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Complete Convergence Criterion for Arrays of Banach Space Valued Random Elements

ABSTRACT. We obtain a criterion for complete convergence for arrays of rowwise independent Banach space valued random elements. In the main result no assumptions are made concerning the existence of expected values or absolute moments of the random elements. Also no assumptions are made concerning the geometry of the underlying Banach space. The corresponding convergence rates are also established.

1. Introduction. A sequence $\{U_n,\ n \geq 1\}$ of random variables is said to *converge completely* to the constant C if $\sum\limits_{n=1}^{\infty} P\{|U_n - C| > \varepsilon\} < \infty$ for all $\varepsilon > 0$. In Hsu and Robbins (1947) it was proved there that the sequence of arithmetic means of i.i.d. random variables converges completely to the expected value if the variance of the summads is finite. This result has been generalized and extended in several directions and we direct the reader to [3] for reference.

Etemadi [2] obtained an elementary proof of the weak law of large numbers for separable Banach space valued random elements. The present note is devoted to obtaining an analogous extension of Hsu-Robbins theorem

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to general triangular arrays of rowwise independent, but not necessarily identically distributed Banach space valued random elements. Rowwise independence means that random elements within each row are independent, but that no indenpendence is assumed between the rows.

In Section 2 we give some inequalities and lemmas which will be used in the proofs of our main results. In Section 3 we obtain the complete convergence of row sums with the corresponding rates of convergence. However, in the main result no assumptions are made concerning the existence of expected values or absolute moments of the random elements. Also no assumptions are made concerning the geometry of underlying Banach space.

2. Definitions and preliminaries. Let $(\Omega, \mathfrak{A}, P)$ be a probability space and $(B, \|\cdot\|)$ be a real separable Banach space with the norm $\|\cdot\|$. A random element is defined as a measurable mapping from Ω with σ-algebra $\mathfrak{A}$ into Banach space B with Borel σ-algebra. The concept of independent distributions has direct extension to B. A detailed account of basic properties of random elements in separable Banach spaces can be found in [8].

Throughout this paper we shall denote $\{(X_{nk},\ 1 \le k \le k_n),\ n \ge 1\}$ as an array of rowwise independent, but not necessarily identically distributed, random elements taking values in B. Here and in the sequel we denote by S_n the sum $\sum_{k=1}^{k_n} X_{nk}$. If $k_n = \infty$ we will assume that the series converges a.s. For $\delta > 0$ define $Y_{nk} = X_{nk} I\{\|X_{nk}\| \le \delta\}$ and write $S_n^\delta = \sum_{k=1}^{k_n} Y_{nk}$.

Before going futher, we first of all introduce some definitions on an array of random elements.

Definition 1. An array $\{(X_{nk},\ 1 \le k \le k_n),\ n \ge 1\}$ is said to be *symmetric* if each X_{nk} is symmetrically distributed for all $1 \le k \le k_n$ and all $n \ge 1$.

Definition 2. An array $\{(X_{nk},\ 1 \le k \le k_n),\ n \ge 1\}$ is said to be *infinitesimal* if for all $\varepsilon > 0$: $\lim_{n\to\infty} \sup_{1 \le k \le k_n} P\{\|X_{nk}\| > \varepsilon\} = 0$.

Now we are able to formulate Etemadi result established in [2].

Degenerate convergence criteria. *Let $\{(X_{nk},\ 1 \le k \le k_n),\ n \ge 1\}$ be an array of rowwise independent random elements and $p \ge 1$. The array is infinitesimal and there exists a (nonrandom) sequence $\{A_n\}$ of vectors in B such that $S_n - A_n \to 0$ in probability as $n \to \infty$ if and only if there exists $\delta > 0$ such that:*

(i) $\lim\limits_{n\to\infty} \sum_{k=1}^{k_n} P\{\|X_{nk}\| > \delta\} = 0,$

(ii) $\lim_{n\to\infty} E\|S_n^\delta - A_n\|^p = 0$.

A_n may be taken as ES_n^δ. Furthermore, if (i) *and* (ii) *are true for some $\delta > 0$, they are also true for all $\delta > 0$.*

Now we shall present some well-known inequalities and lemmas which will be useful in the proof of the main result.

Hoffmann–Jorgensen inequality [4]. *If an array $\{(X_{nk},\ 1 \le k \le k_n),\ n \ge 1\}$ of rowwise independent random elements is symmetric, then for all $t > 0$:*

$$P\{\|S_n\| > 3t\} \le P\left\{\sup_{1\le k\le k_n} \|X_{nk}\| > t\right\} + 4P\{\|S_n\| > t\},$$

Etemadi inequality [1]. *If an array $\{(X_{nk},\ 1 \le k \le k_n),\ n \ge 1\}$ of rowwise independent random elements is symmetric, then for any $\varepsilon > 0$:*

$$\sum_{k=1}^{k_n} P\{\|X_{nk}\| > \varepsilon\} \le P\{\|S_n\| > \varepsilon/8\}/(1 - 8P\{\|S_n\| > \varepsilon/8\})$$

The following lemma is only a slight modification of the well-known result (cf. [6]).

Lemma 1. *Let an array $\{(X_{nk},\ 1 \le k \le k_n),\ n \ge 1\}$ of rowwise independent random elements be symmetric and suppose there exists $\delta > 0$ such that $\|X_{nk}\| \le \delta$ for all $1 \le k \le k_n, n \ge 1$. If $S_n \to 0$ in probability then for any $\varepsilon > 0$, any $p \ge 1$ any all sufficiently large n :*

$$E\|S_n\|^p I\{\|S_n\| > \varepsilon\} \le 2(3\delta)^p \sum_{k=1}^{k_n} P\{\|X_{nk}\| > \varepsilon/3\} + 2\varepsilon^p P\{\|S_n\| > \varepsilon/3\}.$$

Proof. Fix any $\varepsilon > 0$ and $A > 0$. Since $S_n \to 0$ in probability, we can choose N large enough such that $\sup_{n\ge N} P\{\|S_n\| \ge \varepsilon\} \le 1/3^p 8$. Integrating by parts we have:

$$E\|S_n\|^p I\{\|S_n\| > \varepsilon\} = \int_\varepsilon^A P\{\|S_n\| \ge t\}\, dt^p + \varepsilon^p P\{\|S_n\| > \varepsilon\}.$$

Now by the change of variable, setting $t = 3u$, and by Hoffmann–Jorgensen inequality with $n \geq N$, we have:

$$\int_{\varepsilon}^{A} P\{\|S_n\| \geq t\}\, dt^p = 3^p \int_{\varepsilon/3}^{A/3} P\{\|S_n\| \geq 3u\}\, du^p$$

$$\leq 3^p \left(4 \int_{\varepsilon/3}^{A/3} P^2\{\|S_n\| \geq u\}\, du^p + \int_{\varepsilon/3}^{A/3} P\{\sup_{1 \leq k \leq k_n} \|X_{nk}\| \geq u\}\, du^p \right)$$

$$\leq 3^p 4 \int_{\varepsilon/3}^{A/3} \frac{1}{3^p 8} P\{\|S_n\| \geq u\}\, du^p + 3^p \int_{\varepsilon/3}^{\delta} P\{\sup_{1 \leq k \leq k_n} \|X_{nk}\| \geq u\}\, du^p$$

$$\leq \frac{1}{2} \int_{\varepsilon/3}^{A} P\{\|S_n\| \geq u\}\, du^p + 3^p \int_{\varepsilon/3}^{\delta} P\{\sup_{1 \leq k \leq k_n} \|X_{nk}\| \geq u\}\, du^p$$

$$\leq \frac{1}{2} \int_{\varepsilon}^{A} P\{\|S_n\| \geq t\}\, dt^p + \frac{1}{2} \int_{\varepsilon/3}^{\varepsilon} P\{\|S_n\| \geq u\}\, du^p$$

$$+ 3^p \delta^p P\{\sup_{1 \leq k \leq k_n} \|X_{nk}\| \geq \varepsilon/3\}.$$

Hence, we obtain

$$\int_{\varepsilon}^{A} P\{\|S_n\| \geq t\}\, dt^p \leq 3^p 2 \delta^p \sum_{k=1}^{k_n} P\{\|X_{nk}\| \geq \varepsilon/3\} + \varepsilon^p P\{\|S_n\| > \varepsilon/3\}.$$

Finally, letting $A \to \infty$ the proof can be easily completed.

We also need the following simple symmetrization inequalities in the proof of Section 3. For a random element Y, define its symmetrization $Y^s = Y - \widetilde{Y}$, where $\widetilde{Y}$ is an independent copy of Y.

Lemma 2. (a) *If an array* $\{(X_{nk},\ 1 \leq k \leq k_n),\ n \geq 1\}$ *of rowwise independent random elements is infinitesimal then for all* $t > 0$ *sufficiently large* n *and all* $1 \leq k \leq k_n$ *we have:*

$$P\{\|X_{nk}\| > t\} \leq 2P\{\|X^s_{nk}\| > t/2\}.$$

(b) *If a sequence* $\{Y_n,\ n \geq 1\}$ *of random elements is such that* $Y_n \to 0$ *in probability as* $n \to \infty$, *then for all* $t > 0$ *and sufficiently large* n:

$$P\{\|Y_n\| > t\} \leq 2P\{\|Y_n^s\| > t/2\}.$$

(c) *If* $A \in B$ *is nonrandom then for any random element* Y *and* $t > 0$ *we have:*

$$P\{\|Y^s\| > t\} \leq P\{\|Y - A\| > t/2\}.$$

Proof. Denote by m_{nk} the median of a random variable $\|X_{nk}\|$. Since the array is infinitesimal, we have $\sup_{1 \leq k \leq k_n} m_{nk} \to 0$ as $n \to \infty$. Let N be so large that $\sup_{n \geq N} \sup_{1 \leq k \leq k_n} m_{nk} \leq t/2$. Then for $n \geq N$:

$$P\{\|X_{nk}\| > t\} \leq P\{\|X_{nk}\| - m_{nk}) > t/2\} \leq P\{|\|X_{nk}\| - m_{nk})| > t/2\}$$

$$\leq 2P\left\{\left|\|X_{nk}\| - \|\widetilde{X_{nk}}\|\right| > t/2\right\} \leq 2P\{\|X_{nk}^s\| > t/2\}$$

(cf. [7, p.257]). We mention that the proof can be also found in [2, p.249]. Note that part (b) can be proved by the same arguments as the proof of part (a) and (c) can be found in [7, p.257].

The following lemma will be also usefull in the proofs of our main result. Before the proof of Lemma 3, we recall that $S_n^\delta = \sum_{k=1}^{k_n} X_{nk} I\{\|X_{nk}\| \leq \delta\}$

Lemma 3. *Let* $\{(X_{nk},\ 1 \leq k \leq k_n),\ n \geq 1\}$ *be an array of rowwise independent random elements and* $\{A_n\}$ *be a (nonrandom) sequence of vectors in* B *Then for all* $\varepsilon > 0$ *and* $\delta > 0$ *we have*

$$P\{\|S_n - A_n\| \geq \varepsilon\} \leq P\left\{\|S_n^\delta - A_n\| \geq \varepsilon/2\right\} + \sum_{k=1}^{k_n} P\{\|X_{nk}\| > \delta\}$$

and

$$P\left\{\|S_n^\delta - A_n\| \geq \varepsilon\right\} \leq P\{\|S_n - A_n\| \geq \varepsilon\} + \sum_{k=1}^{k_n} P\{\|X_{nk}\| > \delta\}$$

Proof. For any $\delta > 0$ write

$$S_n^{''} = \sum_{k=1}^{k_n} X_{nk} I\{\|X_{nk}\| > \delta\}.$$

We mention that the first part of Lemma can be obtained as follows

$$P\{\|S_n - A_n\| \geq \varepsilon\} \leq P\{\|S_n^\delta - A_n\| \geq \varepsilon/2\} + P\left\{\|S_n''\| \geq \varepsilon/2\right\}$$

$$\leq P\{\|S_n^\delta - A_n\| \geq \varepsilon/2\} + \sum_{k=1}^{k_n} P\{\|X_{nk}\| > \delta\}.$$

For the second part, we shall estimate $P\{\|S_n^\delta - A_n\| \geq \varepsilon\}$. (The proof can be also found in [2, p.249]).

$$P\{\|S_n - A_n\| \geq \varepsilon\} \geq P\left\{\|S_n - A_n\| \geq \varepsilon, \sup_{1\leq k\leq k_n} \|X_{nk}\| \leq \delta\right\}$$

$$\geq P\left\{\|S_n^\delta - A_n\| \geq \varepsilon, \sup_{1\leq k\leq k_n} \|X_{nk}\| \leq \delta\right\}$$

$$\geq P\{\|S_n^\delta - A_n\| \geq \varepsilon\} - P\left\{\sup_{1\leq k\leq k_n} \|X_{nk}\| > \delta\right\}.$$

Hence, we have

$$P\{\|S_n^\delta - A_n\| \geq \varepsilon\} \leq P\{\|S_n - A_n\| \geq \varepsilon\} + \sum_{k=1}^{k_n} P\{\|X_{nk}\| > \delta\}$$

3. The main result. In this section, we shall extend Etemadi's criterion on the convergence in probability (see Degenerate convergence criterion in Section 1) to the complete convergence with the rate of convergence. With the preliminaries accounted for, the main theorem can be now presented.

Theorem. *Let $\{(X_{nk},\ 1 \leq k \leq k_n),\ n \geq 1\}$ be an array of rowwise independent random elements and $\{c_n, n \geq 1\}$ be a sequence of positive constants such that $\sum_{n=1}^{\infty} c_n = \infty$. Let $p \geq 1$. The array is infinitesimal and there exists a (nonrandom) sequence $\{A_n\}$ of vectors in B such that $\sum_{n=1}^{\infty} c_n P\{\|S_n - A_n\| > \varepsilon\} < \infty$ for all $\varepsilon > 0$ if and only if there exists $\delta > 0$ such that for all $\varepsilon > 0$:*

(i) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P\{\|X_{nk}\| > \varepsilon\} < \infty$,

(ii) $\sum_{n=1}^{\infty} c_n E\left\|S_n^\delta - A_n\right\|^p I\{\left\|S_n^\delta - A_n\right\| > \varepsilon\} < \infty$,

where $S_n^\delta = \sum_{k=1}^{k_n} X_{nk} I\{\|X_{nk}\| \leq \delta\}$.

A_n may be taken as ES_n^δ. Futhermore, if (ii) *is true for some $\delta > 0$, it is true for all $\delta > 0$.*

Proof. First of all, in the case $k_n = \infty$, we assume that series $\sum_{k=1}^{k_n} X_{nk}$ converges a.s.

For $\delta > 0$, from our assumptions and by Chebyshev's inequality, we have

$$P\{\|S_n - A_n\| > \varepsilon\} \leq P\{\|S_n^\delta - A_n\| > \varepsilon\} + P\left\{\sup_{1\leq k\leq k_n} \|X_{nk}\| > \delta\right\}$$

$$\leq P\{\|S_n^\delta - A_n\| I\{\|S_n - A_n\| > \varepsilon\} > \varepsilon\} + \sum_{k=1}^{k_n} P\{\|X_{nk}\| > \delta\}$$

$$\leq E\|S_n^\delta - A_n\|^p I\{\|S_n^\delta - A_n\| > \varepsilon\}/\varepsilon^p + \sum_{k=1}^{k_n} P\{\|X_{nk}\| > \delta\}.$$

Hence the sufficiency can be easily established.

For the necessity first of all we shall estimate the l.h.s. in (i). By Lemma 2 (a), Etemadi's inquality and Lemma 2 (c) we have:

$$\sum_{k=1}^{k_n} P\{\|X_{nk}\| > \varepsilon\} \leq 2\sum_{k=1}^{k_n} P\{\|X_{nk}^s\| > \varepsilon\}$$

$$\leq \frac{2P\{\|S_n - A_n\| > \varepsilon/8\}}{1 - 8P\{\|S_n - A_n\| > \varepsilon/8\}}$$

Since $\sum_{n=1}^\infty c_n P\{\|S_n - A_n\| > \varepsilon\} < \infty$ and $\sum_{n=1}^\infty c_n = \infty$, it follows that $P\{\|S_n - A_n\| > \varepsilon\} \to 0$. Let N be so large that for all $n > N$:

$$P\{\|S_n - A_n\| > \varepsilon\} < 1/16.$$

Therefore, $\sum_{k=1}^{k_n} P\{\|X_{nk}\| > \varepsilon\} \leq 4P\{\|S_n - A_n\| > \varepsilon/8\}$.

In order to estimate the l.h.s. in (ii) we mention that by Lemma 3 and assumption (i) we have $\sum_{n=1}^\infty c_n P\{\|S_n^\delta - A_n\| > \varepsilon\} < \infty$ and $S_n^\delta - A_n \to 0$ in probability. By Lemma 2 (b), Lemma 1 and Lemma 2 (c):

$$E\left\|S_n^\delta - A_n\right\|^p I\{\left\|S_n^\delta - A_n\right\| > \varepsilon\}$$

$$= \int_\varepsilon^\infty P\{\left\|S_n^\delta - A_n\right\| \geq t\}\ dt^p + \varepsilon^p P\{\|S_n^\delta - A_n\| > \varepsilon\}$$

So, it is sufficient to estimate the integral $\int_\varepsilon^\infty P\{\left\|S_n^\delta - A_n\right\| \geq t\}\ dt^p$

$$\leq 2^{p+1}\int_{\varepsilon/2}^\infty P\{\left\|S_n^{\delta^s}\right\| \geq t\}\ dt^p \leq 2^{p+1} E\left\|S_n^{\delta^s}\right\|^p I\{\left\|S_n^{\delta^s}\right\| \geq \varepsilon/2\}$$

$$\leq 2^{p+1}\left(6\delta^p P\{\|Y_{nk}^s\| > \varepsilon/6\} + \varepsilon^p P\left\{\|S_n^{\delta^s}\| > \varepsilon/6\right\}\right).$$

$$\leq 2^{p+2}\left(6\delta^p P\{\|X_{nk}\| > \varepsilon/6\} + \varepsilon^p P\{\|S_n^\delta - A_n\| > \varepsilon/6\}\right).$$

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