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Weak Convergence of Spectral Measures

Dedicated to Professor Dominik Szynal
on the occasion of his 60th birthday

ABSTRACT. The property of weak compactness for sequences of finite Borel measures on the real line is extended to a sequence of families of Borel measures on $\mathbb{R}$ and discussed in the study of sequences of bounded self-adjoint operators on a separable real Hilbert space.

One of the fundamental results of probability theory is the property of weak compactness for sequences of finite Borel measures on the real line $\mathbb{R}$: if $\{\mu^{(n)}\}$ is a sequence of Borel measures on $\mathbb{R}$ with $\mu^{(n)}(\mathbb{R}) = c$ for $n \geq 1$, then there exists a subsequence $\{\mu^{(n_{n_k})}\}$ and a Borel measure μ, with $\mu(\mathbb{R}) \leq c$ such that $\int \varphi d\mu^{(n_k)} \to \int \varphi d\mu$ for all $\varphi \in C_K(\mathbb{R})$, the real-valued continuous functions with compact support. If $\mu(\mathbb{R}) = c$ then $\int \varphi d\mu^{(n_k)} \to \int \varphi d\mu$ for all $\varphi \in C_b(\mathbb{R})$, the bounded real-valued continuous functions on $\mathbb{R}$. This will be the case if the sequence $\{\mu^{(n)}\}$ is tight, i.e. $\sup_n \mu^{(n)}(K^c) \downarrow 0$ as $K \uparrow \mathbb{R}$, K compact.

A generalization of these ideas would be the following: suppose $\{\mu_t^{(n)} : t \in T\}$ is a sequence of families of Borel measures on $\mathbb{R}$ such that for each $t \in T$, $\mu_t^{(n)}(\mathbb{R}) = c_t$ for $n \geq 1$. Under what conditions can one affirm the existence of a subsequence $\{\mu_t^{(n_k)} : t \in T\}$ and measures $\mu_t, t \in T$, such that $\mu_t(\mathbb{R}) = c_t$ and $\int \varphi d\mu_t^{n_k)} \to \int \varphi d\mu_t$ for all $t \in T$, $\varphi \in C_b(\mathbb{R})$)? Such a situation arises in the study of sequences $\{A_n\}$ of bounded self adjoint operators on a separable real Hilbert space H with inner product (x, y) for $x, y \in H$. As is well-known one can represent A_n in the form $A_n x = \int \lambda dE_n(\lambda)x$ where $E_n(\lambda), \lambda \in \mathbb{R}$, is a resolution of the identity, i.e. a right- continuous increasing family of orthogonal projections on H satisfying $\lim_{\lambda\to-\infty} E_n(\lambda)x = 0$ and $\lim_{\lambda\to\infty} E_n(\lambda)x = x$. One can then define for $\varphi \in C_b(\mathbb{R})$ a bounded self adjoint operator $R_n(\varphi) : H \to H$ by the formula $R_n(\varphi)x = \int \varphi(\lambda)dE_n(\lambda)x$ for $x \in H$ and one has $||R(\varphi)|| \leq ||\varphi||_\infty$ where $||R(\varphi)||$ is the usual operator norm and $||\varphi||_\infty = \sup\{|\varphi(\lambda)| : \lambda \in \mathbb{R}\}$. The functions $\lambda \to (E_n(\lambda)x, x)$ are increasing and right-continuous on $\mathbb{R}$ and therefore define Borel measures $d\mu_{x,x}^{(n)} = d(E_n(\lambda)x, x)$, the so-called spectral measures associated with A_n and $x \in H$. We can then write

$$(R_n(\varphi)x, x) = \int \varphi(\lambda)d(E_n(\lambda)x, x) = \int \varphi(\lambda)d\mu_{x,x}^{(n)}$$

and by polarization

$$(R_n(\varphi)x, y) = \int \varphi d\mu_{x,y}^{(n)},$$

where

$$\mu_{x,y}^{(n)} = \frac{1}{2}\left(\mu_{x+y,x+y}^{(n)} - \mu_{x,x}^{(n)} - \mu_{y,y}^{(n)}\right)$$

for $x, y \in H$. Note that $(R_n(1)x, x) = (x, x) = \mu_{x,x}^{(n)}(\mathbb{R})$ for all $x \in H$. The question now arises as to when can one say that the measures $\{\mu_{x,x}^{(n)} : x \in H\}$ have weakly convergent subsequences as described above. If would follow, of course, that the sequences $\{\mu_{x,y}^{(n)} : x, y \in H\}$ would also have weakly convergent subsequences. In this note we will show that under a mild condition this can always be done and that the limit measures define self adjoint operators. We recall that if $\{\mu^{(n)}\}$ is a sequence of finite Borel measures such that $\sup_n \int \lambda^2 d\mu^{(n)}(\lambda) < \infty$, then the sequence $\{\mu^{(n)}\}$ is tight.

Theorem. *Let $\{A_n\}$ be a sequence of bounded self-adjoint operators on H. Assume there is a dense linear subspace $D \subseteq H$ such that $\sup_n ||A_n x|| < \infty$ for all $x \in D$. Then there is a family of Borel measures $\{\mu_{x,y} : x, y \in H\}$ with $\mu_{x,x}(\mathbb{R}) = (x, x) = ||x||^2$ and a subsequence $\{n_k\}$ of positive integers such that*

$$\int \varphi(\lambda)d(E_{n_k}(\lambda)x, x) \to \int \varphi(\lambda)d\mu_{x,x}$$

for all $x \in H$, $\varphi \in C_b(\mathbb{R})$. Moreover, the formula $(R(\varphi)x,y) = \int \varphi d\mu_{x,y}$ for $x, y \in H$ defines a bounded self-adjoint operator for each $\varphi \in C_b(\mathbb{R})$ with $||R(\varphi)|| \leq ||\varphi||_\infty$.

Proof. If $x \in H$ we denote by $\mu^{(n)}_{x,x}$ the spectral measure associated with A_n and x, i.e. $\mu^{(n)}_{x,x} = d(E_n(\lambda)x, x)$. If $x \in D$ we have

$$||A_n x||^2 = (A_n^2 x, x) = \int \lambda^2 d\mu^{(n)}_{x,x}$$

and hence $\sup_n \int \lambda^2 d\mu^{(n)}_{x,x} < \infty$. Also

$$\mu^{(n)}_{x,x}(\mathbb{R}) = ||x||^2, \quad n \geq 1,\ x \in H.$$

Let $D_C \subseteq D$ be a countable set in D which is dense in H. Using a diagonal argument we can find subsequences $\{\mu^{(n_k)}_{x,x}\}$ and measures $\mu_{x,x}$ satisfying $\mu_{x,x}(\mathbb{R}) = ||x||^2$ and such that

$$(R_{n_k}(\varphi)x, x) = \int \varphi(\lambda) d\mu^{(n_k)}_{x,x} \to \int \varphi(\lambda) d\mu_{x,x}$$

for all $\varphi \in C_b(\mathbb{R}), x \in D_C$.

We claim that for each $x \in H$ there is a finite measure $\mu_{x,x}$ such that $(R_{n_k}(\varphi)x, x) \to \int \varphi(\lambda) d\mu_{x,x}$ for each $\varphi \in C_K(\mathbb{R})$. By compactness it suffices to show that the sequence $\{(R_{n_k}(\varphi)x, x)\}$ is Cauchy. We write for $x, y \in H$

$$(R_{n_k}(\varphi)x, x) = (R_{n_k}(\varphi)(x - y), x) + (R_{n_k}(\varphi)y, x - y) + (R_{n_k}(\varphi)y, y)$$

to obtain

$$|(R_{n_k}(\varphi)x, x) - (R_{n_l}(\varphi)x, x)| \leq 2||\varphi||_\infty ||x - y||\,||x|| + 2||\varphi||_\infty ||x - y||\,||y|| + |(R_{n_k}(\varphi)y, y) - (R_{n_l}(\varphi)y, y)|.$$

The first two terms can be made arbitrarily small by choosing $y \in D_C$ appropriately; the last term tends to 0 as $k, l \to \infty$ for $y \in D_C$. Consider now the function $L : H \times H \to \mathbb{C}$ defined by

$$L(x, y) = \int \varphi d\mu_{x,y} = \lim_k (R_{n_k}(\varphi)x, y)$$

where $\varphi \in C_K(\mathbb{R})$ and the measures $\mu_{x,y}, x, y \in H$ are defined in the obvious way by polarization. The map L is bilinear symmetric and satisfies $|L(x, y)| \leq ||\varphi||_\infty ||x||\,||y||$; hence there exists a self-adjoint operator

$R(\varphi) : H \to H$ such that $(R(\varphi)x, y) = \int \varphi d\mu_{x,y}$ with $||R(\varphi)|| \le ||\varphi||_\infty$. The formula extends, by continuity, to functions $\varphi \in C_b(\mathbb{R})$ and satisfies $||R(\varphi)|| \le ||\varphi||_\infty$. We have for $x \in D_C$

$$(R(1)x, x) = \int 1 \, d\mu_{x,x} = (x, x).$$

Since $R(1) : H \to H$ is bounded and since $D_C \subseteq H$ is dense, the formula holds by continuity for all $x \in H$. Hence $\mu_{x,x}(\mathbb{R}) = ||x||^2$ for all $x \in H$ and therefore

$$(R_{n_k}(\varphi)x, x) \to \int \varphi \, d\mu_{x,x} \quad \text{for all } \ \varphi \in C_b(\mathbb{R})$$

and the proof is complete.

Remark 1. The map $\varphi \to R(\varphi)$ from $C_b(\mathbb{R})$ to the space of self-adjoint operators on H is clearly linear and positive, i.e. $(R(\varphi)x, x) \ge 0$ for all $x \in H$ if $\varphi \ge 0$.

Remark 2. It can be shown that $A_{n_k} \to Ax$ weakly for each $x \in D$ where $A : D \to H$ is the symmetric operator defined by $(Ax, y) = \int \lambda d\mu_{x,y}$ for $x \in D, y \in H$.

As an example we take $H = L^2([0,1])$ and define the sequence of multiplication operators A_n by the formula $A_n f(t) = (\sin nt) f(t)$ for $f \in L^2([0,1])$. Then for $\varphi \in C_b(\mathbb{R})$ $R_n(\varphi)f(t) = (\varphi(\sin nt)f(t)$ and hence

$$(R_n(\varphi)f, f) = \int_0^1 \varphi(\sin nt) f^2(t) dt.$$

By the Theorem there is a subsequence $\{n_k\}$ and self-adjoint operators $R(\varphi) : H \to H$ such that

$$(R_{n_k}(\varphi)f, f) \to (R(\varphi)f, f) \quad \text{for all } \ \varphi \in C_b(\mathbb{R}), \ f \in L^2([0,1]).$$

But for each $\varphi \in C_b(\mathbb{R})$ the functions $\varphi(sinnt)$ are uniformly bounded and hence form a relatively compact set is the weak* topology of $L^\infty([0,1])$. From this it follows that there is a function $T(\varphi) \in L^\infty([0,1])$ such that

$$(R(\varphi)f, f) = \int_0^1 T\varphi(x) f^2(x) dt \,, \quad f \in L^2([0,1]).$$

The map $T : C_b(\mathbb{R}) \to L^\infty([0,1])$ is linear and positive with $T(1) = 1$. Moreover, $R(\varphi)f = T(\varphi)f$, a multiplication operator. It is interesting to note that the operators $R(\varphi)$ cannot be written in the form $R(\varphi)f = \int \varphi(\lambda)dE_\lambda f$ for some resolution of the identity $E(\lambda), \lambda \in \mathbb{R}$. Indeed, it is well known that such operators satisfy the multiplicative property $R(\varphi_1\varphi_2) = R(\varphi_1)R(\varphi_2)$. But if $\varphi_0 \in C_b(\mathbb{R})$ is a function such that $\varphi_0(\lambda) = \lambda$ on $[-1,1]$ and if $f = 1$, then

$$(R(\varphi_0^2)1, 1) = \lim_k \int_0^1 (\sin nt)^2 dt \neq 0 = (R(\varphi_0)R(\varphi_0)1, 1).$$

Here we use the fact that

$$(R(\varphi_0)f, f) = \lim_k \int \sin nt f^2(x)dt = 0$$

by the Riemann-Lebesgue Lemma, i.e. $R(\varphi_0) = 0$.

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