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Almost Sure Convergence of Projections in L_p-Spaces

Dedicated to Professor Dominik Szynal
on the occasion of his 60th birthday

Abstract. The paper is devoted to the analysis of pointwise convergence of sequences of projections in L_p-spaces. Also some approximation problems for the operators in L_2-spaces are discussed.

1. Monotone sequences of projections in Banach spaces are important objects in both classical and functional analysis. Pointwise convergence theorems for the Fourier expansions with respect to general or special orthonormal systems of functions or martingale convergence theorems are typical classical examples concerning such sequences. One can say that all other results on the pointwise convergence of (monotone) sequences of projections are more or less connected with theorems just mentioned.

This paper is devoted to the analysis of the almost sure convergence of sequences of projections in L_p-spaces. The last sections of the paper are a survey of some results obtained recently by the authors. They concern several special problems arising in L_2 for sequences of orthogonal projections.

In the sequel, we shall also consider an 'unbounded' situation. That is why we adopt the following general definition.

2. Definition. Let (A_n) be a sequence of bounded linear operators in L_p over a probability space, say $(\Omega, \mathcal{F}, \mu)$, and let A be linear (bounded or not). We say that (A_n) converges to A *almost surely* ($A_n \to A$ a.s.) if $A_n f \to Af$ μ-almost everywhere, for all $f \in D(A)$.

Let us start with some generalization of pointwise convergence theorems for conditional expectations.

A natural and important generalization of the classical martingale convergence theorem was obtained by E. Stein [17] who proved the following result.

3. Theorem. *Let (P_n) be an increasing sequence of positive orthogonal projections on $L_2(\Omega, \mathcal{F}, \mu)$. Then (P_n) converges almost surely to its strong limit P.*

The original proof of E. Stein was complicated. Very short and elegant proof was found by R. Duncan [8].

We say that an operator T acting in $L_p(\Omega, \mathcal{F}, \mu)$ is a *positive contractive projection* (p.c.p.) if

1^0 $Tf \geq 0$ a.e. for $f \geq 0$ a.e.,
2^0 $\|T\|_p \leq 1$,
3^0 $T^2 = T$.

A sequence (T_n) of p.c.p. operators is increasing (decreasing, resp.) when $T_n T_m = T_{n \wedge m}$ ($T_n T_m = T_{n \vee m}$, resp.) for all $n, m \in \mathbb{N}$.

4. Theorem. (Martingale-type convergence theorem). *Let $L_p = L_p(\Omega, \mathcal{F}, \mu)$ with $p \geq 1$. Assume that (T_n) is an increasing sequence of p.c.p. operators in L_p. Then $T_n \to T$ a.s. where T is the limit of (T_n) in the strong operator topology.*

Our proof of the above theorem is different for $p > 1$ and for $p = 1$.

In the case $p > 1$ the argument is based on the famous theorem of Akcoglu [1]. We just reduce the problem to the Akcoglu's maximal inequality via the result of Neveu [15] on the connection between ergodic theory and martingales. In the case $p = 1$ we use the structure of positive contractive projections in L_1 and submartingale convergence theorem. Let us remark that in both cases i.e. simply for $p \geq 1$ it is possible to use the characterization of contractive projections in L_p [2, 7, 13]. In our argument we do not use this characterization, being rather advanced, and give a direct simple proof. This is possible because of the *positivity of projections* under consideration.

Let us fix $p > 1$. In this case the proof is based on the following results.

(A) (Theorem of Akcoglu [1]). If $T\colon L_p \to L_p$ is a positive contraction, $p > 1$, then the Dominated Ergodic Estimate holds for T. That is

$$\|Mf\|_p \le \frac{p}{p-1}\|f\|_p, \qquad \text{for each } f \in L_p,$$

where

$$(Mf)(\omega) = \sup_{n \ge 1}\left|\frac{1}{n}\sum_{k=0}^{n-1}(T^k f)(\omega)\right|.$$

(B) Theorem of Neveu [15]. Let (T_n) be p.c.p. operators in L_p, $p \ge 1$. Assume that the sequence (T_n) is decreasing. Let (a_n) be a sequence such that

$$0 = a_0 < a_1 < a_2 < \ldots < a_n < \ldots < 1, \quad \text{and} \quad a_n \to 1.$$

Put $S = \sum_{s=1}^{\infty}(a_s - a_{s-1})T_s$.

Then, obviously, S is a positive contraction in L_p. Moreover, for each $\varepsilon > 0$ one can choose (a_s) in such a way that for some increasing sequence (n_s) of positive integers, we have

$$\sum_{s=1}^{\infty}\left\|\frac{1}{n_s}\sum_{k=0}^{n_s-1}S^k - T_s\right\|_p < \varepsilon. \tag{1}$$

Going back to the proof of our theorem, let us assume that (T_n) is an increasing sequence of p.c.p. operators in L_1.

Let us fix N for a moment and put $T'_n = T_{N-n+1}$, for $n = 1, 2, \ldots, N$, and $T'_n = T_1$, for $n > N$. Then $T'_1 \ge T'_2 \ge \ldots$. Let $0 < \varepsilon_N \to 0$, $\varepsilon_N < 1$, for $N = 1, 2, \ldots$. By (B), for $N = 1, 2, \ldots$, we can choose $(a_s^{(N)})$, $(n_s^{(N)})$ such that for

$$S_N = \sum_{s=1}^{\infty}(a_s^{(N)} - a_{s-1}^{(N)})T'_s,$$

we have

$$\sum_{s=1}^{N}\left\|T'_s - \frac{1}{n_s^{(N)}}\sum_{k=0}^{n_s^{(N)}-1} S_N^k\right\|_p < \varepsilon_N. \tag{2}$$

Let $f \in L_p$, $f \ge 0$. Putting

$$\sigma_{s,N} = \frac{1}{n_s^{(N)}}\sum_{k=0}^{n_s^{(N)}-1} S_N^k f, \qquad \gamma_{s,N} = T'_s f - \sigma_{s,N},$$

we can write $T'_s f = \gamma_{s,N} + \sigma_{s,N}$. Thus $T'_s f \leq g_{s,N} + \sigma_{s,N}$, where $g_{s,N} = |\gamma_{s,N}|$. Consequently, for $f \in L_p$, $f \geq 0$, we have

$$\sup_{1\leq s\leq N} T_s f = \sup_{1\leq s\leq N} T'_s f \leq \sup_{1\leq s\leq N} g_{s,N} + \sup_{1\leq s\leq N} \sigma_{s,N}.$$

By Akcoglu Theorem,

$$\| \sup_{1\leq s\leq N} \sigma_{s,N}\|_p \leq \frac{p}{p-1}\|f\|_p.$$

Moreover, by (2) and since $\varepsilon_N < 1$,

$$\begin{aligned}\| \sup_{1\leq s\leq N} g_{s,N}\|_p^p &= \int \sup_{1\leq s\leq N} (g_{s,N})^p \leq \int \sum_{s=1}^{N} (g_{s,N})^p \\ &= \sum_{s=1}^{N} \int (g_{s,N})^p \leq \sum_{s=1}^{N} \left[\int (g_{s,N})^p\right]^{1/p} = \sum_{s=1}^{N} \|g_{s,N}\|_p < \varepsilon_N .\end{aligned}$$

Finally, we get

$$\| \sup_{1\leq s\leq N} T_s f\|_p \leq \varepsilon_N^{1/p} + \frac{p}{p-1}\|f\|_p, \quad N = 1,2,\ldots .$$

Passing with $N \to \infty$, we obtain

$$\|\sup_{s\geq 1} T_s f\|_p \leq \frac{p}{p-1}\|f\|_p,$$

which means that the Dominate Estimate holds for the sequence (T_s) of our projections. This implies in a standard way, the a.e. convergence of $T_s f$ to Tf, for every $f \in L_p$ (see, for example, [9], Chapter 1]).

Indeed, for functions f of the form $f = f_1 + f_2$, where $f_1 \in \bigcup_{k\geq 1} T_k(L_p)$ and $f_2 \in \bigcap_{k\geq 1} \ker T_k$, obviously, $T_s f \to Tf$ a.e. Clearly, the set of such functions is dense in L_p.

Let us pass to the case $p = 1$. Our proof is mostly based on the analysis of the structure of increasing sequences of p.c.p. operators.

The structure of p.c.p. operators in L_1 is known [2, 7, 13]. Our approach is rather different and seems to be more elementary and better fitting to the situation of positive projections we are just interested in. That is why we reproduce our argument in some details.

Before starting the proof of Theorem 4 we analyse in some details increasing sequences of p.c.p. operators in L_1.

Let us observe that for any positive nilpotent operator N (i.e. satisfying $N^2 f = 0$, $Nf \geq 0$ for $f \geq 0$) and for a set $\Omega_0 = (N1 > 0)$, we have

$$N \mathbf{1}_{\Omega_0^c} = \mathbf{1}_{\Omega_0} N = N. \tag{3}$$

Roughly speaking, any positive nilpotent operator in L_1 'transfers from Ω_0^c into Ω_0'. Indeed, we have, for $f \geq 0$,

$$\mathbf{1}_{(f>0)} = \lim_{n\to\infty} n\Big(\frac{1}{n} \wedge f\Big). \tag{4}$$

Thus, for $A \subset \Omega$, we obtain

$$N\mathbf{1}_A \leq N \lim_{n\to\infty} n\Big(\frac{1}{n} \wedge N1\Big) \leq \lim_{n\to\infty} nN^2 1 = 0.$$

Obviously, the set Ω_0 in (3) is not uniquely determined by N.

It is natural to distinguish a class of regular p.c.p. operators. We adopt the following definition.

5. Definition. Let $\Omega_0 \in \mathcal{F}$. We say that a p.c.p. operator T is Ω_0-regular if $\Omega_0 = (T1 > 0)$ and $T\mathbf{1}_{\Omega_0^c} = 0$. Ω-regular T is said to be regular.

Obviously, $T1 = T\mathbf{1}_{\Omega_0}$ for any Ω_0-regular T.

Clearly, any p.c.p. Ω_0-regular operator can be identified with the regular p.c.p. operator acting in $L_1(\Omega_0, \mathcal{F}_0, \mu_0)$ where $\mathcal{F}_0$ (μ_0, resp.) is restriction of $\mathcal{F}$ (μ, resp.) on Ω_0.

Let us remark that for any p.c.p. Ω_0-regular operator $T^{(r)}$ and any positive contractive nilpotent operator N satisfying (3) and

$$N(L_1) \subseteq T^{(r)}(L_1), \tag{5}$$

the sum $T = T^{(r)} + N$ is a p.c.p. operator.

We are in a position to formulate the following representation theorem

6. Proposition. *Every p.c.p. operator in L_1 is the sum*

$$T = T^{(r)} + N,$$

where $T^{(r)}$ is an Ω_0-regular p.c.p. operator and N is a positive contractive nilpotent operator satisfying (3) and (5) (with $\Omega_0 = (T1 > 0)$). Moreover, the regular part $T^{(r)}$ is of the form

$$T^{(r)} f = \varphi \mathbb{E}^{\mathfrak{A}} f, \tag{6}$$

with a σ-field $\mathfrak{A} \subset \mathcal{F}_0$ such that Ω_0^c is its atom and a function φ satisfying $(\varphi > 0) = \Omega_0$ and $\mathbb{E}^{\mathfrak{A}}\varphi = 1$. Then $\varphi = T\mathbf{1}_{\Omega_0}$ and

$$\mathfrak{A} = \{A \in \mathcal{F} : T(\mathbf{1}_A\varphi) = \mathbf{1}_A\varphi\}.$$

Proof. Let as previously $\Omega_0 = (T1 > 0)$. Put

$$Nf = T\mathbf{1}_{\Omega_0^c}f, \quad T^{(r)}f = T\mathbf{1}_{\Omega_0}f,$$

for $f \in L_1(\Omega, \mathcal{F}, \mu)$. Then N is a positive contraction and, for $A \in \mathcal{F}$,

$$NN\mathbf{1}_A \le T\mathbf{1}_{\Omega_0^c}T1 \le T\mathbf{1}_{\Omega_0^c}(\mathbf{1}_{\Omega_0}T1) = 0,$$

so $N^2 = 0$. Obviously, $T^{(r)}\mathbf{1}_{\Omega_0^c} = 0$, $(T^{(r)}\mathbf{1}_{\Omega_0} > 0) = (T^{(r)}1 > 0) = \Omega_0$.

It remains to show that $T^{(r)}$ is of form (6). Clearly, it is enough to consider the case $\Omega_0 = \Omega$.

Let us assume that T is a regular p.c.p. operator and $\varphi = T1$. Then $\mu(\varphi > 0) = 1$.

Let us put $\mathfrak{A} = \{A \in \mathcal{F} : T(\varphi \cdot \mathbf{1}_A) = \varphi \cdot \mathbf{1}_A\}$. $\mathfrak{A}$ is a σ-field. In fact, let $A, B \in \mathfrak{A}$ which means that $T(\varphi \cdot \mathbf{1}_A) = \varphi \cdot \mathbf{1}_A$ and $T(\varphi \cdot \mathbf{1}_B) = \varphi \cdot \mathbf{1}_B$. But $\varphi \cdot \mathbf{1}_{A\cap B} \le \varphi \cdot \mathbf{1}_A$, so $T(\varphi \cdot \mathbf{1}_{A\cap B}) \le T(\varphi \cdot \mathbf{1}_A) = \varphi \cdot \mathbf{1}_A$. Similarly, $T(\varphi \cdot \mathbf{1}_{A\cap B}) \le \varphi \cdot \mathbf{1}_B$. Consequently, $T(\varphi \cdot \mathbf{1}_{A\cap B}) \le (\varphi \cdot \mathbf{1}_A) \wedge (\varphi \cdot \mathbf{1}_B) = \varphi(\mathbf{1}_A \wedge \mathbf{1}_B) = \varphi \cdot \mathbf{1}_{A\cap B}$. On the other hand, since T preserves the integral, we have $\int T(\varphi \cdot \mathbf{1}_{A\cap B}) = \int \varphi \cdot \mathbf{1}_{A\cap B}$, thus $T(\varphi \cdot \mathbf{1}_{A\cap B}) = \varphi \cdot \mathbf{1}_{A\cap B}$ which means that $A \cap B \in \mathfrak{A}$. The rest is standard.

Now, we observe that for $x \in L_1$ we have $Tx = x$ if and only if x/φ is an $\mathfrak{A}$-measurable function. In fact, let $Tx = x$ and $\alpha \in \mathbb{R}$. Then $(x/\varphi > \alpha) = (x - \alpha\varphi > 0)$ and $T(x - \alpha\varphi) = x - \alpha\varphi$, so it is enough to show $(x > 0) \in \mathfrak{A}$. Let us assume additionally that $x \ge 0$. Using the fact that for every function $z : \Omega \to \mathbb{R}$, $z \ge 0$, $n(z \wedge \frac{1}{n}) \to \mathbf{1}_{(z>0)}$, $n \to \infty$, we get

$$n(x \wedge \varphi/n) = \varphi \cdot n\Big(x/\varphi \wedge \frac{1}{n}\Big) \to \varphi \cdot \mathbf{1}_{(x/\varphi>0)} = \varphi \cdot \mathbf{1}_{(x>0)}, \quad n \to \infty \quad \text{a.e.}\ .$$

But

$$T(x \wedge \varphi/n) = Tx \wedge T(\varphi/n) = x \wedge \varphi/n,$$

so

$$(n(x \wedge \varphi/n) = T(n(x \wedge \varphi/n)) \to T(\varphi \cdot \mathbf{1}_{(x>0)}), \quad n \to \infty.$$

Thus $T(\varphi \cdot \mathbf{1}_{(x>0)}) = \varphi \cdot \mathbf{1}_{(x>0)}$ and consequently, $(x > 0) \in \mathfrak{A}$.

For an arbitrary $x \in L_1$ with $Tx = x$ it suffices to consider the decomposition $x = x^+ - x^-$ (where $x^+ = x \vee 0$) because $Tx^\pm = x^\pm$. Then x/φ is $\mathfrak{A}$-measurable.

Conversely, if x/φ is $\mathfrak{A}$-measurable, then x/φ is the a.e.-limit of simple functions of the form $\sum_k \alpha_k \mathbf{1}_{A_k}$ with $A_k \in \mathfrak{A}$. Thus x is the a.e.-limit of the functions $\sum_k \alpha_k \cdot \varphi \cdot \mathbf{1}_{A_k} = T(\sum_k \alpha_k \cdot \varphi \cdot \mathbf{1}_{A_k})$, so $x = Tx$.

Since $T = T^2$, the above observation implies that Tx/φ is $\mathfrak{A}$-measurable for $x \in L_1$.

Next, let us remark that

(7) $$T\mathbf{1}_A = \varphi \cdot \mathbf{1}_A \qquad \text{for } A \in \mathfrak{A}.$$

In fact, for suitable $0 < \alpha \searrow 0$, $B \nearrow A$, $B \in \mathfrak{A}$, we have $\alpha \mathbf{1}_B \leq \varphi \cdot \mathbf{1}_A$. Thus, $T\mathbf{1}_B \leq \frac{1}{\alpha} T(\varphi \cdot \mathbf{1}_A) = \frac{1}{\alpha}\varphi \cdot \mathbf{1}_A$. But $T\mathbf{1}_B \leq \varphi$, so $T\mathbf{1}_B \leq \varphi \wedge \frac{1}{\alpha}\varphi \cdot \mathbf{1}_A \leq \varphi \cdot \mathbf{1}_A$ for sufficiently small α. Hence $T\mathbf{1}_A \leq \varphi \cdot \mathbf{1}_A$. If the sharp inequality $T\mathbf{1}_A < \varphi \cdot \mathbf{1}_A$ were true on some set C of positive measure then, replacing A by A^c, we would have $\varphi = T1 < \varphi$ on C, a contradiction. Thus, we get $T\mathbf{1}_A = \varphi \mathbf{1}_A$.

The last equality leads immediately to

$$\int_A 1 = \int T(\mathbf{1}_A) = \int \varphi \cdot \mathbf{1}_A = \int_A \varphi, \quad \text{for } A \in \mathfrak{A}$$

which means $\mathbb{E}^{\mathfrak{A}}\varphi = 1$.

Now, let us notice that, for $0 \leq x \leq \varphi$, we have

(8) $$T(x\mathbf{1}_A) = (Tx)\mathbf{1}_A \qquad \text{for } A \in \mathfrak{A}.$$

Indeed, $x \cdot \mathbf{1}_A \leq \varphi \cdot \mathbf{1}_A$ implies $T(x \cdot \mathbf{1}_A) \leq T(\varphi \cdot \mathbf{1}_A)$ and $x \cdot \mathbf{1}_A \leq x$ implies $T(x \cdot \mathbf{1}_A) \leq Tx$. Consequently $T(x \cdot \mathbf{1}_A) \leq Tx \wedge \varphi \cdot \mathbf{1}_A = (Tx \wedge \varphi) \cdot \mathbf{1}_A = (Tx) \cdot \mathbf{1}_A$. This implies (8) in the same way as in the proof of formula (7). Let us notice that the assumption $x \leq \varphi$, used in the proof of (8), is not essential because for $x \geq 0$ one can easily find a sequence (y_s) with $0 \leq y_s \leq \varphi$ and such that $x = \sum_s y_s$.

Finally, for $x \geq 0$, $A \in \mathfrak{A}$ we obtain by (8)

$$\begin{aligned}\int_A x = \int T(x\mathbf{1}_A) = \int (Tx) \cdot \mathbf{1}_A = \int \frac{Tx}{\varphi} \cdot \varphi \cdot \mathbf{1}_A = \int \mathbb{E}^{\mathfrak{A}}\Big(\frac{Tx}{\varphi} \cdot \varphi \cdot \mathbf{1}_A\Big) \\ = \int \frac{Tx}{\varphi} \cdot \mathbf{1}_A \cdot \mathbb{E}^{\mathfrak{A}}\varphi = \int_A \frac{Tx}{\varphi}.\end{aligned}$$

Hence

$$\int_A f = \int_A \frac{Tf}{\varphi} \quad \text{for } f \in L_1$$

which means that $\mathbb{E}^{\mathfrak{A}} f = \dfrac{Tf}{\varphi}$ for $f \in L_1$. ■

We conclude this section describing the inequality $T_1 \leq T_2$ for p.c.p. operators.

7. Proposition. *For any two regular p.c.p. operators in L_1, say $T_1 = \varphi_1\mathbb{E}^{\mathfrak{A}_1}$, $T_2 = \varphi_2\mathbb{E}^{\mathfrak{A}_2}$, the following are equivalent*

(i) $T_1 \le T_2$ *(that is* $\ker T_2 \subseteq \ker T_1$ *and* $T_1(L_1) \subseteq T_2(L_1)$*, or, equivalently,* $T_1T_2 = T_2T_1 = T_1$*);*

(ii) $T_1(L_1) \subseteq T_2(L_1)$*;*

(iii) $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$, $\varphi_2 = \frac{\varphi_1}{\mathbb{E}^{\mathfrak{A}_2}\varphi_1}$.

Proof. (i) $\Rightarrow$ (ii) obvious.

(ii) $\Rightarrow$ (iii). Clearly $T_1 1 = \varphi_1 \in T_1(L_1) \subset T_2(L_1)$, so $\varphi_1 = T_2\varphi_1 = \varphi_2\mathbb{E}^{\mathfrak{A}_2}\varphi_1$. To prove the inclusion $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$, let us take $A \in \mathfrak{A}_1$. That means that $\varphi_1 \mathbf{1}_A = T_1(\varphi_1\mathbf{1}_A)$ and (ii) implies $\varphi_1\mathbf{1}_A \in T_2(L_1)$. Thus $\varphi_1\mathbf{1}_A = \varphi_2\mathbb{E}^{\mathfrak{A}_2}g$, for some g. Consequently,

$$\mathbf{1}_A = \frac{\varphi_2}{\varphi_1}\mathbb{E}^{\mathfrak{A}_2}g = \frac{1}{\mathbb{E}^{\mathfrak{A}_2}\varphi_1}\mathbb{E}^{\mathfrak{A}_2}g,$$

so $\mathbf{1}_A$ is $\mathfrak{A}_2$-measurable.

(iii) $\Rightarrow$ (i). Under asssumption (iii)

$$T_1T_2f = \varphi_1\mathbb{E}^{\mathfrak{A}_1}(\varphi_2\mathbb{E}^{\mathfrak{A}_2}f) = \varphi_1\mathbb{E}^{\mathfrak{A}_1}\mathbb{E}^{\mathfrak{A}_2}(\varphi_2\mathbb{E}^{\mathfrak{A}_2}f) = \varphi_1\mathbb{E}^{\mathfrak{A}_1}f = T_1f,$$

since $\mathbb{E}^{\mathfrak{A}_2}\varphi_2 = 1$,

$$T_2T_1f = \varphi_2\mathbb{E}^{\mathfrak{A}_2}(\varphi_1\mathbb{E}^{\mathfrak{A}_1}f) = \frac{\varphi_1}{\mathbb{E}^{\mathfrak{A}_2}\varphi_1}(\mathbb{E}^{\mathfrak{A}_2}\varphi_1)(\mathbb{E}^{\mathfrak{A}_1}f) = T_1f.$$

It is worth noting that if p.c.p. operators T_1, T_2 are not regular, then the inclusion $T_1(L_1) \subseteq T_2(L_1)$ does not imply the inequality of projections $T_1 \le T_2$. ■

By Proposition 6, every p.c.p. operator T is of the form

$$T = \varphi\mathbb{E}^{\mathfrak{A}}(\mathbf{1}_{\Omega_0}\cdot) + N(\mathbf{1}_{\Omega_0^c}\cdot),$$

so we can write shortly $T = (\Omega_0, \varphi, \mathfrak{A}, N)$.

In the proof of a strong limit theorem for increasing sequence of projections we shall use the following consequence of the inequality between two p.c.p. operators.

8. Proposition. *Let $T_s = (\Omega_s, \varphi_s, \mathfrak{A}_s, N_s)$, $s = 1, 2$ be two arbitrary p.c.p. operators. Then we have that*

(A) *The inclusion $T_1(L_1) \subseteq T_2(L_1)$ implies the following conditions*

1^0 $\Omega_1 \subseteq \Omega_2$;

2^0 $\Omega_1 \in \mathfrak{A}_2$;

3^0 $\mathbf{1}_{\Omega_1}\varphi_2 = \mathbf{1}_{\Omega_1}\frac{\varphi_1}{\mathbb{E}^{\mathfrak{A}_2}\varphi_1}$;

4^0 $\mathfrak{A}_1 \subset \mathfrak{A}_2 \cap \Omega_1$.

(B) *The inequality* $T_1 \le T_2$ *additionally implies*

5^0 $\mathbb{E}^{\mathfrak{A}_1} N_1 \mathbf{1}_{\Omega_2^c} f \ge \mathbb{E}^{\mathfrak{A}_1}(\mathbf{1}_{\Omega_1} N_2 f)$.

Proof. Assume that $T_1(L_1) \subseteq T_2(L_1)$. Then we have

1^0 $\Omega_1 = (T_1 1 > 0) = (T_2 T_1 1 > 0) = (\varphi_2 \mathbb{E}^{\mathfrak{A}_2}(T_1 1) > 0) \subseteq (\varphi_2 > 0) = \Omega_2$.

2^0 $T_1 1 = \varphi_2 \mathbb{E}^{\mathfrak{A}_2}(T_1 1)$, $(\varphi_2 > 0)\Omega_2 \supset \Omega_1$, so $(T_1 1 > 0) = (\mathbb{E}^{\mathfrak{A}_2}(T_1 1) > 0) \in \mathfrak{A}_2$.

3^0 Obviously, $T_2 T_1 = T_1$. Thus $\varphi_2 \mathbb{E}^{\mathfrak{A}_2}(\varphi_1 \mathbb{E}^{\mathfrak{A}_2} \mathbf{1}_{\Omega_1}) = \varphi_1 \mathbb{E}^{\mathfrak{A}_1} \mathbf{1}_{\Omega_1}$ so $\mathbf{1}_{\Omega_1} \varphi_2 \mathbb{E}^{\mathfrak{A}_2} \varphi_1 = \mathbf{1}_{\Omega_1} \varphi_1$.

4^0 Let $A \in \mathfrak{A}_1$, $A \subset \Omega_1$. That means that

$$\mathbf{1}_A \varphi_1 = T_1(\mathbf{1}_A \varphi_1) = T_2(\mathbf{1}_A \varphi_1) = \varphi_2 \mathbb{E}^{\mathfrak{A}_2}(\mathbf{1}_A \varphi_1).$$

Consequently,

$$\mathbf{1}_A = \frac{\varphi_2}{\varphi_1} \mathbb{E}^{\mathfrak{A}_2}(\mathbf{1}_A \varphi_1) = \frac{1}{\mathbb{E}^{\mathfrak{A}_2} \varphi_1} \mathbb{E}^{\mathfrak{A}_2}(\mathbf{1}_A \varphi_1) \text{ is } \mathfrak{A}_2\text{-measurable.}$$

Part (A) is thus proved.

Inequality 5^0 concerning the nilpotents is a consequence of the additional assumption $\ker T_1 \supseteq \ker T_2$. Indeed, for $f \ge 0$, $(f > 0) \subset \Omega_2^c$, we have $N_2 f = T_2 f$ and $T_2(-N_2 f + f) = -N_2 f + T_2 f = 0$. In consequence,

$$\begin{aligned} 0 &= T_1(-N_2 f + f) = -T_1(\mathbf{1}_{\Omega_1} N_2 f + \mathbf{1}_{\Omega_1^c \cap \Omega_2} N_2 f) + T_1 f) \\ &= -T_1^{(r)} \mathbf{1}_{\Omega_1} N_2 f - N_1 \mathbf{1}_{\Omega_1^c \cap \Omega_2} N_2 f + N_1 f \\ &= -\varphi_1 \mathbb{E}^{\mathfrak{A}_1} \mathbf{1}_{\Omega_1} N_2 f - N_1 N_2 f + N_1 f. \end{aligned}$$

Thus, as $N_1 \cdot = (\varphi_1 \mathbb{E}^{\mathfrak{A}_1} \mathbf{1}_{\Omega_1}) N_1 \cdot$, we get 5^0. ∎

Remark. Actually, we have proved the following characterizations (which are interesting themselves, though they will not be used.

(A′) The inclusion $T_1(L_1) \supseteq T_2(L_1)$ is equivalent to 1^0–4^0.

(B′) The inequality $T_1 \le T_2$ is equivalent to 1^0–4^0 and

$$(5^{00}) \qquad \mathbb{E}^{\mathfrak{A}_1} N_1 \mathbf{1}_{\Omega_2^c} f = \mathbb{E}^{\mathfrak{A}_1} \mathbf{1}_{\Omega_1} N_2 f + \mathbb{E}^{\mathfrak{A}_1} N_1 N_2 f.$$

This can be obtained by the use of Theorem 1 and the implication

$$\mathbf{1}_{\Omega_1^c \cap \Omega_2} f \in \ker T_2 \quad \text{implies} \quad \mathbf{1}_{\Omega_1^c \cap \Omega_2} f = 0.$$

Now we go back to the proof of Theorem 4 in the case $p = 1$.

First, let us remark that our theorem holds for the *regular* p.c.p. operators. Indeed, in this case, by Proposition 8 we have that

$$T_n f = \varphi_n \mathbb{E}^{\mathfrak{A}_n} f,$$

with an increasing sequence $(\mathfrak{A}_n)$ of σ-fields and $\varphi_n = \frac{\varphi_1}{\mathbb{E}^{\mathfrak{A}_n}\varphi_1}$. It is enough to apply the martingale convergence theorem.

Now, let $T_1 \leq T_2 \leq \ldots$ be p.c.p. operators. Let, according to Proposition 6, $\Omega_n = (T_n 1 > 0)$ and

$$T_n = T_n^{(r)} + N_n,$$

$$T^{(r)} f = \varphi_n \mathbb{E}^{\mathfrak{A}_n} f$$

with $(\varphi_n > 0) = \Omega_n \in \mathfrak{A}_n$, $\mathbb{E}^{\mathfrak{A}_n}\varphi_n = 1$. Moreover,

$$N_n = \mathbf{1}_{\Omega_n} N_n \mathbf{1}_{\Omega_n^c}.$$

Then, by Proposition 8, $\Omega_1 \subseteq \Omega_2 \subseteq \ldots$. Let us fix arbitrary $n_0 \geq 1$. In the sequel always $n \geq n_0$ and f denotes a fixed positive function. Obviously, $\mathbf{1}_{(\bigcup_{n\geq 1}\Omega_n)^c} T_n f \leq \mathbf{1}_{\Omega_n^c} T_n f = 0$. It is enough to show that $(\mathbf{1}_{\Omega_{n_0}} T_n f)$ converges a.e.

Step 1. $\mathbf{1}_{\Omega_{n_0}} T_n^{(r)} f$ converges a.e. Indeed, for $n \geq n_0$, $\Omega_{n_0} \in \mathfrak{A}_n$ by Proposition 8. This implies that the operators

$$\mathbf{1}_{\Omega_{n_0}} \varphi_n \mathbb{E}^{\mathfrak{A}_n}, \qquad n \geq n_0, \tag{9}$$

are Ω_{n_0}-regular p.c.p. Obviously, they can be treated as regular operators acting in the space $L_1(\Omega_{n_0}, \mathcal{F}_{n_0}, \mu_{n_0})$ being the restriction of $L_1(\Omega, \mathcal{F}, \mu)$ on Ω_{n_0}. By Proposition 7, sequence (9) is increasing in $L_1(\Omega_{n_0})$ so it converges μ-almost everywhere.

Step 2. $\mathbf{1}_{\Omega_{n_0}} N_n f$ converges a.e. Indeed, for a sequence of functions

$$\xi_n = \mathbb{E}^{\mathfrak{A}_n} \mathbf{1}_{\Omega_{n_0}} N_n f, \qquad n \geq n_0, \quad f \geq 0,$$

defined on Ω_{n_0}, one has, by Proposition 8,

$$\begin{aligned}\mathbb{E}^{\mathfrak{A}_n} \xi_{n+1} &= \mathbf{1}_{\Omega_{n_0}} \mathbb{E}^{\mathfrak{A}_n} N_{n+1} f \leq \mathbf{1}_{\Omega_{n_0}} \mathbb{E}^{\mathfrak{A}_n} N_n \mathbf{1}_{\Omega_{n+1}^c} f \\ &\leq \mathbf{1}_{\Omega_{n_0}} \mathbb{E}^{\mathfrak{A}_n} N_n f = \xi_n.\end{aligned}$$

Consequently, (ξ_n) converges a.e. as a supermartingale. Thus the sequence $\mathbf{1}_{\Omega_{n_0}} N_n f = \varphi_n \mathbb{E}^{\mathfrak{A}_n} \mathbf{1}_{\Omega_{n_0}} N_n f = \varphi_n \xi_n$ converges a.e.

For *decreasing* sequence of p.c.p. operators in L_p the a.s. convergence depends heavily on p. Namely, we have the following result.

9. Theorem. *If (T_n) is a decreasing sequence of p.c.p. operators in L_p $(p>1)$, then $T_n \to T$ a.s. but there exists a decreasing sequence of regular p.c.p. operators in L_1 which does not converge almost surely.*

Proof. The case $p>1$ can be considered in the same way as in the proof of Theorem 4, even easier (because we do not need to pass from increasing (T_n) to decreasing (T_n') to use the result of Neveu).

Thus it remains to construct a suitable decreasing sequence (T_n) of regular p.c.p. operators which does not converge almost surely.

To this end we construct a probability space $(\Omega,\mathcal{F},P)$, a decreasing sequence $(\mathfrak{A}_n)$ of sub-σ-ideals of $\mathcal{F}$ and a sequence (φ_n) of strictly positive measurable functions on $(\Omega,\mathcal{F})$ satisfying the conditions

$$\varphi_n = \frac{\varphi_{n+1}}{\mathbb{E}^{\mathfrak{A}_n}\varphi_{n+1}} \tag{10}$$

in particular,

$$\mathbb{E}^{\mathfrak{A}_n}\varphi_n = 1, \qquad n=1,2,\dots, \tag{11}$$

$$\varphi_n \text{ does not converge } P\text{-a.s.} \tag{12}$$

Then, obviously, it is enough to put $T_nf=\varphi_n\mathbb{E}^{\mathfrak{A}_n}f$ for $f\in L_1(\Omega,\mathcal{F},P)$, because by (10) and (11), (T_n) is decreasing sequence of p.c.p. operators, and by (12) $T_n1=\varphi_n \not\to$ a.s.

By $\mathcal{B}[\alpha,\beta]$ we denote the σ-field of Borel subsets of the interval $[\alpha,\beta]$. For $a\in[0,1]$ we set $B_a=\mathcal{B}[0,\alpha]\cup\{[a,1]\}$. Let us consider a product probability space $(\Omega,\mathcal{F},P)=([0,1],\mathcal{B}[0,1],\lambda)^\infty$, where λ denotes the Lebesgue measure on $[0,1]$.

For a sequence $(n_1,n_2,\dots)$ of positive integers (which will be fixed later) we define a decreasing sequence $(\mathfrak{A}_n)$ of σ-fields by putting

$$\begin{aligned}
\mathfrak{A}_1 &= \frac{B_{n_1-1}}{2n_1}\otimes B_1\otimes B_1\otimes\dots\\
\mathfrak{A}_2 &= \frac{B_{n_1-2}}{2n_1}\otimes B_1\otimes B_1\otimes\dots\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
\mathfrak{A}_{n_1} &= \frac{B_0}{2n_1}\otimes B_1\otimes B_1\otimes\dots\\
\mathfrak{A}_{n_1+1} &= B_0\otimes\frac{B_{n_2-1}}{2n_2}\otimes B_1\otimes\dots\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned} \tag{13}$$

$$\mathfrak{A}_{n_1+n_2} = B_0 \otimes \frac{B_0}{2n_2} \otimes B_1 \otimes \dots$$

$$\mathfrak{A}_{n_1+n_2+1} = B_0 \otimes B_0 \otimes \frac{B_{n_3-1}}{2n_3} \otimes B_1 \otimes \dots$$

....

We define on $(\Omega, \mathcal{F}, P)$ a sequence of measurable functions $(\psi_k(\omega_1, \omega_2, \dots)$, $k = 1, 2, \dots)$ by putting

(14)

$$\psi_1(\omega_1, \omega_2, \dots) = \Big(\mathbf{1}_{\left[0, \frac{n_1-1}{2n_1}\right)} + \alpha_1^{(1)} \mathbf{1}_{\left[\frac{n_1-1}{2n_1}, \frac{n_1}{2n_1}\right)} + \frac{1}{2}\mathbf{1}_{\left[\frac{n_1}{2n_1}, 1\right]}\Big)(\omega_1)$$

....

$$\psi_{n_1}(\omega_1, \omega_2, \dots) = \Big(\mathbf{1}_{\left[0, \frac{0}{2n_1}\right)} + \alpha_{n_1}^{(1)} \mathbf{1}_{\left[0, \frac{1}{2n_1}\right)} + \frac{1}{2}\mathbf{1}_{\left[\frac{1}{2n_1}, 1\right]}\Big)(\omega_1)$$

$$\psi_{n_1+1}(\omega_1, \omega_2, \dots) = \Big(\mathbf{1}_{\left[0, \frac{n_2-1}{2n_2}\right)} + \alpha_{n_1}^{(2)} \mathbf{1}_{\left[\frac{n_2-1}{2n_2}, \frac{n_2}{2n_2}\right)} + \frac{1}{2}\mathbf{1}_{\left[\frac{n_2}{2n_2}, 1\right]}\Big)(\omega_2)$$

....

$$\psi_{n_1+n_2}(\omega_1, \omega_2, \dots) = \Big(\mathbf{1}_{\left[0, \frac{0}{2n_2}\right)} + \alpha_{n_2}^{(2)} \mathbf{1}_{\left[0, \frac{1}{2n_2}\right)} + \frac{1}{2}\mathbf{1}_{\left[\frac{1}{2n_2}, 1\right]}\Big)(\omega_2)$$

$$\psi_{n_1+n_2+1}(\omega_1, \omega_2, \dots) = \Big(\mathbf{1}_{\left[0, \frac{0}{n_3-1}\right)} + \alpha_1^{(3)} \mathbf{1}_{\left[\frac{n_3-1}{2n_3}, \frac{n_3}{2n_3}\right)} + \frac{1}{2}\mathbf{1}_{\left[\frac{n_3}{2n_3}, 1\right]}\Big)(\omega_3)$$

....

We postulate that $\int_\Omega \psi_k \, dp = 1$. It is equivalent to the condition

$$\frac{n_m - i}{2n_m} + \alpha_i^{(m)} \frac{1}{2n_m} + \frac{1}{2}\,\frac{n_m + i - 1}{2n_m} = 1. \tag{15}$$

The coefficients $\alpha_i^{(m)}$ are determined by (15). It can be easily seen that then we have

$$\alpha_i^{(m)} > \frac{n_m}{4} \qquad (i = 1, 2, \dots;\ m = 1, 2, \dots). \tag{16}$$

The form of $\mathfrak{A}_k$, ψ_k and the condition $\int \psi_k = 1$ imply

$$\mathbb{E}^{\mathfrak{A}_k} \psi_k = 1 \qquad (k = 1, 2, \dots). \tag{17}$$

Also we have

$$\psi_k \text{ is } \mathfrak{A}_{k-1} \text{ measurable,} \tag{18}$$

which can be easily checked.

Let us put

$$\varphi_k = \psi_1 \psi_2 \ldots \psi_k. \tag{19}$$

By (17) and (18), we have

$$\mathbf{E}^{\mathfrak{A}_k} \varphi_k = \mathbf{E}^{\mathfrak{A}_k} (\psi_1 \ldots \psi_k) = \mathbf{E}^{\mathfrak{A}_k} \psi_k \mathbf{E}^{\mathfrak{A}_{k-1}} \psi_{k-1} \ldots \mathbf{E}^{\mathfrak{A}_2} \psi_2 \mathbf{E}^{\mathfrak{A}_1} \psi_1 = 1$$

and

$$\frac{\varphi_{k+1}}{\mathbf{E}^{\mathfrak{A}_k} \varphi_{k+1}} = \frac{\psi_1 \ldots \psi_k \psi_{k+1}}{\psi_{k+1} \mathbf{E}^{\mathfrak{A}_k} \varphi_k} = \psi_1 \ldots \psi_k = \varphi_k.$$

Thus the conditions (10)–(12) are satisfied. Since the sequence of σ-fields $(\mathfrak{A}_k)$ is decreasing and the functions (φ_k) satisfy (10)–(12), the operators $T_k(\cdot) = \varphi_k \mathbf{E}^{\mathfrak{A}_k}(\cdot)$ form a decreasing sequence of regular p.c.p. operators.

The sequence (n_m) can be fixed in such a way that $P(\omega : \varphi_k(\omega) \nrightarrow) = 1$. More exactly, we will show that for a suitable $(n_m,\ m = 1, 2, \ldots)$

$$\max_{1 \le i \le n_{m+1}} |\varphi_{n_1+\ldots+n_m}(\omega) - \varphi_{n_1+\ldots+n_m+i}(\omega)| > 1 \tag{20}$$

on the set $Z_m = \left\{\omega \in \Omega : \omega_{m+1} \in [0, 1/2]\right\}$, $m = 1, 2, \ldots$.
Obviously, $P(Z_m) = 1/2$ and the cylinders Z_m are independent.

Putting $Z = \limsup Z_m$, by the Borel-Cantelli Lemma we have $P(Z) = 1$, and $\varphi_n(\omega)$ does not satisfy the Cauchy condition on Z.

The sequence (n_m) will be defined by induction. Let $n_1 = 1$. Assume that we have already fixed $n_1, n_2, \ldots, n_{\bar{m}}$ in such a way that (20) holds on the set Z_m, for $m = 1, \ldots, \bar{m} - 1$. We put $\beta(\omega) = \min_\omega \varphi_{n_1+\ldots+n_{\bar{m}}}(\omega)$. Note that $\beta > 0$ a.e. and take $n_{\bar{m}+1}$ large enough to have that

$$\left(\frac{n_{\bar{m}+1}}{4} - 1\right)\beta > 1 \quad \text{a.e.} \tag{21}$$

We shall prove (20) for $m = \bar{m}$. Indeed, let us take $\omega_{\bar{m}+1} \in [0, 1/2]$. In the situation when $\omega_{\bar{m}+1} \in \left(\frac{n_{\bar{m}+1}-i}{2n_{\bar{m}+1}}, \frac{n_{\bar{m}+1}-i+1}{2n_{\bar{m}+1}}\right)$, we have

$$\begin{aligned}
&|\varphi_{n_1+\ldots+n_{\bar{m}}}(\omega) - \varphi_{n_1+\ldots+n_{\bar{m}}+i}(\omega)| \\
&= \varphi_{n_1+\ldots+n_{\bar{m}}}(\omega)|1 - (\psi_{n_1+\ldots+n_{\bar{m}}+1}(\omega)\psi_{n_1+\ldots+n_{\bar{m}}+2}(\omega) \ldots \psi_{n_1+\ldots+n_{\bar{m}}+i}(\omega)| \\
&= \varphi_{n_1+\ldots+n_{\bar{m}}}(\omega)|1 - \underbrace{1 \ldots 1}_{(i-1)\text{ times}} \alpha_i^{\bar{m}+1}| > \beta(\alpha_i^{\bar{m}+1} - 1) > 1
\end{aligned}$$

by (16) and (21). Thus (20) holds for $m = \bar{m}$. The proof is completed. ■

In [11] we described all possible quasi-strong limits of monotone sequences of projections in a Banach space. These limits are always some idempotent operators unbounded, in general. In the case of almost sure convergence, sequences of projections may converge to operators of a very general form. For example the following theorem holds.

10. Theorem [11]. *Let $X = L_2(\Omega, \mathcal{F}, \mu)$ be a separable Hilbert space such that $0 < \mu(Z_n) \to 0$ for some $(Z_n) \subset \mathcal{F}$. Let A be an unbounded closed and densely defined operator in X. Then there exists an increasing sequence (S_n) of finite-dimensional projections in X such that (S_n) converges almost surely to A.*

Obviously, the projections in the above theorem are not selfadjoint, in general. The situation is drastically different if we want to approximate the operators in L_2 by the orthogonal projections. As an example let us consider an unbounded positive selfadjoint operator A in $X = (\Omega, \mathcal{F}, \mu)$. Let $A = \int_0^\infty \lambda e(d\lambda)$ be its spectral representation.

11. Theorem [10, 12]. *The following conditions are equivalent*

(i) *there exists a sequence (P_n) of orthogonal projections and positive coefficients $\Lambda_n \nearrow \infty$, such that $\Lambda_n P_n \to A$ a.s.;*

(ii) *for every $\varepsilon > 0$ and $m > 0$, there exists a normalized vector $f \in X$ such that $f \in e[m, \infty)(X)$ and $\mu(\omega \in \Omega : |f(\omega)| > \varepsilon) < \varepsilon$.*

In condition (i) *finite-dimensional projections P_n can be taken.*

In the last theorem the sequence (P_n) is not monotone.

The proofs of two above results concerning the a.s. approximation of linear operators in L_2 are based, among others, on the following general theorem.

12. Theorem [4, 6, 10]. *Let (A_n) be a sequence of finite dimensional operators acting in $X = L_2(\Omega, \mathcal{F}, \mu)$, satisfying condition*

$$(*) \qquad \text{there exists } (Y_n) \subset \mathcal{F} \quad \text{with } 0 < \mu(Y_n) \to 0.$$

Assume that $A_n \to A$ in the strong operator topology. Then there exists an increasing sequence $(n(s))$ of indices such that $A_{n(s)} \to A$ a.s.

Proof. The theorem is in fact a consequence of the existence in X of increasing sequence of finite dimensional orthogonal projections P_n tending to $\mathbf{1}$ strongly and almost surely as $n \to \infty$. Namely, $B_n = A_n - P_n A \to 0$ strongly and B_n are finite dimensional. Moreover, one can define, by induction, sequences $n(s) \nearrow \infty$, $t(s) \nearrow \infty$ satisfying $t(1) = 1$ and

$$\|B_{n(s)} P_{t(s)}\| < 2^{-s}, \qquad \|B_{n(s)} P_{t(s+1)}^{\perp}\| < 2^{-s}.$$

Then

$$B_{n(s)}f = B_{n(s)}P_{t(s)}f + B_{n(s)}(P_{t(s+1)} - P_{t(s)})f + B_{n(s)}P_{t(s+1)}^{\perp}f$$
$$= \pi_s^{(1)} + \pi_s^{(2)} + \pi_s^{(3)}$$

and

$$\sum_{s=1}^{\infty}\|\pi_s^{(1)}\|^2 < \infty, \quad \sum_{s=1}^{\infty}\|\pi_s^{(3)}\|^2 < \infty, \quad \sum_{n=1}^{\infty}\|\pi_s^{(2)}\|^2 \leq \max_n \|B_n\|^2\|f\|^2 < \infty.$$

Thus $B_{n(s)} \to 0$ a.s. Consequently, $A_{n(s)} \to A$ a.s.

The above theorem can and should be treated as an extension of the following classical theorem of Marcinkiewicz [14].

13. Theorem [14]. *Let (φ_n) be an orthonormal system in $L_2(0,1)$. Put $P_n = \sum_{k=1}^{n}\langle\cdot,\varphi_k\rangle$. Then there exists an increasing sequence $n(k)$ such that $P_{n(k)} \to P$ a.s., P being the strong limit of P_n.*

The proof of Theorem 12 seems to be as short as possible. In comparison with the original proof of Marcinkiewicz [14] and the reasoning of the authors [2], it is much simpler.

It should be stressed here that the assumption in Theorem 12 that the operators A_n are finite dimensional cannot be omitted. Namely, one can construct a sequence (P_n) of orthogonal projections in $L_2(0,1)$ increasing to the identity and such that, for any increasing sequence $(n(s))$ of indices, $(P_{n(s)})$ does not converge almost surely [5].

The counterexample just mentioned has an interesting implication in the ergodic theory. Namely, there exists a unitary operator U in $L_2(0,1)$ such that for every increasing sequence $(n(s))$ of indices, there exists a vector $f \in L_2(0,1)$ such that

$$\frac{1}{n(s)}\sum_{k=1}^{n(s)} U^k f \qquad \text{does not converge a.s. [5].}$$

The assumption on the strong convergence of operators A_n in Theorem 12 cannot be replaced by the assumption that $A_n \to A$ weakly. Indeed, let us consider $H = L_2(-1,1)$ and put $\varphi = 1_{(-1,0)}$, while $\{\varphi_k\}$ is the Rademacher system in $L_2(0,1)$. Put $\psi_k(x) = \frac{1}{\sqrt{2}}(\varphi(x) + \varphi_k(x)), x \in (-1,1)$ (here φ_k equals zero outside the interval $(0,1)$).

Let P (P_k, resp.) stand for the orthogonal projection onto the space generated by φ (ψ_k, resp.), $k = 1,2,\ldots$. Obviously, $P_k \to \frac{1}{2}P$ weakly, as $k \to \infty$.

On the other hand, $P_k\varphi = (\varphi, \psi_k)\psi_k = \psi_k/2$, that is $|(P_k\varphi)(x)| = 1/2$, $x \in (0,1)$, however, $(P\varphi)(x)/2 = 0$, for $x \in (0,1)$. Clearly, for every sequence $\{P_n\} \subset \mathrm{Proj}(H)$, $P_n \to A$ a.s. implies $P_n \to A$ weakly. This implies immediately $P_n \to A$ weakly.

It is worth noting here that from Theorem 12 one can deduce the following corollaries.

14. Corollary. *If $A_n \to A$ in the strong operator topology for some finite dimensional operators A_n in H, then one can choose indices $n(s) \nearrow \infty$ in such a way that $A^k_{n(s)} \to A^k$ a.s. as $s \to \infty$, for any $k = 1, 2, \ldots$.*

Proof is given by diagonal method.

15. Corollary [4]. *Let $0 \le A \le 1$. Then there exists a sequence (P_n) of finite dimensional projections such that $P_n \to A$ a.s.*

16. Corollary [12]. *Let A be a closed densely defined linear operator in H such that, for some finite dimensional A_n, we have $\|A_n f - Af\| \to 0$ for all $f \in \mathcal{D}(A)$. Then $A_{n(s)} \to A$ a.s., for some increasing sequence $(n(s))$.*

Proof. It is enough to consider the operator $B = \int_{[0,\infty)} \min(1, \lambda^{-1}) e(d\lambda)$, where $e(\cdot)$ is the spectral measure of $|A|$, and apply Theorem 11 to the sequence $(A_n B)$.

17. Corollary. *Let A be a normal (unbounded) operator in H. Then there exists a sequence (A_n) of finite dimensional normal operators such that $A^k_n \to A^k$ a.s. as $n \to \infty$, for $k \in \mathbf{Z}$.*

Proof. It is enough to take $A_n = \sum_{s=1}^n A_{ns}$ with finite dimensional normal operators A_{ns} converging strongly to $\int_{(s-1 \le |\lambda| < s)} \lambda e(d\lambda)$ as $n \to \infty$. Then $\|A^k_n f - A^k f\| \to 0$ as $n \to \infty$, for any $f \in \mathcal{D}(A^k)$, $k \in \mathbf{Z}$, and Corollary 16 can be used.

We conclude with few remarks concerning γ-mixing sequences of projections.

Let $0 < \gamma < 1$. A sequence $\{P_n\} \subset \mathrm{Proj}(H)$ is said to be mixing almost surely with the density γ if $P_n@ >$ a.s. $>> \gamma 1$ i.e. $P_n f \to \gamma f$ a.s., for all $f \in H$. In particular, every mixing a.s. with the density γ sequence $\{P_n\}$ is also mixing with the density γ in the sense that $P_n \to \gamma 1$ weakly.

Let us remark that $P_n \xrightarrow{\text{a.s.}} \gamma 1$ implies $P_n f \to \gamma f$ in $L_1(0,1)$ for $f \in H$ (since the functions $f_n = P_n f$ are uniformly integrable).

Let us remark that there are projections P_n such that $P_n \to \gamma 1$ weakly but the sequence $\{P_n\}$ is not mixing almost surely with the density γ. For example, let $\{Z_n\}$ be a sequence of sets which is strongly mixing in the sense of Renyi [16] and put $P_n f = 1_{Z_n} f$ for $f \in H$. Then $P_n f \to \gamma 1$ weakly but $\{P_n\}$ is not mixing a.s. with the density γ. Indeed, if the contrary, then we would have, for $f \in L_\infty$

$$\|P_n f - \gamma f\|_2^2 \leq 2\|f\|_\infty \|P_n f - f\|_1 \to 0$$

which is impossible since $P_n f \to \gamma f$ strongly in H implies that $\gamma = 0$ or 1.

It is easy to give some examples of sequences $\{P_n\} \subset \mathrm{Proj}(H)$ which are mixing with the density γ (i.e. $P_n \to \gamma 1$ weakly) and mixing a.s. with the density γ.

References

[1] Akcoglu, M. A., *A pointwise ergodic theorem in L^p-spaces*, Can. J. Math. **27** (1975), 1075–1082.

[2] Ando, T., *Contractive projections in L_p spaces*, Pac. J. Math. **17** (1966), no. 3, 391–405.

[3] Ando, T. and I. Amemiya, *Almost everywhere convergence of prediction sequence in L_p ($1 < p < \infty$)*, Z. Wahrsch. und Verw. Geb. 4 (1965), 113–120.

[4] Ciach, L. J., R. Jajte, A. Paszkiewicz, *On the almost sure approximation of self-adjoint operators in $L_2(0,1)$*, Proc. Conf. "Ergodic theory and its connections with harmonic analysis, May 24–28, 1993, Alexandria, Egypt (ed. K. Petersen), London Math. Soc. Lecture Notes Series 205, Cambridge Univ. Press (1995), 247-251.

[5] ———, *On the almost sure approximation and convergence of linear operators in L_2-spaces*, Probab. Math. Statist. **15** (1995), 215–225.

[6] ———, *On the almost sure approximation of selfadjoint operators in $L_2(0,1)$*, Math. Proc. Camb. Philos. Soc. **119** (1996), 537–543.

[7] Douglas, R. G., *Contractive projections on an $\mathfrak{L}_1$ space*, Pac. J. Math. **15** (1965), no. 3, 443–462.

[8] Duncan, R., *Some pointwise convergence results in $L^p(\mu)$, $1 < p < \infty$*, Can. Math. Bull. **20** (1977), 277–284.

[9] Garsia, A., *Topics in Almost Everywhere Convergence*, Markham, Chicago, 1970.

[10] Jajte, R., A. Paszkiewicz, *Topics in almost sure approximation of operators in L_2-spaces*, Interaction between functional analysis, harmonic analysis, and probability, ed. N. Kalton, E. Saab, S. Montgomery-Smith, Marcel Dekker, Inc. (1996).

[11] ———, *Convergence of orthogonal series of projections in Banach spaces*, Ann. Polon. Math. **66** (1997), 137–153.

[12] ———, *Almost sure approximation of unbounded operators in $L_2(X, \mathcal{A}, \mu)$* (to appear).

[13] Elton Lacey, H., *The isometric theory of classical Banach spaces*, Die Grundlehren der math. Wissenschaften **208**, Springer-Verlag, Berlin-Heidelberg-New York 1974.

[14] Marcinkiewicz, J., *Sur la convergence de séries orthogonales*, Stud. Math. **6** (1936), 39–45.

[15] Neveu, J., *Deux remarques sur la théorie des martingales*, Z. Wahrsch. und Verv. Gebiete **3** (1964), 122–127.

[16] Renyi, A., *On mixing sequences of sets*, Acta Math. **9** (1959), 1–2.

[17] Stein, E., *Topics in Harmonic Analysis Related to the Littlewood-Paley Theory*, Ann. Math. Studies **63**, Princeton Univ. Press.

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