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Estimate of the Third Coefficient of a Univalent, Bounded, Symmetric and Nonvanishing Function

To Professor Eligiusz Złotkiewicz on His 60th birthday

ABSTRACT. Let $B_0^{(R)}(b)$, $0 < b < 1$, denote the class of functions $F(z) = b + A_1 z + A_2 z^2 + A_3 z^3 + \dots$, analytic and univalent in the unit disk U, which satisfy the conditions $F(U) \subset U$, $0 \notin F(U)$, $\operatorname{Im} F^{(n)}(0) = 0$, $n = 1, 2, \dots$, $A_1 > 0$. The class $B_0^{(R)}(b)$, introduced by the authoress in [8], [9], is a subclass of the class B_u of bounded, nonvanishing, univalent functions in the unit disk. The last class and closely related ones have been studied recently by various authors in [6], [2], [1], [3], [7]. There was found the exact lower bound of the coefficient A_3 in the class $B_0^{(R)}(b)$. The result was obtained by using the estimates of the functional $a_3 + \alpha a_2$ in the family of univalent, bounded and symmetric functions. The lower bound of this functional was found by Jakubowski in [5].

Introduction. Let $B_0^{(R)}(b)$, $0 < b < 1$, denote the class of all functions F that are analytic, univalent in the unit disk U and satisfy the conditions: $F(U) \subset U$, $F(0) = b$, $0 \notin F(U)$, $\operatorname{Im} F^{(n)}(0) = 0$, $n = 1, 2, \dots$, $F'(0) > 0$. Let

$$F(z) = b + A_1 z + A_2 z^2 + A_3 z^3 + \dots \tag{1}$$

and

$$L(z) = K^{-1}\left(\frac{4b}{(1-b)^2}\left(K(z)+\frac{1}{4}\right)\right) \qquad (2)$$
$$= b + B_1 z + B_2 z^2 + B_3 z^3 + \dots,$$

where $K(z) = z/(1-z)^2$,

$$B_1 = \frac{4b(1-b)}{1+b}$$
$$B_2 = \frac{-8b(1-b)(b^2+2b-1)}{(1+b)^3}, \qquad (3)$$
$$B_3 = \frac{4b(1-b)}{(1+b)^5}(3(1+b)^4 - 32b),$$

$L(U) = U \setminus (-1, 0]$. It is known from [7], [8], that

$$0 < A_1 \le \frac{4b(1-b)}{1+b},$$
$$-b(1-b)^2 \le A_2 \le \begin{cases} \dfrac{-8b(1-b)(b^2+2b-1)}{(1+b)^3}, & 0 < b \le \frac{2}{3}\sqrt{3} - 1, \\ \dfrac{1-b^2}{b+2}, & \frac{2}{3}\sqrt{3} - 1 \le b < 1. \end{cases}$$

Let $\mathcal{B}_0^{(R)}(b,T)$, $0 < T \le 1$, denote a subclass of such functions from $\mathcal{B}_0^{(R)}(b)$, that $A_1 = [4b(1-b)/(1+b)]T$. $\mathcal{B}_0^{(R)}(b,T)$ are not empty because $L_T(z) = L(Tz) \in \mathcal{B}_0^{(R)}(b,T)$. Moreover $\mathcal{B}_0^{(R)}(b) = \bigcup_{0<T\le 1} \mathcal{B}_0^{(R)}(b,T)$ and $\mathcal{B}_0^{(R)}(b,T)$ is a compact family. Hence there exists in this family a function with the smallest coefficient A_3 and

$$\inf_{\mathcal{B}_0^{(R)}(b)} A_3 = \inf_{0<T\le 1}\left(\min_{\mathcal{B}_0^{(R)}(b,T)} A_3\right).$$

Let now $S_1^{(R)}(T)$ denote the class of all functions of the form

$$f(z) = T(z + a_2 z^2 + a_3 z^3 + \dots), \qquad (4)$$

that are analytic and univalent in U and satisfy the conditions $f(U) \in U$, $\operatorname{Im} a_n = 0$ $n = 2, \dots$. The class $\mathcal{B}_0^{(R)}(b,T)$ is related with the class $S_1^{(R)}(T)$ through the function (2).

In fact, if $f \in S_1^{(R)}(T)$, then $L \circ f \in \mathcal{B}_0^{(R)}(b,T)$ and conversely if $F \in \mathcal{B}_0^{(R)}(b,T)$, then $L^{-1} \circ F \in S_1^{(R)}(T)$. The relation $F = L \circ f$, the formulas (1), (2), (3), (4) and an application of the formula

$$A_3 = B_1 T\left(a_3 - \frac{4(b^2+2b-1)}{(1+b)^2} T a_2 + \left(3 - \frac{32b}{(1+b)^4}\right) T^2\right) \tag{5}$$

allow us to express the coefficient A_3 of a function from $\mathcal{B}_0^{(R)}(b,T)$ through the coefficients a_2 and a_3 of the function from $S_1^{(R)}(T)$.

1. Estimation of the coefficient A_3 in the class $\mathcal{B}_0^{(R)}(b,T)$. To find the lower bound of the right-hand side of (5) we will use the Jakubowski Theorem [5], p. 213:

Theorem. *Let R_T, $0 < T \le 1$, denote the family of functions analytic and univalent in U of the form*

$$f(z) = b_1(z + a_2 z^2 + a_3 z^3 + \dots),$$

where $f(U) \in U$, $\operatorname{Im} a_n = 0$, $n = 1, 2, \dots$, $b_1 \ge T$. Let

$$G(f) = a_3 + \alpha a_2, \tag{6}$$

where $\alpha \ge 0$. For each function $f \in R_T$:

$$G(f) \ge \begin{cases} 3 - 2\alpha + 2(\alpha - 4)T + 5T^2 & \text{for } \alpha > 4(1-T), \\ -1 - \frac{1}{4}\alpha^2 + T^2 & \text{for } 0 \le \alpha \le 4(1-T). \end{cases} \tag{7}$$

This estimate is sharp and the extremal functions $w = f(z)$ satisfy the equations

$$\frac{w}{(1+w)^2} = \frac{Tz}{(1+z)^2}, \tag{8}$$

$$T(w + w^{-1}) = z + z^{-1} + \frac{1}{2}\alpha, \tag{9}$$

respectively.

Remark 1. In the case $\alpha < 0$ the lower bound of $G(f)$ is given by

$$G(f) \ge \begin{cases} 3 + 2\alpha - 2(\alpha + 4)T + 5T^2 & \text{for } \alpha < -4(1-T), \\ -1 - \frac{1}{4}\alpha^2 + T^2 & \text{for } -4(1-T) \le \alpha < 0. \end{cases} \tag{10}$$

This estimate is sharp and the extremal functions $w = f(z)$ satisfy

$$\frac{w}{(1-w)^2} = \frac{Tz}{(1-z)^2}, \tag{11}$$

$$T(w + w^{-1}) = z + z^{-1} + \frac{1}{2}\alpha, \tag{12}$$

respectively.

In fact, let us notice that if f belongs to R_T then also $-f(-z)$ is in R_T, and hence the sets of values of the functionals $a_3 + \alpha a_2$ and $a_3 - \alpha a_2$ coincide and if f minimizes the functional $a_3 + \alpha a_2$ then $-f(-z)$ minimizes $a_3 - \alpha a_2$ and conversely.

Remark 2. Since for the extremal functions (8), (9) or (11), (12) we have $b_1 = T$, it follows that the bounds (7) and (10) occur also in the class $S_1^{(R)}(T)$ which is a subclass of the class R_T.

Let us put $\alpha = [-4(b^2 + 2b - 1)/(1+b)^2]T$ in (6). If $0 < b \le \sqrt{2} - 1$ then $\alpha \ge 0$, hence, according to (5) and (7), we have following inequalities in the class $\mathcal{B}_0^{(R)}(b, T)$:

$$A_3 = B_1 T\left(G(f) + \left(3 - \frac{32b}{(1+b)^4}\right)T^2\right)$$

$$\ge \begin{cases} B_1 T\left(3 - \dfrac{16}{(1+b)^2}T + \dfrac{16(1+b)^2}{(1+b)^4}T^2\right) & \text{for } \dfrac{(1+b)^2}{2} \le T \le 1, \\ B_1 T\left(\dfrac{16b^2}{(1+b)^4}T^2 - 1\right) & \text{for } 0 < T \le \dfrac{(1+b)^2}{2}, \end{cases} \tag{13}$$

and the extremal functions are compositions of the function L with the functions $w = f(z)$, where

$$\frac{w}{(1+w)^2} = \frac{Tz}{(1+z)^2} \quad \text{or} \quad T(w + w^{-1}) = z + z^{-1} - \frac{2(b^2 + 2b - 1)}{(1+b)^2}T,$$

respectively.

If, on the contrary, $\sqrt{2} - 1 \le b < 1$ then $\alpha \le 0$, and hence, according to (5) and (10), the following inequalities hold in the class $\mathcal{B}_0^{(R)}(b, T)$:

$$A_3 = B_1 T\left(G(f) + \left(3 - \frac{32b}{(1+b)^4}\right)T^2\right)$$

$$\geq \begin{cases} B_1 T\left(3 - \dfrac{16b(b+2)}{(1+b)^2}T + \dfrac{16b^2(b^2+4b+5)}{(1+b)^4}T^2\right) & \text{for } \dfrac{(1+b)^2}{2b(b+2)} \leq T \leq 1, \\ B_1 T\left(\dfrac{16b^2}{(1+b)^4}T^2 - 1\right) & \text{for } 0 < T \leq \dfrac{(1+b)^2}{2b(b+2)}, \end{cases} \tag{14}$$

and the extremal functions are compositions of the function L with the functions $w = f(z)$, where

$$\frac{w}{(1-w)^2} = \frac{Tz}{(1-z)^2} \quad \text{or} \quad T(w + w^{-1}) = z + z^{-1} - \frac{2(b^2+2b-1)}{(1+b)^2}T,$$

respectively.

2. Estimation of the coefficient A_3 in the class $\mathcal{B}_0^{(R)}(b)$.

Theorem. *In the class $\mathcal{B}_0^{(R)}(b)$*

$$A_3 \geq \begin{cases} -2b(1-b^2)\dfrac{10 - 54b^2 + (7-9b^2)^{3/2}}{27(1+b^2)^2} & \textit{for } 0 < b \leq \dfrac{1}{2\sqrt{3}}, \quad (15) \\ -\dfrac{2}{3\sqrt{3}}(1-b^2) & \textit{for } \dfrac{1}{2\sqrt{3}} \leq b < 1. \quad (16) \end{cases}$$

These inequalities are sharp. The extremal functions are compositions of the function L and the functions $w = f(z)$, where

$$\frac{w}{(1+w)^2} = \frac{Tz}{(1+z)^2}, \quad T = \frac{4 + \sqrt{7-9b^2}}{12(1+b^2)}(1+b)^2, \tag{17}$$

$$T(w + w^{-1}) = z + z^{-1} - \frac{2(b^2+2b-1)}{(1+b)^2}T, \quad T = \frac{(1+b)^2}{4\sqrt{3}b}, \tag{18}$$

respectively.

The first function maps the disk U onto $U \setminus (-1, 0] \setminus [c, 1)$, where c is a smaller root of the equation

$$c^2 - \frac{1}{b}\left(b^2 - \frac{4 + \sqrt{7-9b^2}}{12(1+b^2)}(1-b^2)^2\right)c + 1 = 0,$$

and the second one maps the disk U onto $U \setminus (-1, c] \setminus [d, 1)$, where c is a smaller root of the equation

$$\frac{c}{(1-c)^2} = \frac{1}{(1-b)^2}\frac{2\sqrt{3}b - 1}{2\sqrt{3}+b+2},$$

and d is a smaller root of the equation

$$\frac{d}{(1-d)^2} = \frac{1}{(1-b)^2}\frac{2\sqrt{3}b + 1}{2\sqrt{3}-b-2}.$$

Proof. In order to find the infimum of the coefficient A_3 in the class $\mathcal{B}_0^{(R)}(b)$ it is necessary to calculate the infimum with respect to $T \in (0, 1]$ of the functions that are on the right-hand side of the inequality (13) in the case $0 < b \le \sqrt{2} - 1$, and inequality (14) in the case $\sqrt{2} - 1 \le b < 1$.

Let $0 < b \le \sqrt{2} - 1$. Let us put:

$$P(T) = \begin{cases} 3T - \dfrac{16}{(1+b)^2}T^2 + \dfrac{16(1+b^2)}{(1+b)^4}T^3, & \dfrac{(1+b)^2}{2} \le T \le 1, \\ \dfrac{16b^2}{(1+b)^4}T^3 - T, & 0 < T \le \dfrac{(1+b)^2}{2}. \end{cases}$$

First, we are going to find the infimum of $P(T)$ in the interval $[(1+b)^2/2, 1]$. Since $T_{1,2} = [(4 \pm \sqrt{7 - 9b^2})/12(1+b^2)](1+b)^2$ are zeros of the derivative $P'(T)$, $T_1 < (1+b)^2/2$ for $0 < b \le \sqrt{2} - 1$ and $T_2 < (1+b)^2/2$ for $\frac{1}{2\sqrt{3}} \le b \le \sqrt{2} - 1$ as well as $T_2 \in [(1+b)^2/2, 1]$ for $0 < b \le \frac{1}{2\sqrt{3}}$, then

$$\inf_{[(1+b)^2/2, 1]} P(T) = \begin{cases} P(T_2) = -\dfrac{(1+b)^2}{2}\dfrac{10 - 54b^2 + (7-9b^2)^{3/2}}{12(1+b^2)^2} \\ \qquad \text{for} \quad 0 < b \le \dfrac{1}{2\sqrt{3}} & (19) \\ P\left(\dfrac{(1+b)^2}{2}\right) = -\dfrac{(1+b)^2}{2}(1 - 4b^2) \\ \qquad \text{for} \quad \dfrac{1}{2\sqrt{3}} \le b \le \sqrt{2} - 1. & (20) \end{cases}$$

Analogously, looking for the infimum of $P(T)$ in the interval $(0, (1+b)^2/2]$, we notice that for $0 < b \le 1/2\sqrt{3}$ we have

$$[0, (1+b)^2/2] \subset [T_1, T_2],$$

where $T_{1,2} = \pm(1+b)^2/(4\sqrt{3}b)$ are zeros of the derivative $P'(T)$, and $T_2 \in (0,(1+b)^2/2)$ for $\frac{1}{2\sqrt{3}} \le b \le \sqrt{2}-1$. Hence

$$\inf_{(0,(1+b)^2/2]} P(T) = \begin{cases} P\left(\frac{(1+b)^2}{2}\right) = -\frac{(1+b)^2}{2}(1-4b^2) & \text{for} \quad 0 < b \le \frac{1}{2\sqrt{3}}. \quad (21) \\ P(T_2) = -\frac{(1+b)^2}{6\sqrt{3}b} & \text{for} \quad \frac{1}{2\sqrt{3}} \le b \le \sqrt{2}-1. \quad (22) \end{cases}$$

Comparing the estimates (19) and (21), we come to a conclusion that for $0 < b \le \frac{1}{2\sqrt{3}}$ the coefficient A_3 of an arbitrary function from the class $\mathcal{B}_0^{(R)}(b)$ satisfies the inequality (15) and the equality occurs for the composition of L and the function (17). Comparing the estimates (20) and (22), we conclude that for $\frac{1}{2\sqrt{3}} \le b \le \sqrt{2}-1$ the coefficient A_3 of an arbitrary function from the class $\mathcal{B}_0^{(R)}(b)$ satisfies the inequality (16) and the equality occurs for the composition of the function L and the function (18). Let now $\sqrt{2}-1 \le b < 1$. Let us put

$$P(T) = \begin{cases} 3T - \frac{16b(b+2)}{(1+b)^2}T^2 + \frac{16b^2(b^2+4b+5)}{(1+b)^4}T^3, & \frac{(1+b)^2}{2b(b+2)} \le T \le 1, \\ \frac{16b^2}{(1+b)^4}T^3 - T, & 0 < T \le \frac{(1+b)^2}{2b(b+2)}. \end{cases}$$

First, looking for the minimum $P(T)$ in the interval $[(1+b)^2/(2b(b+2)), 1]$, we conclude that

$$T_{1,2} = \frac{4(b+2) \pm \sqrt{7b^2+28b+19}}{12b(b^2+4b+5)}(1+b)^2$$

are zeros of the derivative $P'(T)$ and $T_2 < (1+b)^2/(2b(b+2))$. Hence

$$\inf_{[(1+b)^2/(2b(b+2)),1]} P(T) = P\left(\frac{(1+b)^2}{2b(b+2)}\right) = -\frac{(1+b)^2(b+4)}{2(b+2)^3}. \quad (23)$$

Analogously, looking for the minimum $P(T)$ in $(0,(1+b)^2/(2b(b+2))]$, we conclude that

$$P'(T) = 0 \quad \text{for} \quad T_{1,2} = \pm\frac{(1+b)^2}{4\sqrt{3}b} \quad \text{and} \quad T_2 < \frac{(1+b)^2}{2b(b+2)}.$$

Hence

$$\inf_{(0,(1+b)^2/(2b(b+2))]} P(T) = P\left(\frac{(1+b)^2}{4\sqrt{3b}}\right) = -\frac{(1+b)^2}{6\sqrt{3b}}. \quad (24)$$

Comparing (23) and (24) we come to a conclusion that for $\sqrt{2}-1 \leq b < 1$ the coefficient A_3 of an arbitrary function from the class $\mathcal{B}_0^{(R)}(b)$ satisfies the inequality (16) and the equality occurs for the composition of the function L with the function (18). The theorem is thus proved.

Remark 3. There is also known the upper bound of the functional $a_3 + \alpha a_2$ in R_T [4], but by using this estimate we can not find the exact upper bound of A_3 in the explicit form in the whole class $\mathcal{B}_0^{(R)}(b)$.

References

[1] Duren, P. and G. Schober, *Nonvanishing univalent functions*, Math. Z. **170** (1980), 195-216.

[2] Hummel, J., S. Scheinberg and L. Zalcman, *A coefficient problem for bounded non-vanishing functions*, J. Anal. Math. **31** (1977), 169-190.

[3] Hummel, J. A., B. Pinchuk, *Variations for bounded nonvanishing univalent functions*, J. Anal. Math. **44** (1984-85), 183-199.

[4] Jakubowski, Z. J., *Maksimum funkcjonalu $A_3 + \alpha A_2$ w rodzinie funkcji jednolistnych o współczynnikach rzeczywistych*, Zeszyty Naukowe U.L. Seria II **20** (1966), 147-160.

[5] ———, *Sur les coefficientes des fonctions univalentes dans le cercle unité*, Ann. Polon. Math. **19** (1967), 207-233.

[6] Krzyż, J. G., *Coefficient problem for bounded nonvanishing functions*, Ann. Polon. Math. **70** (1968), 314.

[7] Prokhorov, D. V. and J. Szynal, *Coefficient estimates for bounded nonvanishing functions*, Bull. Polish Acad. Sci. Math. **29** (1981), 223-230.

[8] Śladkowska, J., *On the univalent, bounded, non-vanishing and symmetric functions in the unit disk*, Ann. Polon. Math. **64(3)** (1996), 291-299.

[8] ———, *Estimations of the second coefficient of the univalent, bounded, symmetric and non-vanishing functions by means of Loewner's parametric method*, Ann. Polon. Math. (to appear).

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