(x) := \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \mathbf{1}_{(0,\infty)}(x) \tag{2.1}$$

($a > 0, \lambda > 0$ are positive parameters, Γ is Euler's gamma function). They belong to L because their Lévy spectral measures are of the form $dM(x) = \alpha e^{-\lambda x}/x dx$, $x > 0$ and therefore satisfy (1.6) or (1.7). Their BDLP processes are compounded Poisson processes $Y_{\alpha,\lambda}(t) := Y_\lambda(\alpha t)$, $t \geq 0$ and

$$Y_\lambda(t) :\stackrel{d}{=} \sum_{j=1}^{N_\lambda(t)} \rho_j \tag{2.2}$$

where $\rho_l, \rho_2, \ldots$ are i.i.d exponential rv's $\gamma_{1,\lambda}$, independent of the stationary Poisson process $N_\lambda(\cdot)$, i.e., for $k = 0, l, 2, \ldots$ we have $P[N_\lambda(t) - N_\lambda(s) = k] = e^{-\lambda(t-s)}(\lambda(t-s))^k/k!, t > s > 0$. Cf. [12, Corollary 1 and Remark 1]

(b) *Chi-square distributions* $\chi^2(r)$ *are in* L.

It is obvious because by definition $\chi^2(r) := \gamma_{r/2,1/2}$, for $r \in \mathbb{N}$.

(c) $\log \gamma_{\alpha,\lambda}$ *are in* L.

Since we have

$$\mathbb{E}[\exp(it \log \gamma_{\alpha,\lambda})] = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1+it} e^{-\lambda x} dx = e^{-it\log\lambda} \frac{\Gamma(\alpha+it)}{\Gamma(\alpha)}$$

and from [7, p. 249],

$$\frac{\Gamma(\alpha+\theta)}{\Gamma(\alpha)} = \exp\left[c_\alpha \theta + \int_{-\infty}^0 \left(e^{\theta t} - 1 - \frac{\theta t}{1+t^2}\right) e^{\alpha t} (|t|(1-e^t))^{-1} dt\right]$$

for all $\theta \in \mathbb{C}$ with $\operatorname{Re}\theta > -\alpha$, and a constant c_α, putting $\theta = it$ we obtain

$$\frac{\Gamma(\alpha+it)}{\Gamma(\alpha)} = \exp\left\{itc_\alpha + \int_{-\infty}^0 \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) \frac{e^{\alpha x}}{(|x|(1-e^x)} dx\right\}$$

and hence we conclude it is infinitely divisible characteristic function with Lévy spectral measure $dM(x) = e^{\alpha x}[|x|(1-e^x)]^{-1} 1_{(-\infty,0)}(x)dx$ (note that $\int_{-\infty}^{+\infty}(1 \wedge x^2)dM(x) < \infty$). Furthermore, $xM'(x)$ is non-decreasing on the negative half-line and therefore it satisfies (1.6) which proves claim (c). See: [15] - [17].

Corollary 2.1. *Symmetrization of the logarithm of rv with gamma distribution gives beta (generalized logistic)* β_α *distribution, i.e., it has probability density function* $B(\alpha,\alpha)^{-1} e^{\alpha s}(1+e^s)^{-2\alpha}$, $-\infty < s < \infty$ *and* $\alpha > 0$.

Proof. From the above we get

$$|\mathbb{E}\exp(it\log\gamma_{\alpha,\lambda})|^2 = \exp\left\{\int_{\mathbb{R}\setminus\{0\}} (\cos tx - 1) \frac{e^{-\alpha|x|}}{|x|(1-e^{-|x|})} dx\right\}$$

$$= \frac{B(\alpha+it, \alpha-it)}{B(\alpha,\alpha)},$$

by Corollary 5 in [12]. The BDLP for β_α *rv* is computed in [12].

Corollary 2.2 [15]. *For standard normal rv* Z, $\log|Z| \in L$. *Similarly,* $\log|t| \in L$ *and* $\log F \in L$, *for Student t-distribution and Snedecor's F-distribution.*

Proof. Since Z^2 is $\chi^2(1)$, i.e., chi-square distributed with one degree of freedom, (b) with (1.8) gives $\log|Z| \in L$. Similarly F-distribution is given by ratio of two independent chi-square distribution and again (b) with (1.8) justifies that $\log F$ is in L. Finally, for independent rv's Z and $\chi^2(r)$, Student t-distribution is given by the ratio $Z/((\chi^2(r)/r)^{1/2}$, which in distribution is equal to the most frequently used form $(\overline{X}-\mu)/(S/\sqrt{k})$ where $\overline{X}$ and S^2 are the sample mean value and the sample variance form the normal population $N(\mu,\sigma^2)$, respectively.

Corollary 2.3 [15]. *The (cumulative) probability distributions functions*

$$G_1(x) := 1-\exp(-\exp)\,, \qquad G_2(x) = \exp(-\exp(-x))\,,\ x\in\mathbb{R}$$

are in L and correspond to plus and minus of the logarithm of standard exponential rv's.

Proof. Note that

$$\Gamma(1+it) = \int_{-\infty}^{\infty} e^{itu}\exp(u-e^u)du = \int_{-\infty}^{\infty} e^{itu}dG_1(u)\,.$$

Similarly we have $\Gamma(1-it) = \int_{-\infty}^{\infty} e^{itx}dG_2(x)$.

(d) *Hyperbolic sine and cosine characteristic functions* $\pi t(\sinh\pi t)^{-1}$, $(\cosh(\pi t/2))^{-1}$ *are in L.*

This is from [12], where the BDLP's are found as well.

(e) *Generalized Inverse Gaussian distributions are in L.* [10]

These are positive rv's with the probability density given by

$$\frac{(\psi/\chi)^{\lambda/2}}{2K_\lambda(\sqrt{\chi/\psi})}x^{\lambda-1}\exp(-1/2(\chi x^{-1}+\psi x)),\ x>0\,,$$

λ, ψ, χ are parameters, (cf. [10, p.14] and K_λ is the modified Bessel function of the third kind with index λ; cf. [7]. The BDLP for these *rv's* were computed by O. Barndorff-Nielsen (oral communication).

(f) *(Barndorff-Nielsen) Generalized hyperbolic distributions are in L.* Cf. [1] - [3].

These are rv's whose density are of the following form

(2.3)

$$\frac{[\chi+(x-\mu)^2]^{(\lambda-1/2)/2}K_{\lambda-1/2}(\sqrt{(\beta^2+\psi)(\chi+(x-\mu)^2)}\,e^{\beta(x-\mu)}}{\sqrt{2\pi}(\chi/\psi)^{\lambda/2}(\beta^2+\psi)^{(\lambda-1/2)/2}K_\lambda(\sqrt{\chi\psi})},\ x\in\mathbb{R}.$$

and μ, β, ψ, χ, λ are parameters in an appropriate range. They were introduced by 0. Barndorff-Nielsen in [1], [2] studying normal variance-mean mixtures, when mixing has generalized inverse Gausian distribution form (e). Characteristic function is given in [3, p. 149].

(h) *Student t-distributions are in L.*

This follows from (f) by taking $\mu = \beta = \psi = 0$ and $-2\lambda = \chi = r, r$ being the degree of freedom of t-distribution.

(i) *Beta distributions $\beta_{a,b}$ are in L.* [3, p. 153].

These are rv's with densities: $B(a,b)^{-1} \exp ax(1 + \exp x)^{-(a+b)}$, $x \in \mathbb{R}$, where B is the beta function and a, b are positive parameters a. Note that $\beta_{\alpha,\alpha}$ is the β_α in Corollary 2.1, and $\beta_{a,b}$ has the characteristic function $B(a+it, b+it)/B(a,b)$, $t \in \mathbb{R}$.

Remark 2.1. In contrast to (a) and (c), there are $\beta_{a,b}$ (in some range of parameters) such that $\log \beta_{a,b}$ is infinitely divisible but not in L ! ([17, Remark 2] and [5, p. 143/144]).

(j) *Log-normal and F distributions are in L.* [4] (or [6, p. 112]).

Let us remark here that $\log F$ is also in L (Corollary 2.2) and that the question of infinite divisibility of log-normal distribution, used frequently in insurance, lead Olof Thorin to the class of convolution of gammas; cf. [5, p. 48 and 67].

(k) *Generalized gamma distributions (Thorin class $\mathfrak{T}$) are in L.*

Thorin class $\mathfrak{T}$ is defined as the smallest class of distributions on $(0, \infty)$ which contains all gamma distributions and is closed under convolutions and weak limits. Let us remark that (e) is proved by showing that generalized inverse Gaussian distributions are in $\mathfrak{T}$. For the claim (k) we refer to the very interesting monograph [5], by L. Bondesson. In fact, the title of this paper is inspired by his ... apology on page 160 !

Class L distributions possess some properties which are of great theoretical and practical importance. At the same time these properties allows us to check quickly that some (classes) of ID laws do not belong to class L. For convenience we collect some of those properties in the following

Theorem 2.1. (1) *All class L distributions are unimodal, i.e., their distribution functions are convex and concave on half-lines $(-\infty, a)$ and (a, ∞) respectively, for some $a \in \mathbb{R}$.*
(2) *All class L distributions are absolutely continuous with respect to Lebesgue measure.*
(3) *All class L distributions have infinite Lévy spectral measures, provided they are non-zero.*

Proof. Unimodality in full generality is proved by M. Yazamoto [20]. See there for a long history of "comedy of errors". Absolute continuity is due to the work of K. Sato and M. Yamazato (see: [13, p. 162]). However, in Section 3 of this note, we present the proof based on random integral representation (1.3), as given in Jacod [11]. Finally, property (3) trivially follows from (1.5), which is also a consequence of the random integral representation.

(i) *Compound Poisson distributions are not in L.*

These are distributions of random sums $\sum_{k=1}^{N_\lambda} \xi_k$, (where N_λ is Poisson rv independent of i.i.d rv $\xi_i's$), which have Lévy spectral measure M of the form $\lambda \cdot m$ and $m \stackrel{d}{=} \xi_i$. (Recall that convolutions of compound Poisson distributions and weak limits generate the whole class ID.)

(ii) *Compound geometric distributions (of type 1) are not in L.*

These are the distributions of (2.2) with Poisson rv N_λ replaced by *geometric rv* $\widetilde{G}_p$ *(waiting time* for the first succes in Bernoulli trials; p=probability of success), that is $P[\widetilde{G}_p = j] = (1-p)^j p$, $j = 0, 1, 2, \dots$ (as in (2.2) sums $\sum_1^0 \xi_i$ means zero rv!). If $F \stackrel{d}{=} \xi_i$ then for $ge\tilde{o}m(p, F) := \sum_{i=1}^{\widetilde{G}_p} \xi_i$, its characteristic function is of the form

$$\varphi_{ge\tilde{o}m(p,F)}(t) = \sum_{k=0}^{\infty} \mathbb{E}[\mathbb{E}\exp\Big(it\sum_{i=1}^{k}\xi_i\Big)|\widetilde{G}_p = k]$$

$$= P[\widetilde{G}_p = 0] + \sum_{k=1}^{\infty}(\varphi_F(t))^k P[\widetilde{G}_p = k] = \frac{1-q}{1-q\varphi_F(t)},$$

where $q = 1 - p$ and φ_F is the characteristic function of $\xi_i \stackrel{d}{=} F$. Using the formula $\log(1-u) = -\sum_{k=1}^{\infty} u^k/k$ we infer that

$$\log\varphi_{ge\tilde{o}m(p,F)}(t) = \sum_{k=1}^{\infty}\frac{q^k}{k}(\varphi_F^k(t) - 1)$$

$$= \sum_{k=1}^{\infty}\int_{-\infty}^{\infty}(e^{itx} - 1)\frac{q^k}{k}\,dF^{*k}(x) = \int_{-\infty}^{\infty}(e^{itx} - 1)dM_{q,F}(x),$$

where

$$dM_{q,F}(-\infty, x] := \Big(\sum_{k=1}^{\infty} q^k F^{*k}(x)/k\Big)dx \le \sum_{k=1}^{\infty}\frac{q^k}{k} = -\log(1-q)$$

is a finite Lévy spectral measure, and therefore $ge\tilde{o}m(p,F) \notin L$, by Theorem 2.1 (3).

(From the above formula we see that compound geometric (of type 1) distributions are compound Poisson distributions. Furthermore, for $\lambda > 0$ and $n \to \infty$ we get

$$\varphi_{(\widetilde{geom}(\lambda/n,F))^{*n}}(t) = \left[\frac{1-\lambda/n}{1-\lambda/n\varphi_F(t)}\right]^n \to \frac{\exp(-\lambda)}{\exp(-\lambda\varphi_F)}$$
$$= \exp\left(-\lambda\left(\int_{-\infty}^{\infty}(e^{itx}-1)dF(x)\right)\right),$$

which means that compound geometric distributions generate (by convolutions and limits) all compound Poisson distributions. Consequently, compound geometric (of type 1) distributions generate whole class ID, in particular L distributions although themselves are not members of L.)

Remark 2.2. Compound Poisson and compound geometric distributions have atoms at zero and thus are not absolutely continuous (cf. Theorem 2.1(2)).

In [19] there are examples of geometric sums that give class L and not class L distributions. However, the assumption on moments, on p.232, is unnecessary. Some applications to queueing theory are given.

(iii) *Compound geometric distributions (of type 2) might be or not to be in L.*

These are the probability distributions of the random sums as in (ii), but $\widetilde{G}_p$ is replaced by $G_p := \widetilde{G}_p + 1$ (moment of the first success !). Denoting such sums by $geom(p, F)$ (without tilda) we get

$$\varphi_{geom(p,F)}(t) = \varphi_F(t)\,\frac{1-q}{1-q\varphi_F(t)}\,. \tag{2.4}$$

Hence, if for instance $\varphi_F(t) = 0$ for some t, then $geom(p, F) \notin ID$. If F is infinite divisible then so is $geom(p, F)$ and its Lévy spectral measure is

$$dM^{p,F} = dM + dM_{p,F} \tag{2.5}$$

where M is spectral measure of F and $M_{p,F}$ is from part (ii).

Example. (I) If $F = \gamma_{2,\lambda}$ (gamma) then the compound geometric distribution (of type 2) gives $geom(p, \gamma_{2,\lambda}) \in L$. Furthermore

$$\gamma_{1,\lambda_1} * \gamma_{1,\lambda_2} = geom\left(\frac{4\lambda_1\lambda_2}{(\lambda_1+\lambda_2)^2}, \gamma_{2,(\lambda_1+\lambda_2)/2}\right). \tag{II}$$

Proof. (I) From Subsection (a), (2.5) and (ii) we obtain

$$\begin{aligned}\frac{dM^{p,F}}{dx} &= 2\,\frac{e^{-\lambda x}}{x} + \sum_{k=1}^{\infty} \frac{q^k}{k}\,\gamma_{2,\lambda}^{*k}(x)\\ &= 2\,\frac{e^{-\lambda x}}{x} + e^{\lambda x}\sum_{k=1}^{\infty}\frac{q^k}{k}\,x^{2k-1}\lambda^{2k}/\Gamma(2k)\\ &= 2\,\frac{e^{-\lambda x}}{x}\left(1+\sum_{k=1}^{\infty}\frac{(q^{1/2}x\lambda)^{2k}}{(2k)!}\right)\\ &= \frac{e^{-\lambda x}}{x}\left(\exp(q^{1/2}\lambda x)+\exp(-q^{1/2}\lambda x)\right)\\ &= \left[\exp(-\lambda(1-q^{1/2})x)+\exp(-\lambda(1+q^{1/2})x)\right]/x\end{aligned}$$

and this is a sum of Lévy spectral measures corresponding to convolution of $\gamma_{1,\lambda(1-q^{1/2})}$ and $\gamma_{1,\lambda(1+q^{1/2})}$. Thus by (1.8) the proof of (I) is complete.

(II) Since the just proved identity means that

$$geom(p,\gamma_{2,\lambda}) = \gamma_{1,\lambda(1-q^{1/2})} * \gamma_{1,\lambda(1+q^{1/2})}\,,$$

and $q = 1-p$, taking $\lambda_1 := \lambda(1-q^{1/2})$, $\lambda_2 := \lambda(1+q^{1/2})$ gives (II). (Note that the above equality can be also easily checked by comparing characteristic functions !).

3. Autoregression, moments and integral representations.

(a) Let $X_0 := X$ be given *rv* and define an *autoregressive sequence* (X_n) of first order by equality

$$X_{n+1} = cX_n + \varepsilon_n\,,\quad n\geq 1\,, \tag{3.1}$$

where $0<c<1, (\varepsilon_n)_{n=1}^{\infty}$ are i.i.d (so called innovation process) and independent of $(X_n)_{n=1}^{\infty}$ that is identically distributed. In distribution one has $X_0 \stackrel{d}{=} cX_0+\varepsilon_1$, $X\sim\varepsilon_1$. So, if (3.1) happens to be true for all $0<c<1$ then $X_0\in L$. In other words, selfdecomposable distributions can be realized as the marginal distributions of autoregressive sequences. Cf. Cox (1981), in particular p. 111. In fact, by (1.2) the innovation process has the following distribution

$$\varepsilon_1 \stackrel{d}{=} \int_{(0,-lnc]} e^{-s}dY(s),\quad \text{with } Y \text{ as the BDLP of } X_0\,. \tag{3.2}$$

Example 3.1. For autoregression driven by $\gamma_{\alpha,\lambda}$ (i.e. $X_0 = \gamma_{\alpha,\lambda}$ in (3.1)) its innovation process has representation

$$\varepsilon_1 \stackrel{d}{=} \sum_{k=1}^{N} e^{-\tau_k/\alpha} J_k ,$$

where N is a Poisson rv with intensity $-\alpha\lambda \log c$, $\tau_1, \tau_2, \ldots$ (inter arrival times) and $J_1, J_2, \ldots$ (values of jumps) are exponentially distributed $\gamma_{1,\lambda}$ and all appearing rv's are independent. In particular, ε_1 has compound Poisson distribution. (cf. [14]).

Proof. From part (a) in Section 2 we know that BDLP process for $\gamma_{\alpha,\lambda}$ is given by $Y_{\alpha,\lambda}(t) := Y_\lambda(\alpha t)$, where Y_λ is defined by (2.2). Denoting the moments of jumps of Y_λ by $\tau_1 < \tau_2 < \ldots$ we conclude from (3.2)

$$\begin{aligned}\varepsilon_1 &\stackrel{d}{=} \int_{(0,-\ln c]} e^{-s} dY_\lambda(\alpha s) = \int_{(0,-\alpha \ln c]} e^{-r/\alpha} dY_\lambda(r) \\ &= \sum_{j=1}^{N_\lambda(-\alpha \ln c)} e^{-\tau_j/\alpha}(Y_\lambda(\tau_j) - Y_\lambda(\tau_j - 0)),\end{aligned}$$

which completes the proof.

(b) Now we return to the proof of Part 2 of Theorem 2.1 using random integral representation. Suppose that $X = \int_0^\infty e^{-s} dY(s)$ and write $Y(t) = Y_1(t) + (Y_2(t) - Y_2(t,A)) + Y_2(t,A)$, where Y_1 is the continuous (Gaussian) part of Y, Y_2 is the discontinuous part of Y and finally

$$Y_2(t,A) := \sum_{s \le t} \triangle Y_2(s) 1_A(\triangle Y_2(s))$$

is the jump process with the jumps $\triangle Y_2(s) := Y_2(s) - Y_2(s-0)$ in a set $A \subseteq (|x| \ge \varepsilon)$, for some $\varepsilon > 0$; (cf. [9, Chapter 4]). Since the three processes are independent Lévy processes, the law of X is a convolution of three probability measures. If $Y_1 \neq 0$ then X has a Gaussian factor and thus is absolutely continuous. Assume $Y_2 \neq 0$ and choose $A \subseteq [\varepsilon, \infty)$, $\varepsilon > 0$, such that $Y_2(\cdot, A)$ is non-zero, i.e., its Lévy spectral measure M satisfies $0 < M(A) < \infty$, and let $\tau_A := \inf\{t > 0 : Y_2(t,A) \neq 0\}$ be Markow time of the first jump. Note further that

$$\begin{aligned}\widetilde{X} :&= \int_0^\infty e^{-s} dY_2(s,A) = e^{-\tau_A} \triangle Y_2(\tau_A, A) + \int_{(\tau_A,\infty)} e^{-s} dY_2(s,A) \\ &= e^{-\tau_A} \triangle Y_2(\tau_A, A) + e^{-\tau_A} \int_{(0,\infty)} e^{-s} dY_2(s+\tau_A, A)).\end{aligned} \tag{3.3}$$

Furthermore

$$\widetilde{X} \sim \int_{(0,\infty)} e^{-s} dY_2(s+\tau_A, A) =: \widetilde{X'} \stackrel{d}{=} \int_{(0,\infty)} e^{-s} dY_2(s, A),$$

because we have the following property of stopped Lévy processes

$$Y_2(\cdot\,, A) \sim Y_2(\cdot + \tau_A, A) - Y_2(\tau_A, A) \stackrel{d}{=} Y_2(\cdot\,, A).$$

(Here: $W \sim V$ means independence of rv' s W and V.) Finally recall that $\tau_A \sim Y_2(\cdot + \tau_A, A) - Y_2(\tau_A)$, we have

$$\widetilde{X} = e^{-\tau_A}(\triangle Y_2(\tau_A, A) + \widetilde{X'}) \tag{3.4}$$

with the three terms on the right hand side stochastically independent. Since $Y_2(\cdot\,, A)$ is a pure jump process, τ_A has the exponential distribution with $\lambda := M(A)$, and $\triangle Y(\tau_A, A)$ has the distribution $\lambda^{-1} M|_A(\cdot)$ (restriction of M to set A). Denoting by $\nu \stackrel{d}{=} \widetilde{X}$ and using (3.4) we get

$$\begin{aligned}
\mathbb{E}[g(\widetilde{X})] &= \int_0^\infty \left(\int_A \left(\int_0^\infty g(e^{-s}(x+y))\lambda e^{-\lambda s} ds \right) \lambda^{-1} M(dx) \right) \nu(dy) \\
&= \int_0^\infty \left(\int_A \left(\int_0^{x+y} g(z) z^{\lambda-1}(x+y)^{-\lambda} dz \right) M(dx) \right) \nu(dy) \\
&= \int_0^\infty z^{\lambda-1} g(z) \left(\int_A \left(\int_0^{\max(z-x,0)} (x+y)^{-\lambda} \nu(dy) \right) M(dx) \right) dz
\end{aligned}$$

for all g measurable and bounded. Taking $g = 1_B$ we get the formula for the density of ν, and the proof is complete.

(c) We complete this Section with yet another application of random integral representations - this time for identifying moments.

Proposition 3.1. *For $r > 0$, and Lévy process Y we have*

$$\left[\mathbb{E} \| \int_0^\infty e^{-t} dY(t) \|^r < \infty \right] \quad \textit{iff} \quad [\mathbb{E} \| Y(1) \|^r < \infty].$$

Proof. For subadditive functions integrability with respect to infinitely divisible law is equivalent to integrability of such functions with respect to corresponding Lévy spectral measures on sets $(\|x\| \geq a)$, cf. for instance [13, p. 36]. If M is Lévy spectral measure of the integral and N of $Y(1)$

then they are related by (1.5). The proof follows from the equalities (for any $a > 0$)

$$\int_{(||x||\geq a)} ||x||^r M(dx) = \int_0^\infty \int_{(||y||\geq e^t a)} ||e^{-t}y||^r N(dy)dt$$
$$= \int_{(||y||\geq a)} \left(\int_0^{\log(||y||/a)} e^{-rt}||y||^r dt \right) N(dy)$$
$$= r^{-1} \int_{||x||>a} (||y||^r - a^r)N(dy)$$
$$= r^{-1} \int_{||x||>a} ||y||^r dN(y) - r^{-1}a^r N(||x|| > a).$$

This means that Lévy measures M and N simutenously integrate $||x||^r$ (over $(||x|| > a)$), which completes the proof.

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