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On a Stability in Renyi's Sense

ABSTRACT. The aim of this note is to give a construction of the stable sequence of random elements with a given density function.

Let X_S be the set of all random elements (r.e.) defined on a probability space $(\Omega, \mathcal{A}, P)$ with values in a separable, complete metric space (S, ρ), i.e. $X_S = \{X : \Omega \to S;\, X^{-1}(\mathcal{B}) \subset \mathcal{A}\}$, where $\mathcal{B}$ stands for the σ-field generated by the open subsets of S.

Now, let T denote the set of positive integer-valued random variables (r.v.) and $\mathcal{P}(S)$-the set of all probability measures defined on $(S, \mathcal{B})$.

On $\mathcal{P}(S)$ the Levy-Prokhorov metric is defined as follows:

$$L(P_X, P_Y) = \inf\{\varepsilon > 0 : P_X(A) < P_Y(A^\varepsilon)+\varepsilon\,,\ P_Y(A) < P_X(A^\varepsilon)+\varepsilon\,,\ A \in \mathcal{B}\},$$

where P_X is the probability distribution of r.e. X, $A^\varepsilon = \{x : d(x, A) = \inf_{y \in A} \rho(x, y) < \varepsilon\}$, $P_X(B) = P[X \in B]$, $B \in \mathcal{B}$.

It is known [2] that the convergence of a sequence of probability measures in the Levy-Prokhorov metric and the weak convergence of this sequence coincide.

The concept of stability of a sequence of events was introduced by A. Renyi [6]. A survey of stable limit theorems is given in [1].

Definition 1 [6]. The sequence $\{A_n, n \geq 1\}$ of events $(A_n \in \mathcal{A}, n \geq 1)$ is said to be stable, if the limit $\lim_{n\to\infty} P(A_n \cap D) = Q(D)$ exists for every $D \in \mathcal{A}$.

In this case Q is a bounded measure on $(\Omega, \mathcal{A})$ which is absolutely continuous with respect to the measure P and thus $Q(D) = \int_D \alpha dP$, for every $D \in \mathcal{A}$, where $\alpha = \alpha(\omega)$ is a measurable function on Ω such that $0 \leq \alpha(\omega) \leq 1$. The function α is called the local density of the stable sequence of events $\{A_n, n \geq 1\}$.

It is well known [5] that any sample space Ω can be represented as

$$(1)\quad \Omega = D \cup \bigcup_{k=1}^{\infty} D_k\,,\ D_m \cap D_n = \emptyset\,,\ m \neq n\,,\ B \cap B_n = \emptyset\,,\ m = 1, 2, ...,$$

where each D_k is an atom or an empty set and D has the property that, for any given $A \in \mathcal{A}$ such that $A \subset D$ and any ε, $0 < \varepsilon < P(A)$, there exists $C \subset A, C \in \mathcal{A}$, such that $P(C) = \varepsilon$.

Lemma 1. *Let $(\Omega, \mathcal{A}, P) = (<0,1>, \mathcal{B}^0, \nu)$, where $\mathcal{B}^0$–denotes the Borel σ–field of subsets of the interval $< 0, 1 >$ and ν denotes the Lebesgue measure. For every measurable real function α such that $0 \leq \alpha \leq 1$ there exists a stable sequence of events $\{A_n, n \geq 1\}$ such that*

$$\lim_{n\to\infty} \nu(A_n \cap B) = \int_B \alpha d\nu = \mu_B \cdot \nu(B), \text{ where } 0 \leq \mu_B \leq 1.$$

Proof. We put

$$(2)\qquad A_n = \bigcup_{k=0}^{n-1} \left(\frac{k}{n}, \frac{k}{n} + \int_{k/n}^{(k+1)/n} \alpha d\nu \right).$$

Then for every interval $I \subset < 0, 1 >$ we have $\lim_{n\to\infty} \nu(A_n \cap I) = \int_I \alpha d\nu$, which proves that the sequence $\{A_n, n \geq 1\}$ is stable with density α.

Corollary 1. *Let $(\Omega, \mathcal{A}, P)$ be a probability space. If there exists a r.e. $Z : \Omega \to < 0, 1 >$ such that $Z^{-1}(\mathcal{B}^0) = \mathcal{A}$ and $P_Z = \nu$ then for every measurable real function α satisfying $0 \leq \alpha \leq 1$ there exists a stable sequence of events $\{A_n, n \geq 1\}$ for which*

$$\lim_{n\to\infty} P(A_n \cap B) = \int_B \alpha dP = \mu_B \cdot P(B),$$

where $0 \leq \mu_B \leq 1$.

$$P_Z(A) = P[Z \in A] = \nu(A).$$

By Lemma 1 we see that, if α' and α are measurable real functions satisfying $0 \leq \alpha' \leq \alpha \leq 1$ then there exist stable sequences $\{A'_n, n \geq 1\}$

and $\{A_n, n \geq 1\}$ with density α' and α respctively, such that $A'_n \subset A_n$, $n = 1, 2, \ldots$. It is obvious that the sequence $\{A_n \setminus A'_n, n \geq 1\}$ is stable with density $\alpha - \alpha'$.

If α' and α are nonnegative measurable real functions such that $0 \leq \alpha' + \alpha \leq 1$ then there exist stable sequences $\{A'_n, n \geq 1\}$ and $\{A_n, n \geq 1\}$ with density α' and α respectively which satisfy $An' \cap A_n = \emptyset, n = 1, 2, \ldots$.

Definition 2. A sequence $\{X_n, n \geq 1\}$ of r.e. is said to be stable if for every $B \in \mathcal{A}_+ = \{B \in \mathcal{A} : P(B) > 0\}$ there exists a probability measure μ_B, defined on $(S, \mathcal{B})$ such that

$$\lim_{n \to \infty} P([X_n \in A] \mid B) = \mu_B(A) \tag{3}$$

for every $A \in \mathcal{C}_{\mu_B} = \{A \in \mathcal{A} \colon \mu_B(\partial A) = 0\}$ where ∂A denotes the boundary of A.

If $\mu_B(A) = \mu(A)$ for every $B \in \mathcal{A}_+$ and $A \in \mathcal{B}$ then a sequence $\{X_n, n \geq 1\}$ of r.e. is said to be μ-mixing.

Let $Q_A(B) = \mu_B(A)P(B)$. Obviously Q_A is a measure absolutely continuous with respect to P and, by Radon-Nikodym Theorem, there exists a nonnegative function $\alpha_A : \Omega \to \mathbb{R}^+$ such that $Q_A(B) = \int_B \alpha_A dP$. The function α_A is called the density of a stable sequence $\{X_n, n \geq 1\}$.

The set $\mathcal{P}_{\mathcal{A}}(S) = \{\mu_B : B \in \mathcal{A}_+\}$ of all probability measures defined by (3) satisfies the following condition:

$$P\left(\bigcup_{i=1}^{n} B_i\right) \mu_{\cup_{i=1}^n B_i}(A) = \sum_{i=1}^{n} \mu_{B_i}(A) P(B_i)$$

for every $B_i \in \mathcal{A}_+, i = 1, 2, \ldots, n, B_i \cap B_j = \emptyset, i \neq j$. Moreover, it is known [9] that a sequence $\{X_n, n \geq 1\}$ of r.e. converges in probability to a r.e. X iff $X_n, n \geq 1\}$ is stable and $\mathcal{P}_{\mathcal{A}}(S)$ satisfies the following condition

(*) If $\mu_B(A) > 0$ then there exists a set $B' \in \mathcal{A}_+, B' \subset B$ such that $\mu_{B'}(A) = 1$.

Theorem 1. *Let $(\Omega, \mathcal{A}, P) = (< 0, 1 >, \mathcal{B}', \nu)$. If the set $\mathcal{P}_{\mathcal{B}'}(S) = \{\mu_B : B \in \mathcal{B}'\}$ of probability measures defined on $(S, \mathcal{B})$ satisfies the condition*

$$\nu\left(\bigcup_{i=1}^{n} B_i\right) \mu_{\cup_{i=1}^n B_i}(A) = \sum_{i=1}^{n} \mu_{B_i}(A) \nu(B_i), \tag{I}$$

$$B_i \in \mathcal{B}'_+, i = 1, 2, \ldots, n, B_i \cap B_j = \emptyset, i \neq j,$$

then there exists a stable sequence $\{X_n, n \geq 1\}$ such that

$$\lim_{n \to \infty} \nu\Big([X_n \in A] \cap B\Big) = \mu_B(A) \nu(B), \ A \in \mathcal{B}, B \in \mathcal{B}'_+.$$

Proof. Let $Q_A(B) = \mu_B(A)\nu(B)$. Obviously Q_A is a measure absolutely continuous with respect to ν and $Q_A(B) = \int_B \alpha_A d\nu$, $0 \leq \alpha \leq 1$, a.e..Then there exists a variant $\lambda(A, \cdot)$ of $\alpha(A, \cdot)$ such that, with probability 1, $\lambda(\cdot, \omega)$ is a probability measure on $(S, \mathcal{B}), (\nu\{\omega : \alpha(A, \omega) \neq \lambda(A, \omega)\} = 0$ for every $A \in \mathcal{B})$, [8].

Let us choose a sequence of Borel subsets $S_{i_1, i_2, \ldots, i_k} \in \mathcal{C}_\nu$ satisfying the following conditions [7]:

a) $S_{i_1, i_2, \ldots, i_k} \cap S_{i'_1, i'_2, \ldots, i'_k} = \emptyset$ if $i_s \neq i'_s$ for some $1 \leq s \leq k$

b) $\bigcup_{i_k=1}^{\infty} S_{i_1, i_2, \ldots, i_{k-1}, i_k} = S_{i_1, i_2, \ldots, i_{k-1}}$, $\bigcup_{i_1=1}^{\infty} S_{i_1} = S$

c) $d(S_{i_1, i_2, \ldots, i_k}) < 2^{-k}$ where $d(B)$ denotes the diameter of the set $B \subset S$.

By Lemma 1 for every $S_{i_1, i_2, \ldots, i_k}$ there exists a stable sequence $\{A^n_{i_1, i_2, \ldots, i_k}, n \geq 1\}$ with density $\alpha(S_{i_1, i_2, \ldots, i_k}, \cdot)$ such that

a′) $\cap A^n_{i'_1, i'_2, \ldots, i'_k} = \emptyset$ if $i_s \neq i'_s$ for some $1 \leq s \leq k$ and

b′) $A^n_{i_1, i_2, \ldots, i_{k+1}} \subset A^n_{i_1, i_2, \ldots, i_k}$, $n \geq 1, k \geq 1$ and

c′) $\bigcup_{i_{k+1}=1}^{\infty} A^n_{i_1, i_2, \ldots, i_{k+1}} = A^n_{i_1, i_2, \ldots, i_k}$, $\bigcup_{i_1=1}^{\infty} A^n_{i_1} = < 0, 1 >, n \geq 1$.

If $z_{i_1, i_2, \ldots, i_k} \in S_{i_1, i_2, \ldots, i_k}$ and

$$X^k_n(\omega) = z_{i_1, i_2, \ldots, i_k} \quad \text{for } \omega \in A^n_{i_1, i_2, \ldots, i_k}, n \geq 1, \tag{4}$$

then for every ω the sequence $\{X^k_n, k \geq 1\}$ satisfies the Cauchy condition and therefore convergences to some r.e. $X_n \in \mathcal{X}_S$.

Moreover, for every k, the sequence $\{X_n, n \geq 1\}$ is stable.

Let $B \in \mathcal{B}^0$ and $\varepsilon > 0$. We choose $\delta > 0$ such that

$$\int_B \alpha\left(S^{2\delta}_{i_1, i_2, \ldots, i_l}, \cdot\right) d\nu < \int_B \alpha(S_{i_1, i_2, \ldots, i_l}, \cdot) d\nu + \varepsilon,$$

where $A^\delta = \left\{x : \int_{y \in A} \rho(x, y) < \delta\right\}$.

Hence

$$\nu\left([X_n \in S_{i_1, i_2, \ldots, i_l}] \cap B\right) \leq \nu\left(\left[X^k_n \in S^\delta_{i_1, i_2, \ldots, i_l}\right] \cap B\right)$$

$$\leq \nu\left(\left[X^k_n \in \bigcup_{\{i_1, i_2, \ldots, i_s : S_{i_1, i_2, \ldots, i_s} \cap S_{i_1, i_2, \ldots, i_l} \neq \emptyset\}} S_{i_1, i_2, \ldots, i_s}\right] \cap B\right)$$

$$\underset{n \to \infty}{\longrightarrow} \int_B \alpha\left(\bigcup_{\{i_1, i_2, \ldots, i_s : S_{i_1, i_2, \ldots, i_s} \cap S_{i_1, i_2, \ldots, i_l} \neq \emptyset\}} S_{i_1, i_2, \ldots, i_s}, \cdot\right) d\nu$$

$$\leq \int_B \alpha\left(S^{2\delta}_{i_1, i_2, \ldots, i_l}, \cdot\right) d\nu \leq \int_B \alpha\left(S_{i_1, i_2, \ldots, i_l}, \cdot\right) d\nu + \varepsilon$$

$$\text{for } k > s > \log_2(1/\delta).$$

Similarly

$$\lim_{n\to\infty} \nu([X_n \in S_{i_1,i_2,\dots,i_l}] \cap B) \geq \int_B \alpha(S_{i_1,i_2,\dots,i_l}, \cdot)\, d\nu - \varepsilon$$

which proves that

$$\lim_{n\to\infty} \nu([X_n \in S_{i_1,i_2,\dots,i_l}] \cap B) = \int_B \alpha(S_{i_1,i_2,\dots,i_l}, \cdot)\, d\nu\,.$$

Since the set $S_{i_1,i_2,\dots,i_l}$ form the convergence-determining class this completes the proof.

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