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A Note on the Strong Tightness in $C[0,1]$

Abstract. A sequence $\{X_n, n \geq 1\}$ of random elements is called strongly tight if for any $\epsilon > 0$ there exists a compact set K such that

$$P\left\{\bigcap_{n=1}^{\infty}[X_n \in K]\right\} > 1-\epsilon.$$

A new type of convergence of r.e. was indroduced in [4]. With this kind of convergence some criteria of strong tightness in $C_{[0,1]}$ are given. Also almost sure convergence of random functions in $C_{[0,1]}$ is investigated.

1. Notation and definitions. Let (Ω, A, P) be a probability space and (S, ρ) a separable and complete metric space (Polish space). A random element with values in S is a measurable map X from the probability space (Ω, A, P) into S equipped with its Borel σ-algebra B_ρ , $(X^{-1}(B_\rho) \subset A)$.

The distribution of X is the probability measure $P_X : B_\rho \to [0,1]$ defined by the formula

$$\forall_{B\in B_\rho} P_X(B) = P\{\omega : X(\omega) \in B\}.$$

Definition 1. We say a sequence $\{X_n, n \geq 1\}$ of random elements *converges in distribution* to the random element X , and write $(X_n \xrightarrow{D} X,\ n \to \infty)$, if

$$\forall_{B\in C_{P_X}} \lim_{n\to\infty} P_{X_n}(B) = P_X(B),$$

where $C_{P_X} = \{B \in B_\rho : P_X(\partial B) = 0\}$ and ∂B denotes the boundary of B.

Definition 2. A sequence $\{X_n, n \geq 1\}$ of random elements is said to be *essentially convergent in law* to a r.e. X ($X_n \xrightarrow{ED} X,\ n \to \infty$) , if

$$P(\limsup_{n\to\infty}[X_n \in A]) = P(\liminf_{n\to\infty}[X_n \in A]) = P([X \in A])$$

for every $A \in C_{P_X}$.

Definition 3. A probability measure P on (S, ρ) *is tight* if

$$\forall_{\varepsilon>0} \exists_{\substack{K\subset S \\ K-\text{compact}}} \quad P(K) > 1 - \varepsilon$$

Definition 4. A sequence $\{P_n : n \geq 1\}$ of probability measures defined on (S, ρ) *is tight* if

$$\forall_{\varepsilon>0} \exists_{\substack{K\subset S \\ K-\text{compact}}} \forall_{n\geq 1} \ P_n(K) > 1 - \varepsilon$$

A sequence $\{X_n, n \geq 1\}$ of random elements is tight if the sequence of distributions $\{P_{X_n} : n \geq 1\}$ is tight.

Theorem 1 ([1; Th. 8, p. 241]). *Suppose that* $X_n \xrightarrow{D} X,\ n \to \infty$. *Then* $\{P_{X_n} : n \geq 1\}$ *is tight.*

Let $C_{[0,1]}$ denote the metric space of continuous functions on $[0, 1]$ with the metric defined by formula

$$\rho(x, y) = \sup\{|x(t) - y(t)| : t \in [0, 1]\}.$$

The modulus of continuity of $x \in C_{[0,1]}$ is defined by

$$\omega_x(\delta) = \sup\{|x(t_1) - x(t_2)| : |t_1 - t_2| < \delta \leq 1\}.$$

We define C_ρ as the σ-field generated by the open subsets of $C_{[0,1]}$.

Theorem 2 ([1, p. 54]). *Let $\{X_n, n \geq 1\}$ and X be random elements with values in $C_{[0,1]}$. If*

$$\{X_n(t_1), X_n(t_2), ..., X_n(t_k)\} \xrightarrow{D} \{X(t_1), X(t_2), ..., X(t_k)\},\ n \to \infty,$$

for every k and $0 \leq t_1 < t_2 < ... < t_k \leq 1$, and if $\{P_{X_n} : n \geq 1\}$ is tight, then $X_n \xrightarrow{D} X,\ n \to \infty$.

2. Almost sure convergence. Now, we give some conditions which assure almost sure convergence of random elements in $C_{[0,1]}$.

It seems to be worth mentioning that every almost surely convergent sequence converges in probability.

Definition 5. We say that a sequence $\{X_n, n \geq 1\}$ of r.e. *is strongly tight* iff

$$(T_1) \qquad \forall_{\varepsilon>0}\ \exists_{K \subset C_{[0,1]}}\ P\left\{\bigcap_{n=1}^{\infty}[\omega : X_n(\omega) \in K]\right\} > 1 - \varepsilon.$$

Obviously, if a sequence $\{X_n, n \geq 1\}$ is strongly tight then it is tight, but the reverse implication does not hold. (For instance sequences of i.i.d. real r.v's having a standard normal distribution are tight but not strongly tight).

By T we denote a collection of all bounded stopping times relative to the sequence $\{\sigma(X_1, X_2, ..., X_n) : n \geq 1\}$, where $\sigma(X_1, X_2, ..., X_n)$ denotes the smallest σ-algebra with respect to which $X_1, X_2, ..., X_n$ are measurable .

Theorem 3. *If $X_n \xrightarrow{a.s.} X$, $n \to \infty$, then the sequence $\{X_n, n \geq 1\}$ is strongly tight.*

Proof. Since $X_n \xrightarrow{a.s.} X$ for $n \to \infty$, X_n is randomly convergent in probability to X. This means that for any $\epsilon > 0$ there exists $\tau_0 \in T$, such that for every $\tau \geq \tau_0(a.s.)$, $\pi(X_\tau, X) < \epsilon$, where π denotes the Prokhorov distance. Now, we will show that the family $\{P_{X_\tau}, \tau \in T\}$ of measures is tight. Let $\{x_i, i \in N\}$ be a countable dense subset of S and fix $\delta > 0$.
Define $B_m(\delta) = \bigcup_{i=1}^{m} K(x_i, \delta)$, where $K(x_i, \delta)$ is the ball of radius δ centered at x_i. We have to show that for any $\epsilon > 0$ there exists $m \in \mathrm{N}$ such that $P[X_\tau \in B_m(\delta)] > 1 - \epsilon$ for every $\tau \in T$. Suppose that it is not true. Then there exists $\epsilon > 0$ such that for any $m \in \mathrm{N}$ we can choose $\tau_m \in T$ that $P[X_{\tau_m} \notin B_m(\delta)] \geq \epsilon$.

Since $\{x_i, i \in \mathbb{N}\}$ is dense, for any $n \in \mathbb{N}$ there exists a sequence $\{m_n\}$ such that $P(\bigcup_{i=1}^{n}[X_i \notin B_{m_n}(\delta)]) \leq \epsilon/2$. We can also assume that $\{m_n\}$ is strictly increasing and $m_n > n$. For $\tau'_{m_n} = \max\{\tau_{m_n}, (n+1)\}$ it is easy to see that $P([X_{\tau'_{m_n}} \notin B_{m_n}(\delta)]) \geq \epsilon/2$.

By [1, Th. 2.1] we get for any n

$$
\begin{aligned}
(1) \qquad P_X(B_{m_n}(\delta)) &\leq \liminf_{k\to\infty} P_{X_{\tau'_{m_k}}}(B_{m_n}(\delta)) \\
&\leq \liminf_{k\to\infty} P_{X_{\tau'_{m_k}}}(B_{m_k}(\delta)) \leq 1 - \epsilon/2.
\end{aligned}
$$

On the other hand, since $B_m(\delta) \uparrow \Omega$ as $m \to \infty$, $\lim_{n\to\infty} P_X(B_{m_n}(\delta)) = 1$, which contradicts (1). For any $k \in \mathbb{N}$ and $\epsilon > 0$ there exists m_{n_k} such that $P[X_\tau \notin B_{m_{n_k}}(1/k)] < \epsilon/2^k$. Put $K = \overline{\bigcap_{k=1}^{\infty} B_{m_{n_k}}(1/k)}$. Obviously, K is compact and $P[X_\tau \in K] > 1 - \epsilon$ for all $\tau \in T$. Thus, the family $\{P_{X_\tau}, \tau \in T\}$ is tight.

Suppose now that the sequence $\{X_n, n \geq 1\}$ is not strongly tight, i.e. there exists $\epsilon > 0$ such that for any compact set K

$$P(\bigcap_{n=1}^{\infty}[X_n \in K]) \leq 1 - 2\epsilon.$$

On the other hand, we know that there exists K_ϵ such that $P([X_\tau \in K_\epsilon]) > 1 - \epsilon$ for all $\tau \in T$. Define $\tau = \inf\{n : X_n \notin K_\epsilon\}$ and $\tau_n = \tau \wedge n \in T$, then

$$P(\bigcup_{n=1}^{\infty}[X_n \notin K_\epsilon]) \leq \lim_{n\to i} P([X_{\tau_n} \notin K_\epsilon]) \leq \epsilon.$$

This contradicts (2) and completes the proof. $\square$

It is easy to observe that this theorem is not true in the case of convergence in probability.

Example. Let $(\Omega, A, P) = (<0,1), B, \mu)$ and

$$X_n(\omega) = \begin{cases} s & \text{for } \omega \in < k/2^s, (k+1)/2^s) \\ 0 & \text{otherwise}, \end{cases}$$

where $s = \max\{i : 2^i \leq n\}$ and $k = n - 2^s$.

It is easy to see that $X_n \xrightarrow{P} 0, n \to \infty$, but the sequence $\{X_n, n \geq 1\}$ is not strongly tight.

Theorem 4. *If $\{X_n, n \geq 1\}$ is strongly tight and $X_n(t) \overset{a.s.}{\to} X(t), n \to \infty$ for every $t \in [0,1]$, then $X_n \overset{a.s.}{\to} X, n \to \infty$.*

Proof. We assume that $X_n \overset{a.s.}{\not\to} X, n \to \infty$ and define

$$N = \{\omega : \lim_{n\to\infty} \rho(X_n(\omega), X(\omega)) \neq 0\} \ \text{ i } \ P(N) = \eta > 0.$$

Let $\{t_1, t_2, \dots\}$ be a dense subset of $[0,1]$.
Define $\Omega_0 = \{\omega : \lim_{n\to\infty} X_n(t_i, \omega) = X(t_i), i = 1,2,\dots\}$. Obviously, $P(\Omega_0) = 1$. For $\epsilon = \eta/2 > 0$ there exists compact set $K \subset C_{[0,1]}$ such that

$$P\left\{\bigcap_{n=1}^{\infty}[\omega : X_n(\omega) \in K]\right\} > 1 - \epsilon.$$

Let $\Omega_1 = \Omega_0 \cap \bigcap\limits_{n=1}^{\infty} [\omega : X_n(\omega) \in K]$. Let us notice that $N \cap \Omega_1 \neq \emptyset$. If $\omega \in N \cap \Omega_1$, there exist subseqences $\{X_{n_k}, k \geq 1\}$ and $\{X_{n_s}, s \geq 1\}$ such that $X_{n_k}(\omega) \to X_1(\omega), k \to \infty$ and $X_{n_s}(\omega) \to X_2(\omega), s \to \infty$ and $\rho(X_1(\omega), X_2(\omega)) > 0$. By the definition of ρ, there exists $t \in [0,1]$ such that $|X_1(t,\omega) - X_2(t,\omega)| > 0$. On the other hand, $\omega \in \Omega_1$ and the functions $X_1(t,\omega)$ and $X_2(t,\omega)$ are continuous and coincide on a dense subset of $[0,1]$ which proves that $X_1 = X_2$ and this completes the proof. □

The following theorem will be needed throughout the paper.

Theorem (Arzela-Ascoli, cf. [1, Appendix]**).** *A subset A of $C_{[0,1]}$ has compact closure if and only if*

(I) $$\sup_{x\in A} |x(0)| < \infty$$

and

(II) $$\lim_{\delta\to 0} \sup_{x\in A} w_x(\delta) = 0$$

By Arzela-Ascoli theorem we see that the sequence $\{X_n, n \geq 1\}$ of r.e. is tight if and only if
(1) $\forall_{\eta>0}\ \exists a\ P_{X_n}\{x : |x(0)| > a\} \leq \eta$, for $n \geq 1$ and
(2) $\forall \epsilon > 0\ \exists \eta > 0\ \exists 0 < \delta < 1\ \exists n_0\ P_{X_n}\{x : w_x(\delta) \geq \epsilon\} \leq \eta$, for $n \geq n_0$
It is easy to observe

Corollary. *If* $\{X_n, n \geq 1\}$ *and* $\{Y_n, n \geq 1\}$ *are strongly tight in* $C_{[0,1]}$ *then* $\{X_n + Y_n, n \geq 1\}$ *and* $\{\alpha X_n, n \geq 1\}$, $\alpha \in \mathbb{R}$, *are strongly tight.*

The sequence $\{X_n, n \geq 1\}$ is uniformly continuous if

(W_1) $\forall \epsilon > 0 \, \exists \delta > 0 \; \sup_{n \geq 1} w_{X_n}(\delta) < \epsilon$, a.e.

The sequence $\{X_n, n \geq 1\}$ is almost uniformly continuous if

(W_2) $\forall \eta > 0 \, \exists_{\substack{\Omega_\eta \\ P(\Omega_\eta) > 1-\eta}} \; \forall \epsilon > 0 \, \exists \delta > 0 \; \sup_{n \geq 1} w_{X_n(\omega)}(\delta) < \epsilon$ dla $\omega \in \Omega_\eta$.

Condition (W_2) is equivalent to the

(W_2') $\lim_{\delta \to 0} \sup_{n \geq 1} w_{X_n}(\delta) = 0$, a.e.

It is easy to see that $(W_1) \Rightarrow (W_2) \Rightarrow (W_2')$. The implication $(W_2') \Rightarrow (W_2)$ follows by the Egoroff Theorem ([2, p. 88]). The implication $(W_1) \Rightarrow (W_2)$ does not hold.

Examples. Let $\{x_n(t), n \geq 1\}$ be defined by the formula

$$x_n(t) = \begin{cases} 2nt & \text{for } 0 \leq t < 1/(2n) \\ -2nt + 2 & \text{for } 1/(2n) \leq t < 1/n \\ 0 & \text{for } 1/n \leq t \leq 1. \end{cases}$$

If

(E_1) $\qquad X_n(t,\omega) = x_n(t)$ a.s., then $X_n(t) \xrightarrow{a.s.} X(t) \overset{a.s.}{=} 0, n \to \infty$,

for every $t \in [0,1]$ but $X_n \overset{a.s.}{\not\to} X, n \to \infty$, in $C_{[0,1]}$.

Let $A_n \in \mathcal{A}$ be a sequence of events such that $0 < P(A_n) \to 0, n \to \infty$ and $A_{n+1} \subset A_n$. We define

(E_2) $$Y_n(t,\omega) = \begin{cases} x_n(t) & \text{dla } \omega \in A_n \\ 0 & \text{dla } \omega \notin A_n. \end{cases}$$

The sequence $\{Y_n, n \geq 1\}$ satisfies condition (W_2) , but not (W_1).
Let $A_n \in \mathcal{A}$ be a sequence of independent events such that $0 < P(A_n) \to 0$, $n \to \infty$, and $\sum_{n=1}^{\infty} P(A_n) = \infty$. We define

(E_3) $$Z_n(t,\omega) = \begin{cases} x_n(t) & \text{dla } \omega \in A_n \\ 0 & \text{dla } \omega \notin A_n . \end{cases}$$

A sequence $\{Z_n, n \geq 1\}$ of $C_{[0,1]}$ -valued r.e.s converges in probability but not almost surely.

Theorem 5. *A sequence $\{X_n, n \geq 1\}$ of $C_{[0,1]}$ -valued r.e's is strongly tight if and only if it satisfies*

$$(R_1) \qquad \forall \eta > 0 \; \exists a \; P\left\{\omega : \sup_{n\geq 1} |X_n(\omega, 0)| > a\right\} \leq \eta$$

and

$$(W_2') \qquad \lim_{\delta\to 0} \sup_{n\geq 1} w_{X_n}(\delta) = 0, \text{ a.e.}$$

Proof. Let $\{X_n, n \geq 1\}$ be strongly tight sequence. Then for every $\eta > 0$ there exists a compact set $A \subset C_{[0,1]}$ such that

$$P\{\bigcap_{n=1}^{\infty}[X_n \in A]\} > 1 - \eta.$$

By the compactness of A we see that there exists $a \in \mathbb{R}$ such that $\sup_{n\geq 1} |X_n(\omega, 0)| = a < \infty$ for $\omega \in \Omega_\eta = \{\bigcap_{n=1}^{\infty}[X_n \in A]\}$ and for every $\epsilon > 0$, there exist $\delta(\epsilon) > 0$ such that $\sup_n w_{X_n(\omega)}(\delta) < \epsilon$ for $\omega \in \Omega_\eta$.

On the other hand, by (R_1) and (W_2') , for every $\eta > 0$ there exists Ω_η , such that $P(\Omega_\eta) > 1 - \eta$ and for $\omega \in \Omega_\eta$ we have $\sup_n |X_n(\omega, 0)| < a$, for some $a \in \mathbb{R}$ and

$$\forall \epsilon > 0 \; \exists \delta(\epsilon) > 0 \; \sup_{n\geq 1} w_{X_n}(\delta(\epsilon)) < \epsilon.$$

It means that X_n belong to the compact set $A \subset C_{[0,1]}$ described by a and the function $\delta(\epsilon)$. □

It is easy to observe that the convergence $(X_n(t_1), X_n(t_2), ..., X_n(t_k)) \xrightarrow{a.s.} (X(t_1), X(t_2)...X(t_k))$ for every $(t_1, t_2, ..., t_k)$ is equivalent to the convergence $X_n(t) \xrightarrow{a.s.} X(t), n \to \infty$, for every $t \in [0,1]$.

Theorem 6. *If $\{\xi_n, n \geq 1\}$ are independent identically distributed random variables with mean 0 and finite variation σ^2, then the random function*

$$X_n(t, \omega) = \frac{1}{n} \cdot S_{[nt]}(\omega) + (nt - [nt]) \cdot \frac{1}{n} \cdot \xi_{[nt]+1}(\omega),$$

where $S_n(\omega) = \sum_{k=1}^{n} \xi_k(\omega)$, converges almost surely to the $X(t, \omega) \equiv 0$ a.s. in $C_{[0,1]}$.

Proof. First, observe that by the Kolmogorof Theorem

$$(3) \qquad \lim_{n\to\infty} \frac{1}{n} S_{[nt]} = \lim_{n\to\infty} \frac{[nt]}{n} \frac{1}{[nt]} S_{[nt]} = 0 \text{ a.s. for every } t \in [0,1]$$

and

$$\sum_{k=1}^{\infty} P\left[\left|\frac{\xi_k}{k}\right| > \epsilon\right] = \sum_{k=1}^{\infty} P[|\xi_k| > k\epsilon] \le \sum_{k=1}^{\infty} \frac{\sigma^2}{\epsilon^2 k^2} < \infty$$

which implies that

$$\lim_{n\to\infty} (nt - [nt]) \cdot \frac{1}{n} \cdot \xi_{[nt]+1}(\omega) = 0 \text{ a.s.} \tag{4}$$

By (3) and (4) we have $X_n(t,) \xrightarrow{a.s.} 0$, as $n \to \infty$, for every $t \in [0,1]$.

Now we prove that the sequence $\{X_n(t,\omega), n \ge 1\}$ is strongly tight. We only need to show that $\sup_t |X_n(t)| \to 0$ *a.s.*

It is easily to seen that $\sup_t |X_n(t,\omega)| = \max_{1\le k\le n} \left|\frac{1}{n} \cdot S_k(\omega)\right|$. By the Kolmogorov inequality

$$P\left\{\omega : \max_{1\le k\le n} \left|\frac{1}{n} \cdot S_k(\omega)\right| > \epsilon\right\} = P\left[\max_{1\le k\le n} |S_k(\omega)| > n\cdot\epsilon\right]$$
$$\le \frac{\sigma^2 S_n}{n^2\cdot\epsilon^2} = \frac{n\cdot\sigma^2}{n^2\cdot\epsilon^2}$$

Hence we have $\sup_t |X_n(t)| \to 0$ a.s. □

References

[1] Billingsley, P., *Convergence of probability measures*, Wiley-Interscience, New York, 1968.

[2] Halmos, P. R., *Measure Theory*, D. Van Nostrand Company, Inc., Toronto, New York, London, 1950.

[3] Neveu, J., *Discrete-Parameter Martingales*, North-Holland Elsevier, Amsterdam-New York, 1975.

[4] Szynal, D. and W. Zięba, *On some type of convergence in law*, Bull. Polish. Acad. Sci. Math. **22** (1974), 1143–1149.

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