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Rényi-Mixing in the Generalized Arc-Sine Law*

Dedicated to Professor Dominik Szynal
on the occasion of his 60th birthday

ABSTRACT. It is shown that the proportion of positive sums of independent and identically distributed random variables in the generalized arc-sine law is Rényi-mixing under Spitzer's classical necessary and sufficient condition for the law. Some consequences for the number of positive sums in a random number of games are derived, along with an extension of the original law for Révész-dependent sequences of random variables.

1. Introduction. Following Rényi [18] and Rényi and Révész [20], we say that a sequence $\{\xi_n\}_{n=1}^{\infty}$ of random variables, given on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, is mixing with the limiting distribution function $G(\cdot)$ if $\mathbb{P}\{\{\xi_n \leq y\} \cap A\} \to G(y)\mathbb{P}\{A\}$ at every continuity point y of G on the real line $\mathbb{R}$, for each event $A \in \mathcal{A}$, where an unspecified convergence relation is meant to hold as $n \to \infty$ throughout this note. To separate the notion clearly from many other types of mixing, for which the word is more customary, we shall refer to it as *Rényi-mixing*. Since G is assumed to

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be a proper non-degenerate distribution function, in general Rényi-mixing of $\{\xi_n\}_{n=1}^{\infty}$ with the limiting distribution function G demands more than convergence in distribution, contained for the choice of $A = \Omega$. In fact, according to Rényi's [18] own characterization, given also in Sections 5.7 and 5.8 of his book [19], it happens if and only if $\mathbb{P}\{\xi_n \le y\} \to G(y)$ and $\mathbb{P}\{\{\xi_n \le y\} \cap \{\xi_k \le y\}\} \to G(y)\mathbb{P}\{\xi_k \le y\}$ for every continuity point y of G, for each fixed $k \in \mathbb{N} := \{1, 2, \ldots\}$.

Besides its intrinsic interest, one motivation for Rényi's mixing extension of a limit theorem is the preservation of that limit theorem under the change of the probability measure $\mathbb{P}$ to another probability measure $\mathbb{Q} \ll \mathbb{P}$, absolutely continuous with respect to $\mathbb{P}$. It also turned out that Rényi-mixing in a limit theorem is the proper framework in which that same limit theorem may be transferred to extended versions for randomly selected subsequences of the original underlying sequence of random variables; see for example work by the author [4] and more recently by Rychlik [22] and numerous references therein. Both types of consequences will be discussed below for the number of positive sums that we consider in the present note.

Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables given on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and let $F(x) := \mathbb{P}\{X \le x\}$, $x \in \mathbb{R}$, denote the common distribution function. Set $S_j := X_1 + \cdots + X_j$, $j \in \mathbb{N}$, for the partial sums and, for any event $A \in \mathcal{A}$, let $I(A)$ denote its indicator, defined for all $\omega \in \Omega$ as $I_\omega(A) = 1$ if $\omega \in A$ and $I_\omega(A) = 0$ if $\omega \notin A$. Then $\sum_{j=1}^{n} I(S_j > 0)$, the occupation time of the positive half-line by the first n sums, is often thought of as the number of times a gambler with cumulative gains $S_1, \ldots, S_n$, which may very well be losses, is ahead of his opponent in n games. Initiated by Lévy [16]** and subsequently enriched in contributions by Erdős and Kac [11], Chung and Feller (particularly for the exact discrete arc-sine law for coin tossing described by Feller [12] including his own combinatorial improvements) and, sensationally in its time, by Sparre Andersen [23], the study of the proportion $\sum_{j=1}^{n} I(S_j > 0)/n$ of leading times has culminated in Spitzer's [24] generalized arc-sine law: The proportion has a non-degenerate limiting distribution if and only if

$$(1) \qquad \frac{1}{n}\sum_{j=1}^{n} \mathbb{P}\{S_j > 0\} \to \rho \quad \text{for some } \rho \in (0,1),$$

** Takács [26] offers a masterful analysis of Lévy's heuristic idea, providing several extensions and variants and delineating its limitations.

and if this condition is satisfied for $\rho \in (0,1)$, then

$$\mathbb{P}\left\{\frac{1}{n}\sum_{j=1}^{n} I(S_j > 0) \le y\right\} \to G_\rho(y) := \frac{\sin(\rho\pi)}{\pi}\int_0^y \frac{dt}{t^{1-\rho}(1-t)^{\rho}} \tag{2}$$
$$= \frac{\int_0^y t^{\rho-1}(1-t)^{[1-\rho]-1}\,dt}{\Gamma(\rho)\Gamma(1-\rho)}, \quad 0 \le y \le 1,$$

with the usual gamma function $\Gamma(s) := \int_0^\infty u^{s-1}e^{-s}ds$, $s > 0$, that is, with the Beta$(\rho, 1-\rho)$ distribution function in the limit, where, for every $r \in \mathbb{N}$,

$$\int_0^1 y^r dG_\rho(y) = \frac{\Gamma(r+\rho)}{\Gamma(r+1)\Gamma(\rho)} = \frac{\rho(\rho+1)\cdots(\rho+r-1)}{r!} \tag{3}$$
$$= \lim_{n\to\infty} \mathbb{E}\left(\left[\frac{1}{n}\sum_{j=1}^{n} I(S_j > 0)\right]^r\right)$$

for the limiting r-th moments. The original arc-sine law is the case of $\rho = 1/2$, when

$$G_{\frac{1}{2}}(y) = \frac{1}{\pi}\int_0^y \frac{dt}{\sqrt{t(1-t)}} = \frac{2}{\pi}\arcsin\sqrt{y}, \quad 0 \le y \le 1.$$

The latter obtains trivially if $F(\cdot)$ is continuous on $\mathbb{R}$ and $F(-x) = 1 - F(x)$ for all $x \ge 0$, for then S_n is also symmetric about zero and so $\mathbb{P}\{S_n > 0\} = 1/2$ for all $n \in \mathbb{N}$. But the case $\rho = 1/2$ occurs in (1) and (2) for asymmetric distributions as well, for example whenever $\mathbb{E}(X) = 0$ and F is in the domain of attraction of the normal law, or, more generally, whenever F is in the domain of attraction of a symmetric stable law. In general, for every $\rho \in [0,1]$ there are families of F for which the convergence in (1) is satisfied, where the extreme cases $\rho = 0$ and $\rho = 1$ determine the corresponding asymptotically degenerate cases in (2). In fact, every $\rho \in (0,1)$ in (1) can be realized by families of F that belong to the domain of attraction of a suitable stable law, but, except for completely asymmetric cases, F need not be in a domain of attraction for (1) to hold. Also, what was unclear in 1956 when [24] was published, Spitzer [25] later constructed families of F for which the averages in (1) do not converge. While there are several later derivations of the generalized arc-sine law itself in the literature, by Spitzer [25] and Bingham, Goldie and Teugels [1] for instance, a most recent one is by Getoor and Sharpe [13], the analysis of Spitzer's condition (1) is difficult and incomplete. Interesting results in this connection were given by Emery [9], Doney [6, 7] and Bingham and Hawkes [2]; for a brief discussion see pp. 379–380, 396–397 in [1]. We also note that, just as the original theorem of Spitzer, the theorem below may be stated for the proportions $\sum_{j=1}^{n} I(S_j \geq 0)/n$ of non-negative sums, instead of those of positive sums, as well; this is because $n^{-1}\sum_{j=1}^{n} \mathbb{P}\{S_j = 0\} \to 0$ for any F not degenerating at zero.

Rényi-mixing of positive sums in the arc-sine law of Erdős and Kac [11] for independent but not necessarily identically distributed random variables *all* having zero mean and unit variance, and the sums themselves satisfying the central limit theorem, has been established by Dzhamirzaev [8]. It is pointed out at the end of the next section that our short and easy proof is also applicable in that situation.

Due to constraints on time at this writing, the related problem of Rényi-mixing of the occupation times $\sum_{j=1}^{n} I(S_j \in B)$ for bounded Borel sets B will be considered in a subsequent note.

2. Rényi-mixing of positive sums. The main result of the present note is the following

Theorem. *The sequence* $\left\{\sum_{j=1}^{n} I(S_j > 0)/n\right\}_{n=1}^{\infty}$ *is Rényi-mixing with a non-degenerate limiting distribution if and only if Spitzer's condition (1) holds, in which case*

$$\mathbb{P}\left\{\left\{\frac{1}{n}\sum_{j=1}^{n} I(S_j > 0) \leq y\right\} \cap A\right\} \to G_\rho(y)\mathbb{P}\{A\}, \quad 0 \leq y \leq 1,$$

for every event $A \in \mathcal{A}$ *and*

$$\mathbb{E}\left(\frac{1}{n^s}\left[\sum_{j=1}^{n} I(S_j > 0)\right]^s \,\middle|\, A\right) \to \frac{\Gamma(s+\rho)}{\Gamma(s+1)\Gamma(\rho)}, \quad s > 0,$$

for every event $A \in \mathcal{A}$ *of positive probability.*

Proof. Taking $A = \Omega$ in the definition of Rényi-mixing for the sequence $\xi_n := n^{-1}\sum_{j=1}^{n} I(S_j > 0) \in [0,1]$, the necessity of (1) follows by its necessity in Spitzer's theorem. (The latter is trivial since if the bounded sequence $\{\xi_n\}$ converges in distribution, then $\mathbb{E}(\xi_n) = n^{-1}\sum_{j=1}^{n} \mathbb{P}\{S_j > 0\} \in [0,1]$ must also converge to the mean of the limiting distribution function G, the support of which is contained in $[0,1]$. The limit ρ must then be in $(0,1)$, otherwise G degenerates at 0 or 1.)

Conversely, suppose that (1) holds. Then, of course, $F(\cdot)$ cannot be degenerate. Standard estimates of the concentration function of S_n imply that

$$\sum_{j=1}^{n} \mathbb{P}\{-x < S_j \leq x\} \leq C_F\, x\sqrt{n}, \quad x > 0,$$

for any non-degenerate F, for some constant C_F depending on F; this follows from Corollary 1 of Kesten [14], for example. Hence both

$$(4) \qquad \frac{1}{n}\sum_{j=1}^{n} \mathbb{P}\{-x < S_j \le 0\} \to 0 \quad \text{and} \quad \frac{1}{n}\sum_{j=1}^{n} \mathbb{P}\{0 < S_j \le x\} \to 0$$

for every $x > 0$, or, what is the same, (1) implies that $n^{-1}\sum_{j=1}^{n}\mathbb{P}\{S_j > x\} \to \rho$ for each fixed $x \in \mathbb{R}$. Then by the Markov inequality both

$$\frac{1}{n}\sum_{j=1}^{n} I(-x < S_j \le 0) \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad \frac{1}{n}\sum_{j=1}^{n} I(0 < S_j \le x) \xrightarrow{\mathbb{P}} 0$$

for every $x > 0$, where $\xrightarrow{\mathbb{P}}$ denotes convergence in probability. Hence we see by Slutsky's elementary theorem that Spitzer's condition (1) in fact implies that

$$(5) \qquad \mathbb{P}\left\{\frac{1}{n}\sum_{j=1}^{n} I(S_j > x) \le y\right\} \to G_\rho(y), \quad 0 \le y \le 1,$$

for each fixed $x \in \mathbb{R}$.

Since $n^{-1}\sum_{j=1}^{k} I(S_j > 0)$ converges to zero in probability for each fixed $k \in \mathbb{N}$, by Rényi's characterization of his mixing and by Slutsky's theorem again, it suffices to show that

$$p_{n,k}(y) := \mathbb{P}\left\{\left\{\frac{1}{n}\sum_{j=k+1}^{n} I(S_j > 0) \le y\right\} \cap \left\{\frac{1}{k}\sum_{j=1}^{k} I(S_j > 0) \le y\right\}\right\}$$

$$\to G_\rho(y)\mathbb{P}\left\{\frac{1}{k}\sum_{j=1}^{k} I(S_j > 0) \le y\right\} =: p_k(y), \quad 0 \le y \le 1,$$

for each fixed $k \in \mathbb{N}$. Let $J(x) = 1$ or $J(x) = 0$ according as $x > 0$ or $x \le 0$, consider the Borel set $B_k(y) := \{\mathbf{x}_k := (x_1, \ldots, x_k) : \sum_{j=1}^{k} J(x_j) \le ky\} \subset \mathbb{R}^k$ and, for typographical convenience, set also $\mathbf{S}_k := (S_1, \ldots, S_k)$. Understanding vectorial inequalities to hold if and only if they hold for all respective components, using the law of total probability, the independence of the terms of the underlying sequence and the stationarity of the whole sequence, for every $y \in [0,1]$ and $k \in \mathbb{N}$ we have

$$p_{n,k}(y) = \int_{B_k(y)} \mathbb{P}\left\{\frac{1}{n} \sum_{j=k+1}^{n} I(S_k + X_{k+1} + \cdots + X_j > 0) \le y \,\middle|\, S_k = x_k\right\} \times d\mathbb{P}\{S_k \le x_k\}$$

$$= \int_{B_k(y)} \mathbb{P}\left\{\frac{1}{n} \sum_{j=k+1}^{n} I(x_k + X_{k+1} + \cdots + X_j > 0) \le y\right\} d\mathbb{P}\{S_k \le x_k\}$$

$$= \int_{B_k(y)} \mathbb{P}\left\{\left(1 - \frac{k}{n}\right) \frac{1}{n-k} \sum_{j=1}^{n-k} I(S_j > -x_k) \le y\right\} \times d\mathbb{P}\{S_1 \le x_1, \dots, S_k \le x_k\}$$

$$\to G_\rho(y)\mathbb{P}\{(S_1, \dots, S_k) \in B_k(y)\} = p_k(y),$$

the convergence in the last step is by (5) and the bounded convergence theorem.

If $\mathbb{P}\{A\} > 0$ for an $A \in \mathcal{A}$, then the first statement just proved means of course nothing but $\mathbb{P}\{\xi_n \le y \mid A\} \to G_\rho(y)$, $0 \le y \le 1$, and since, skipping the second equation, (3) holds for any $s > 0$ replacing $r \in \mathbb{N}$, the conditional moments must converge as stated. ■

Now let $Y_1, Y_2, \dots$ be independent, but not necessarily identically distributed random variables such that $\mathbb{E}(Y_j) = 0$ and $\mathbb{E}(Y_j^2) = 1$ for all $j \in \mathbb{N}$, and that the asymptotic distribution of $[Y_1 + \cdots + Y_n]/\sqrt{n}$ is standard normal. Then the arc-sine law of Erdős and Kac [11] holds, and in a somewhat lengthy and complicated proof Dzhamirzaev [8] shows that in fact the sequence $\{n^{-1} \sum_{j=1}^{n} I(Y_1 + \cdots + Y_j > 0)\}$ is Rényi-mixing with the limiting distribution function $G_{1/2}(\cdot)$. (At one point in his proof, Dzhamirzaev refers for justification to an other result of Erdős and Kac [10] for the asymptotic distribution of the maximum partial sums of independent and *identically distributed* random variables. Replacing this reference by one to Theorem 9 of Rényi [17] which extends the result in question to the non-identically distributed case, the proof appears to be correct.) Since, for any $k \in \mathbb{N}$, the Erdős and Kac [11] conditions above also hold for the sequence $Y_{k+1}, Y_{k+2}, \dots$, and hence their arc-sine law is valid for the partial sums $\{S_j(k) := Y_{k+1} + \cdots + Y_j\}_{j=k+1}^{\infty}$ just as well, and since asymptotic normality implies that $n^{-1} \sum_{j=k+1}^{n} \mathbb{P}\{-x < S_j(k) \le x\} \to 0$, $x > 0$, as an analogue of (4), we see that

$$\mathbb{P}\left\{\frac{1}{n} \sum_{j=k+1}^{n} I(S_j(k) > x) \le y\right\} \to G_{1/2}(y), \quad 0 \le y \le 1, \tag{6}$$

for each fixed $x \in \mathbb{R}$ and $k \in \mathbb{N}$. Using this in place of (5), the rest of the proof above works and hence Dzhamirzaev's theorem follows.

3. Extension of Spitzer's law for Révész-dependent sequences. Let $\mathbb{Q}$ be any probability measure on the σ-algebra $\mathcal{A}$ of the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that $\mathbb{Q} \ll \mathbb{P}$. Then Theorems 1 and 2 of Rényi [18] and the theorem above imply that under condition (1) we also have (2) with $\mathbb{P}$ replaced by $\mathbb{Q}$, and hence also (3) for the expectation $\mathbb{E}_{\mathbb{Q}}(\cdot)$ with respect to $\mathbb{Q}$, and of course Spitzer's condition (1) is necessary for this to happen, or, what is the same, (1) must hold for $\mathbb{Q}$ as well. This fact offers the possibility to extend the original limit theorem to sequences of weakly dependent random variables, the distributions of which are absolutely continuous with respect to the distribution of the original sequence of independent random variables.

One such situation is known to the present author: We say that a sequence of (not necessarily identically distributed) random variables $V_1, V_2, \ldots$ is *Révész-dependent* on $(\Omega, \mathcal{A}, \mathbb{P})$ (or almost independent in the sense of Révész) if there exists a sequence $r_1, r_2, \ldots$ of non-negative numbers such that for an arbitrary system of intervals $(a_1, b_1], (a_2, b_2], \ldots$, where $a_j \leq b_j$ and $a_j, b_j \in \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$, $j \in \mathbb{N}$, we have

$$\left|\mathbb{P}\{\cap_{j=1}^{n} \{a_j < V_j \leq b_j\}\} - \mathbb{P}\{\cap_{j=1}^{n-1} \{a_j < V_j \leq b_j\}\}\mathbb{P}\{a_n < V_n \leq b_n\}\right|$$
$$\leq r_n \mathbb{P}\{\cap_{j=1}^{n-1} \{a_j < V_j \leq b_j\}\}\mathbb{P}\{a_n < V_n \leq b_n\} \qquad \text{for every } n \in \mathbb{N}$$

and $\sum_{n=1}^{\infty} r_n^2 < \infty$. Then, if $V := \{V_j\}_{j=1}^{\infty}$ is Révész-dependent, the proof of Theorem 2 of Révész [21] demonstrates that the distribution $\mathcal{L}_V$ of the sequence V is absolutely continuous with respect to the distribution $\mathcal{L}_W$ of a sequence $W := \{W_j\}_{j=1}^{\infty}$ of independent random variables such that $\mathbb{P}\{W_j \leq x\} = \mathbb{P}\{V_j \leq x\}$, $x \in \mathbb{R}$, for all $j \in \mathbb{N}$. Hence we have the following implication of the Theorem:

Corollary 1. *Let $X_1^*, X_2^*, \ldots$ be identically distributed Révész-dependent random variables with partial sums $S_j^* := X_1^* + \cdots + X_j^*$, $j \in \mathbb{N}$. If a sequence $X_1, X_2, \ldots$ of independent random variables with the same common marginal distribution as that of $X_1^*, X_2^*, \ldots$ satisfies condition* (1), *then* (2) *and* (3) *hold true for the sequence $\{S_j^*\}_{j=1}^{\infty}$ as well.*

Of course, Révész's almost independence is a rather weak form of dependence. Considering how trivial is the Rényi-mixing of positive sums of independent random variables once we have (5) or (6), one cannot honestly hope for more from this source. However, even the slight departure from independence in Corollary 1 is perhaps of some interest if indeed it is the first such departure; the present author is unaware of any other in the generalized arc-sine law.

If $Y_1^*, Y_2^*, \ldots$ are not necessarily identically distributed Révész-dependent random variables with zero mean and unit variance such that the corresponding independent sequence $Y_1, Y_2, \ldots$, with $\mathbb{P}\{Y_j \le x\} = \mathbb{P}\{Y_j^* \le x\}$, $x \in \mathbb{R}$, $j \in \mathbb{N}$, satisfies the Lindeberg condition, then by Dzhamirzaev's theorem (demonstrated above) it also follows that

$$\mathbb{P}\{n^{-1} \sum_{j=1}^{n} I(Y_1^* + \cdots + Y_j^* > 0) \le y\} \to G_{1/2}(y), \quad 0 \le y \le 1,$$

as an extension of the arc-sine law of Erdős and Kac [11]. In this case, however, presumably better extensions must follow directly from generalizations of Prohorov's weak convergence theorem for the partial sum process, with the limit being the distribution of Brownian motion, to various weakly dependent sequences; we did not explore the literature in this connection.

Révész [21] stated his result in the context of limit distributions for *sums* of random variables, which was at the time the only known example of Rényi-mixing of random variables for independent terms. Since then this mixing property has been shown to accompany theorems on limiting distributions for various other functions of independent random variables. Every such theorem extends to Révész-dependent sequences. Examples are Gnedenko's theorems for maxima and the four theorems of Erdős and Kac [10] for maxima of partial sums and of moduli of partial sums and for sums of moduli and squares of partial sums, discussed in [4] with earlier references. (It is a natural question in itself whether Révész's form of weak dependence for V is the best to ensure $\mathcal{L}_V \ll \mathcal{L}_W$; any form would work with Rényi-mixing.)

4. The generalized arc-sine law for a random number of games. Preserving the notation of the first two sections, let additionally $\{\nu_n\}_{n=1}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$ be two sequences of random variables, which may be chosen to be the same and are given on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ where the basic sequence $\{X_n\}_{n=1}^{\infty}$ of independent and identically distributed variables is defined, such that ν_n and μ_n for each $n \in \mathbb{N}$ take on only positive integers as possible values and, for the same non-decreasing sequence $\{d_n\}_{n=1}^{\infty}$ of positive numbers such that $d_n \to \infty$, both

$$\frac{\nu_n}{d_n} \xrightarrow{\mathbb{P}} \nu \quad \text{and} \quad \frac{\mu_n}{d_n} \xrightarrow{\mathbb{P}} \mu$$

for some random variables ν and μ for which $\mathbb{P}\{\nu > 0\} = 1$ and $\mathbb{P}\{\mu > 0\} = 1$.

A special case of Theorem 1 in [4] says that if the sequence of ratios of the terms of an almost surely non-decreasing sequence of random variables to the corresponding terms of a regularly varying norming sequence of positive constants is Rényi-mixing with a non-degenerate limiting distribution function $G(\cdot)$, as $n \to \infty$, then so is the randomly selected subsequence of ratios indexed by ν_n as above. Very recently Kruglov and Zhang [15] proved that (for a sequence $\{\nu_n\}$ being fixed) the converse is also true, at least when the regularly varying norming sequence has a positive exponent and is non-decreasing itself: Rényi-mixing of the randomly indexed sequence of ratios with a limiting distribution function $G(\cdot)$ implies the same for the original sequence of ratios.

Putting together the theorem above and these results, along with a consequence of Rényi-mixing for the randomly indexed sequence specialized from Theorem 2 in [4], we obtain

Corollary 2. *The sequence* $\left\{ \sum_{j=1}^{\nu_n} I(S_j > 0)/\nu_n \right\}_{n=1}^{\infty}$ *is Rényi-mixing with a non-degenerate limiting distribution if and only if Spitzer's condition (1) holds, in which case*

$$\mathbb{P}\left\{ \left\{ \frac{1}{\mu_n} \sum_{j=1}^{\nu_n} I(S_j > 0) \le y \right\} \cap A \right\} \to \int_A G_\rho\left(\frac{\nu}{\mu} y \right) d\mathbb{P}, \quad y \ge 0,$$

for every event $A \in \mathcal{A}$, where $G_\rho(t) = 1$ for $t \ge 1$, and

$$\mathbb{E}\left(\frac{1}{\nu_n^s} \left[\sum_{j=1}^{\nu_n} I(S_j > 0) \right]^s \middle| A \right) \to \frac{\Gamma(s+\rho)}{\Gamma(s+1)\Gamma(\rho)}, \quad s > 0, \tag{7}$$

for every event $A \in \mathcal{A}$ of positive probability.

In particular, to spell out some unconditional special cases, if (1) holds, then

$$\mathbb{P}\left\{ \frac{1}{\mu_n} \sum_{j=1}^{\nu_n} I(S_j > 0) \le y \right\} \to \mathbb{E}\left(G_\rho\left(\frac{\mu}{\nu} y \right) \right), \quad y \ge 0.$$

Under condition (1), this statement contains of course that

$$\mathbb{P}\left\{ \frac{1}{\nu_n} \sum_{j=1}^{\nu_n} I(S_j > 0) \le y \right\} \to G_\rho(y), \quad 0 \le y \le 1, \tag{8}$$

which is the essence of Corollary 2, and the entertaining variants

$$\mathbb{P}\left\{ \frac{1}{d_n} \sum_{j=1}^{\nu_n} I(S_j > 0) \le y \right\} \to \mathbb{P}\{\nu \le y\} + \int_y^{\infty} G_\rho\left(\frac{y}{x} \right) d\mathbb{P}\{\nu \le x\}, \quad y \ge 0,$$

and

$$\mathbb{P}\left\{\frac{1}{\mu_n}\sum_{j=1}^{\lfloor d_n\rfloor} I(S_j>0)\le y\right\}\to\mathbb{P}\left\{\mu>\frac{1}{y}\right\}+\int_0^{1/y} G_\rho(xy)d\mathbb{P}\{\mu\le x\}\,,\ y\ge 0\,.$$

More interesting than these last two limit statements is to close with some concrete examples for (8), in which $\{\nu_n\}$ is constructed from the sequence $X_1, X_2, \ldots$ itself. Besides (1), suppose also that $m_\gamma := \mathbb{E}(|X|^\gamma) \in (0,\infty)$ for some $\gamma > 0$. (For example, if $\gamma \ge 2$, then (1) holds with $\rho = 1/2$. On the other hand, if F is in the domain of attraction of a stable law with characteristic exponent $\alpha \in (0,2]$, skewness parameter $\beta \in (-1,1)$ and zero location parameter, where we take $\beta = 0$ if $\alpha = 1$ and assume that $\mathbb{E}(X) = 0$ if $\alpha \in (1,2]$, then (1) is satisfied with $\rho = 1/2$ for $\alpha = 1$ and with $\rho = \frac{1}{2} + \frac{1}{\alpha\pi}\arctan\left(\beta\tan\frac{\alpha\pi}{2}\right)$ for $\alpha \ne 1$, and any $\gamma \in (0,\alpha)$ works; see [5] for instance concerning the details here.) Let $R_k^\gamma := |X_1|^\gamma + \cdots + |X_k|^\gamma, k \in \mathbb{N}$, and for each $n \in \mathbb{N}$, define $\nu_n(1) := R_n^\gamma$, $\nu_n(2) := \min\{k \in \mathbb{N} : R_k^\gamma > n\}$, $\nu_n(3) := \max\{k \in \mathbb{N} : R_k^\gamma \le n\}$ and $\nu_n(4) := \min\{k \in \mathbb{N} : R_k^\gamma > nk^\delta\}$ for some $\delta \in (0,1)$. Then the strong law of large numbers implies that $\nu_n(1)/n \to m_\gamma$, $\nu_n(2)/n \to m_\gamma^{-1}$, $\nu_n(3)/n \to m_\gamma^{-1}$ and $\nu_n(4)/n^{1/(1-\delta)} \to m_\gamma^{1/(\delta-1)}$, in each case almost surely; see Section 5.4 in [3] for example. Thus we have (7) and (8) for all four choices of $\nu_n \equiv \nu_n(l)$, $l = 1,2,3,4$.

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