## ANNALES

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## Starlikeness Properties for Convolutions Involving Hypergeometric Series

Dedicated to Professor Eligiusz Zlotkiewicz on the occasion of his 60th birthday

Abstract. For $\beta<1$ let $\mathcal{R}(\beta)$ denote the family of normalized analytic functions $f$ defined in the unit disc $\Delta$ such that for some $\eta \in(-\pi / 2, \pi / 2)$ $\operatorname{Re}\left[e^{i \eta}\left(f^{\prime}(z)-\beta\right)\right]>0$ for $z \in \Delta$. Further, let $g$ be univalent, or in $\mathcal{R}(\beta)$. Given the Gaussian hypergeometric function

$$
F(a, b ; c ; z)=\sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)(1, n)} z^{n},
$$

we find conditions on the parameters $a, b, c$ and $\beta$ so that the Hadamard product $f(z)=z F(a, b ; c ; z) * g(z)$ will have the starlikeness property given by $\left|z f^{\prime}(z) / f(z)-1\right|<\lambda$ where $\lambda>0$.

1. Introduction. Throughout, $\mathbb{C}$ denotes the complex plane and we denote the unit disc in $\mathbb{C}$ by $\Delta$. Let $\mathcal{A}$ denote the family of all functions $f$ analytic

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in $\Delta$ with the usual normalization $f(0)=f^{\prime}(0)-1=0$. The following subfamilies of $\mathcal{A}$ have been studied extensively:

$$
\begin{gathered}
\mathcal{S}=\{f \in \mathcal{A} \mid f \text { is univalent in } \Delta\} \\
\mathcal{S}^{*}(\beta)=\left\{f \in \mathcal{A} \mid \operatorname{Re}\left(z f^{\prime}(z) / f(z)\right)>\beta, z \in \Delta\right\}
\end{gathered}
$$

where $\beta<1$. The functions in $\mathcal{S}^{*}(\beta)$ are called starlike of order $\beta$. It is well-known that $\mathcal{S}^{*}(\beta) \subset \mathcal{S}^{*}(0) \equiv \mathcal{S}^{*} \subset \mathcal{S}$ for $0 \leq \beta<1$, but for $\beta<0$ the functions in $\mathcal{S}^{*}(\beta)$ need not be univalent. In addition to these basic classes of starlike functions, several subclasses and generalizations of $\mathcal{S}^{*}(\beta)$ have been defined and studied over the years. In this paper we will in particular be concerned with the classes $\mathcal{S}_{\lambda}^{*}, \lambda>0$, defined by

$$
\mathcal{S}_{\lambda}^{*}=\left\{\left.g \in \mathcal{A}| | \frac{z g^{\prime}(z)}{g(z)}-1 \right\rvert\,<\lambda, z \in \Delta\right\} .
$$

It is long known that a sufficient condition for $g(z)=z+\sum_{n=2}^{\infty} b_{n} z^{n}$ to belong to the class $\mathcal{S}^{*}$ is that $\sum_{n=2}^{\infty} n\left|b_{n}\right| \leq 1$. In fact, the same condition is also sufficient for $g \in \mathcal{S}_{1}^{*} \subset \mathcal{S}^{*}$, a result which has been proved by several authors, among others by Goodman [2]. A simple extension of this result is the following, by Silverman [12]:

$$
\begin{equation*}
\sum_{n=2}^{\infty}(n+\lambda-1)\left|b_{n}\right| \leq \lambda \Longrightarrow g \in \mathcal{S}_{\lambda}^{*} \tag{1.1}
\end{equation*}
$$

For $\lambda=1 / 2$ this was proved earlier by Schild [11].
We note that $\mathcal{S}_{\lambda}^{*} \subset \mathcal{S}^{*}(1-\lambda)$. The classes $\mathcal{S}_{\lambda}^{*}$ have turned out to be especially important in working with functions of the type

$$
\begin{equation*}
g(z)=z-\sum_{n=2}^{\infty} b_{n} z^{n}, b_{n} \geq 0 \tag{1.2}
\end{equation*}
$$

When we restrict to functions with real and negative coefficients as in (1.2) we actually have $S_{\lambda}^{*}=\mathcal{S}^{*}(1-\lambda)$, and then the coefficient condition (1.1) becomes both necessary and sufficient for membership in $S_{\lambda}^{*}$ [12], [13]. In the case $\lambda=1$, the equivalence in (1.1) for functions of the type (1.2) was first proved by Ozaki [4, Theorem 1'] in 1934.

For $\beta<1$ define

$$
\mathcal{R}(\beta)=\left\{f \in \mathcal{A}|\exists \eta \in(-\pi / 2, \pi / 2)| \operatorname{Re}\left[e^{i \eta}\left(f^{\prime}(z)-\beta\right)\right]>0, \quad z \in \Delta\right\} .
$$

Note that when $\beta \geq 0$, we have $\mathcal{R}(\beta) \subset \mathcal{S}$, but for each $\beta<0, \mathcal{R}(\beta)$ contains also non-univalent functions. For any complex number $a$, we define the ascending factorial notation $(a, n)=a(a+1) \cdots(a+n-1)=a(a+1, n-1)$ for $n \geq 1$ and $(a, 0)=1$ for $a \neq 0$. When $a$ is neither zero nor a negative integer, we can write $(a, n)=\Gamma(n+a) / \Gamma(a)$.

We shall consider the Gaussian hypergeometric function (series)

$$
F(a, b ; c ; z)=\sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)(1, n)} z^{n}, a, b, c \in \mathbb{C},
$$

where $c$ is neither zero nor a negative integer. The following well-known formula

$$
\begin{equation*}
F(a, b ; c ; 1)=\frac{\Gamma(c-a-b) \Gamma(c)}{\Gamma(c-a) \Gamma(c-b)}, \quad \operatorname{Re}(c-a-b)>0 \tag{1.3}
\end{equation*}
$$

will be used frequently. We recall that the function $F(a, b ; c ; z)$ is bounded if $\operatorname{Re}(c-a-b)>0$, and has a pole at $z=1$ if $\operatorname{Re}(c-a-b) \leq 0$, see e.g. [1]. Univalence, starlikeness and convexity properties of $z F(a, b ; c ; z)$ have been studied extensively e.g. in [7] and [10]. For $f \in \mathcal{A}$, we define the operator $\mathcal{I}_{a, b ; c}(f)$ by $\left[\mathcal{I}_{a, b ; c}(f)\right](z)=z F(a, b ; c ; z) * f(z)$, where * denotes the usual Hadamard product (convolution) of power series. Using the integral representation

$$
F(a, b ; c ; z)=\frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_{0}^{1} t^{b-1}(1-t)^{c-b-1} \frac{d t}{(1-t z)^{a}}, \operatorname{Re} c>\operatorname{Re} b>0
$$

we can write

$$
\begin{equation*}
\left[\mathcal{I}_{a, b ; c}(f)\right](z)=\frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_{0}^{1} t^{b-1}(1-t)^{c-b-1} \frac{f(t z)}{t} d t * \frac{z}{(1-z)^{a}} \tag{1.4}
\end{equation*}
$$

Some special cases of this operator should be mentioned explicitly. For $a=1, b=1+\gamma$ and $c=2+\gamma$ with $\operatorname{Re} \gamma>-1,(1.4)$ turns into the so-called Bernardi operator

$$
\mathcal{B}_{f}(z) \equiv\left[\mathcal{I}_{1, \gamma+1 ; \gamma+2}(f)\right](z)=(1+\gamma) \int_{0}^{1} t^{\gamma-1} f(t z) d t
$$

which for $\gamma=1,2$ reduces to the Alexander and Libera transforms, respectively. We remark that these three operators are all examples of the situation where $c=a+b$ in $\mathcal{I}_{a, b ; c}(f)$. Further, we have

$$
\frac{z}{(1-z)^{\delta+1}} * f(z)=\left[\mathcal{I}_{1, \delta+1 ; 1}(f)\right](z), \quad \delta>-1,
$$

which is known as the Ruscheweyh transform (derivative) studied in [8] (see also [9]), and later by a large number of authors. We observe that the Ruscheweyh transform represents the case $c<a+b$ with $a=1, b=1+\delta$ and $\mathrm{c}=1$. For a survey of special cases of the operator $\mathcal{I}_{a, b ; c}$, and more general operators, we refer to the article by Srivastava [14], where also a long list of other references can be found. Thus, the operator $\mathcal{I}_{a, b ; c}(f)$, and hence the Gaussian hypergeometric function, is a natural object for studying inclusion properties related to the convolution product.

The investigations in this paper are mainly motivated by the desire for more information about how the operator $\mathcal{I}_{a, b ; c}(f)$ acts on various classes of analytic functions. Problems of this type naturally fall into one of three categories, depending on whether $\operatorname{Re}(c-a-b)$ is larger than, equal to or less than zero.

In each of these categories, with $a, b, c>0$, examples of univalence as well as non-univalence of the function $z F(a, b ; c ; z)$ can be found in the recent paper of Ponnusamy and Vuorinen [7]. Some other special cases of the operator $\mathcal{I}_{a, b ; c}(f)$ may also be found in [6], [10].

In these papers the methods applied are entirely different from the ones in the present paper. Our results here all deal with the case $c>\operatorname{Re}(a+b)$, and we find conditions on $a, b$ and $c$ such that when $\mathcal{F}_{1}=\mathcal{R}(\beta)$ or $\mathcal{S}$ and $\mathcal{F}_{2}=S_{\dot{\lambda}}^{*}$ the operator $\mathcal{I}_{a, b ; c}$ maps $\mathcal{F}_{1}$ into $\mathcal{F}_{2}$.

We remark that our methods cannot be used in the case $c \leq \operatorname{Re}(a+b)$, and the reason for this is that then the series $F(a, b ; c ; 1)$ diverges. Starlikeness properties of integral transforms have also been studied by the authors in the recent paper [5] using duality theory for convolutions. We will develop some of the results from [5] a little further, and also compare these results with the results that we get with other methods.
2. Main results. The first results that we present deal with the case $\mathcal{F}_{1}=\mathcal{R}(\beta)$ and $\mathcal{F}_{2}=\mathcal{S}_{\lambda}^{*}$.
Theorem 2.1.
Let $a, b \in \mathbb{C} \backslash\{0\}, c>0,|a| \neq 1,|b| \neq 1, c \neq 1$ and $c>|a|+|b|$. For $\lambda \in(0,1]$, assume that

$$
\begin{align*}
\frac{\Gamma(c-|a|-|b|) \Gamma(c)}{\Gamma(c-|a|) \Gamma(c-|b|)} & {\left[1+\frac{(\lambda-1)(c-|a|-|b|)}{(|a|-1)(|b|-1)}\right] } \\
& \leq \lambda\left\{1+\frac{1}{2(1-\beta)}\right\}+\frac{(\lambda-1)(c-1)}{(|a|-1)(|b|-1)} . \tag{2.1}
\end{align*}
$$

Then the operator $\mathcal{I}_{a, b ; c}(f)$ maps $\mathcal{R}(\beta)$ into $\mathcal{S}_{\lambda}^{*}$.
In the case $a=1$ we can formulate the following result.

Theorem 2.2. Let $b \in \mathbb{C} \backslash\{0\}, c>0,|b| \neq 1$ and $c>1+|b|$. For $\lambda \in(0,1]$, assume that

$$
\frac{c-1}{c-|b|-1}+(\lambda-1) \frac{c-1}{b-1}(\psi(c-1)-\psi(c-b))<\lambda\left(1+\frac{1}{2(1-\beta)}\right),
$$

where $\psi(x)=\Gamma^{\prime}(x) / \Gamma(x)$. Then the operator $\mathcal{I}_{1, b ; c}(f)$ maps $\mathcal{R}(\beta)$ into $\mathcal{S}_{\lambda}^{*}$.
When $\lambda=1$, we see from the proof of the above theorems that we do not need to treat the case $a=1$ separately, neither do we need the restriction $|b| \neq 1$ and $c \neq 1$. In this case we get the following result, unifying Theorems 2.1 and 2.2.

Corollary 2.3. Let $a, b \in \mathbb{C} \backslash\{0\}$ and $c>|a|+|b|$. Assume that

$$
\begin{equation*}
\frac{\Gamma(c-|a|-|b|) \Gamma(c)}{\Gamma(c-|a|) \Gamma(c-|b|)} \leq 1+\frac{1}{2(1-\beta)} \tag{2.2}
\end{equation*}
$$

Then the operator $\mathcal{I}_{a, b ; c}(f)$ maps $\mathcal{R}(\beta)$ into $\mathcal{S}_{1}^{*}$.
We now briefly present the situtation which we have studied more closely in [5]. Let $\pi:[0,1] \rightarrow \mathbb{R}$ be a non-negative function normalized so that $\int_{0}^{1} \pi(t) d t=1$, and define

$$
\left[V_{\pi}(f)\right](z)=\int_{0}^{1} \pi(t) \frac{f(t z)}{t} d t, f \in \mathcal{A}
$$

Let $\Pi(t)=\int_{t}^{1} \pi(s) d s / s$, and assume that $t \Pi(t) \rightarrow 0$ when $t \rightarrow 0^{+}$. It is easy to verify that the class $\mathcal{S}_{\lambda}^{*}(\lambda>0)$ can be characterized in terms of convolutions in the following way,

$$
f \in S_{\lambda}^{*} \Leftrightarrow \frac{f(z)}{z} * \frac{h_{\lambda}(z)}{z}
$$

where $h_{\lambda}(z)=z\left(1+\frac{x-\lambda}{\lambda}\right) /(1-z)^{2},|x|=1$. For $\lambda=1$ this result is stated in [9]. An application of Theorem 2.1 in [5] gives the following result.

Theorem 2.4. Let $\beta$ be given by

$$
\frac{\beta}{1-\beta}=-\int_{0}^{1} \pi(t)\left[\frac{2-\lambda(1+t)}{\lambda(1+t)}+\frac{2(\lambda-1)}{\lambda} \frac{\log (1+t)}{t}\right] d t
$$

Then

$$
V_{\pi}(\mathcal{R}(\beta)) \subset \mathcal{S}_{\lambda}^{*} \Longleftrightarrow L_{\Pi}\left(e^{-i \theta} h_{\lambda}\left(e^{i \theta} z\right)\right) \geq 0, z \in \Delta,
$$

where

$$
L_{\Pi}(h)=\inf _{z \in \Delta} \int_{0}^{1} \Pi(t)\left[\operatorname{Re} \frac{h(t z)}{t z}-\frac{1+t(1-1 / \lambda)}{(1+t)^{2}}\right] d t .
$$

We observe that the operator $\mathcal{I}_{1, b ; c}(f)$ corresponds to $V_{\pi}(f)$ with

$$
\pi(t)=\pi_{b, c}(t)=\frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} t^{b-1}(1-t)^{c-b-1}
$$

To apply Theorem 2.4 we would like to have conditions on $b$ and $c$ such that $L_{\Pi_{b, c}}\left(h_{\lambda}\right) \geq 0$ where

$$
\begin{aligned}
\Pi_{b, c}(t) & =\int_{t}^{1} \pi_{b, c}(s) d s / s \\
& =\frac{c-1}{b-1}-\frac{\Gamma(c)}{(b-1) \Gamma(b) \Gamma(c-b)} t^{b-1} F(b-1,1+b-c ; b ; t)
\end{aligned}
$$

$b \neq 1$. Such conditions are not easy to establish in general, but we will look at some examples.

In [5, Theorem 4.2] it is proved that for $\Pi_{2,6}(t)=5(1-t)^{4}$ we have

$$
L_{\Pi_{2,6}}\left(e^{-i \theta} h_{1}\left(e^{i \theta} z\right)\right) \geq 0
$$

We now extend this result to some other cases.
Proposition 2.5. For $b=2,3,4$ and 5 we have $L_{\Pi_{b, c}}\left(h_{1}\right) \geq 0$ when $c=$ $2 b+2$.

Remark 1. Theorem 2.4 seems to give sharp values of $\beta$. We cannot prove that this is always the case, but in all the examples we have seen, it is so. We will give one such example later. However, the condition $L_{\Pi}\left(e^{-i \theta} h_{\lambda}\left(e^{i \theta} z\right)\right) \geq 0$ is often difficult to check, and that makes the result less applicable. Theorems 2.1 and 2.2 have conditions that are easy to check, but the $\beta$-values that we obtain from these results are expected to be not as good as the ones we get from Theorem 2.4. Also, keep in mind that Theorem 2.4 is more restricted in the sense that it only deals with the case $a=1$. Combining Theorem 2.4 with Proposition 2.5 we can make a direct
comparison with the results from Theorem 2.2. Based on this comparison we present the following table.

| b | c | $\beta$ obtained from Theorem 2.4 | $\beta$ obtained from Theorem 2.2 |
| :---: | :---: | :---: | :---: |
| 2 | 6 | $\frac{667-960 \log 2}{664-960 \log 2}=-1.11 \ldots$ | 0.25 |
| 3 | 8 | $\frac{6720 \log 2-4657}{6720 \log 2-4659}=-0.903 \ldots$ | $0.333 \ldots$ |
| 4 | 10 | $\frac{111793-161280 \log 2}{111788-161280 \log 2}=-0.800 \ldots$ | 0.375 |
| 5 | 12 | $\frac{887040 \log 2-614848}{887040 \log 2-614851}=-0.739 \ldots$ | 0.4 |

As we can see, the values obtained from Theorem 2.2 are much larger than the ones obtained from Theorem 2.4, which we believe to be sharp.

Remark 2. Here we prove that the $\beta$ value obtained from Theorem 2.4 is sharp in the case $b=2, c=6$. Let

$$
f(z)=z+2(1-\beta) \sum_{k=2}^{\infty} \frac{z^{k}}{k} .
$$

Then $f \in \mathcal{R}(\beta)$, and with $F(z):=\left[V_{\pi_{2,6}}(f)\right](z)$ we get

$$
\begin{aligned}
F(z) & =z+40(.1-\beta) \sum_{k=2}^{\infty} \int_{0}^{1} \frac{(1-t)^{3} t^{k} z^{k}}{k} d t \\
& =z+40(1-\beta) \sum_{k=2}^{\infty}\left(\frac{1}{k(k+1)}-\frac{3}{k(k+2)}+\frac{3}{k(k+3)}-\frac{1}{k(k+4)}\right) z^{k}
\end{aligned}
$$

and

$$
z F^{\prime}(z)=z+40(1-\beta) \sum_{k=2}^{\infty}\left(\frac{1}{k+1}-\frac{3}{k+2}+\frac{3}{k+3}-\frac{1}{k+4}\right) z^{k} .
$$

Substituting $z=-1$ we get, using $1-\beta=1 /(320 \log 2-(664 / 3))$ from Theorem 2.4,

$$
z F^{\prime}(z)=-1+40 \frac{-\frac{83}{15}+8 \log 2}{320 \log 2-\frac{664}{3}}=0 .
$$

Together with $F(-1) \neq 0$, this shows the sharpness.
In our next theorem we obtain a result corresponding to Theorems 2.1 and 2.2 by replacing the assumption $f \in \mathcal{R}(\beta)$ by $f \in \mathcal{S}$.

Theorem 2.6. Let $a, b \in \mathbb{C} \backslash\{0\}, c>2+|a|+|b|$. Suppose that $a, b$ and $\lambda \in(0,1]$ satisfy the condition

$$
\begin{align*}
\frac{\Gamma(c-|a|-|b|) \Gamma(c)}{\Gamma(c-|a|) \Gamma(c-|b|)} & {\left[\frac{|a(a+1) b(b+1)|}{(c-1-|a|-|b|)(c-2-|a|-|b|)}\right.}  \tag{2.3}\\
& \left.+\frac{(\lambda+2)|a b|}{c-1-|a|-|b|}+\lambda\right] \leq 2 \lambda .
\end{align*}
$$

Then the operator $\mathcal{I}_{a, b ; c}(f)$ maps $\mathcal{S}$ into $\mathcal{S}_{\lambda}^{*}$.

We do not have a result corresponding to Theorem 2.4 with $\mathcal{R}(\beta)$ replaced by $\mathcal{S}$ because the duality theory cannot be applied to $\mathcal{S}$ in the same way as to $\mathcal{R}(\beta)$.
3. Proofs. We first state the following lemma which will be used in proving the theorems.

Lemma 3.1. Let $a, b, c>0$. Then we have the following:
(i) For $c>a+b+1$,

$$
\sum_{n=0}^{\infty} \frac{(n+1)(a, n)(b, n)}{(c, n)(1, n)}=\frac{\Gamma(c-a-b) \Gamma(c)}{\Gamma(c-a) \Gamma(c-b)}\left[\frac{a b}{c-1-a-b}+1\right] .
$$

(ii) For $c>a+b+2$,

$$
\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(n+1)^{2}(a, n)(b, n)}{(c, n)(1, n)} \\
& \quad=\frac{\Gamma(c-a-b) \Gamma(c)}{\Gamma(c-a) \Gamma(c-b)}\left[1+\frac{(a, 2)(b, 2)}{(c-2-a-b, 2)}+\frac{3 a b}{c-1-a-b}\right] .
\end{aligned}
$$

(iii) For $a \neq 1, b \neq 1$ and $c \neq 1$ with $c>\max \{0, a+b-1\}$,

$$
\sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)(1, n+1)}=\frac{1}{(a-1)(b-1)}\left[\frac{\Gamma(c+1-a-b) \Gamma(c)}{\Gamma(c-a) \Gamma(c-b)}-(c-1)\right] .
$$

(iv) For $b \neq 1$ and $c>1+b$,

$$
\sum_{n=0}^{\infty} \frac{(b, n)}{(c, n)} \frac{1}{n+1}=\frac{c-1}{b-1}(\psi(c-1)-\psi(c-b)) .
$$

Proof. (i) Using the ascending factorial notation, we can write

$$
\begin{aligned}
S_{1} & =\sum_{n=0}^{\infty} \frac{(n+1)(a, n)(b, n)}{(c, n)(1, n)}=\sum_{n=1}^{\infty} \frac{(a, n)(b, n)}{(c, n)(1, n-1)}+\sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)(1, n)} \\
& =\frac{a b}{c} \sum_{n=0}^{\infty} \frac{(a+1, n)(b+1, n)}{(c+1, n)(1, n)}+\sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)(1, n)} .
\end{aligned}
$$

The formula (1.3) and the condition $c>1+a+b$ immediately give

$$
\begin{aligned}
S_{1} & =\frac{a b}{c}\left[\frac{\Gamma(c-a-b-1) \Gamma(c+1)}{\Gamma(c-a) \Gamma(c-b)}\right]+\left[\frac{\Gamma(c-a-b) \Gamma(c)}{\Gamma(c-a) \Gamma(c-b)}\right] \\
& =\frac{\Gamma(c-a-b) \Gamma(c)}{\Gamma(c-a) \Gamma(c-b)}\left[\frac{a b}{c-1-a-b}+1\right]
\end{aligned}
$$

and the conclusion follows.
(ii) Using the ascending factorial notation and by adjusting the coefficients suitably, we can write

$$
\begin{aligned}
S_{2} & =\sum_{n=0}^{\infty} \frac{(n+1)^{2}(a, n)(b, n)}{(c, n)(1, n)}=\sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)(1, n)} \\
& +\sum_{n=1}^{\infty} \frac{n(n+3-1)(a, n)(b, n)}{(c, n)(1, n)} \\
& =\sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)(1, n)}+\sum_{n=1}^{\infty} \frac{(n-1)(a, n)(b, n)}{(c, n)(1, n-1)}+3 \sum_{n=1}^{\infty} \frac{(a, n)(b, n)}{(c, n)(1, n-1)} \\
& =\sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)(1, n)}+\sum_{n=1}^{\infty} \frac{(a, n+1)(b, n+1)}{(c, n+1)(1, n-1)} \\
& +\frac{3 a b}{c} \sum_{n=1}^{\infty} \frac{(a+1, n-1)(b+1, n-1)}{(c+1, n-1)(1, n-1)} \\
& =\sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)(1, n)}+\frac{a(a+1) b(b+1)}{c(c+1)} \sum_{n=0}^{\infty} \frac{(a+2, n)(b+2, n)}{(c+2, n)(1, n)} \\
& +\frac{3 a b}{c} \sum_{n=0}^{\infty} \frac{(a+1, n)(b+1, n)}{(c+1, n)(1, n)} .
\end{aligned}
$$

Therefore, by the formula (1.3), it follows that

$$
\begin{aligned}
S_{2} & =\frac{\Gamma(c-a-b) \Gamma(c)}{\Gamma(c-a) \Gamma(c-b)}+\frac{a(a+1) b(b+1)}{c(c+1)} \frac{\Gamma(c-2-a-b) \Gamma(c+2)}{\Gamma(c-a) \Gamma(c-b)} \\
& +\frac{3 a b}{c} \frac{\Gamma(c-1-a-b) \Gamma(c+1)}{\Gamma(c-a) \Gamma(c-b)}
\end{aligned}
$$

which after simplification gives the required conclusion.
(iii) Let $a, b$ be positive real numbers such that $a \neq 1, b \neq 1$ and $c \neq 1$ with $c>\max \{0, a+b-1\}$. Then we find that

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)(1, n+1)} & =\frac{c-1}{(a-1)(b-1)} \sum_{n=0}^{\infty} \frac{(a-1, n+1)(b-1, n+1)}{(c-1, n+1)(1, n+1)} \\
& =\frac{c-1}{(a-1)(b-1)}\left[\frac{\Gamma(c+1-a-b) \Gamma(c-1)}{\Gamma(c-a) \Gamma(c-b)}-1\right]
\end{aligned}
$$

and the result follows.
(iv) Let $b \neq 1$, and $c>1+b$. We can write

$$
\sum_{n=0}^{\infty} \frac{(b, n)}{(c, n)} \frac{1}{n+1} z^{n}=\frac{1}{z} \int_{0}^{z} \frac{f(\zeta)}{\zeta} d \zeta
$$

where

$$
\begin{aligned}
f(z) & =\sum_{n=0}^{\infty} \frac{(b, n)}{(c, n)} z^{n+1}=z F(1, b ; c ; z) \\
& =\frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_{0}^{1} t^{b-1}(1-t)^{c-b-1} \frac{z}{1-t z} d t
\end{aligned}
$$

This gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{(b, n)}{(c, n)} \frac{1}{n+1} & =\int_{0}^{1} \frac{f(\zeta)}{\zeta} d \zeta \\
& =\frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_{0}^{1}\left(\int_{0}^{1} t^{b-1}(1-t)^{c-b-1} \frac{1}{1-t \zeta} d t\right) d \zeta \\
& =-\frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_{0}^{1} t^{b-2}(1-t)^{c-b-1} \log (1-t) d t \\
& =\frac{c-1}{b-1}(\psi(c-1)-\psi(c-b))
\end{aligned}
$$

Proof of Theorem 2.1. Let $a, b \in \mathbb{C} \backslash\{0\}$ and $c>|a|+|b|,|a| \neq 1,|b| \neq 1$, $c \neq 1$. Suppose that $f(z)=z+\sum_{n=2}^{\infty} a_{n} z^{n} \in \mathcal{R}(\beta)$. Then, from [3] we know that

$$
\begin{equation*}
\left|a_{n}\right| \leq \frac{2(1-\beta)}{n} \tag{3.1}
\end{equation*}
$$

Consider $z F(a, b ; c ; z) * f(z)=z+\sum_{n=2}^{\infty} B_{n} z^{n}$ where $B_{1}=1$ and for $n \geq 1$

$$
B_{n}=\frac{(a, n-1)(b, n-1)}{(c, n-1)(1, n-1)} a_{n}
$$

Therefore, according to (1.1), we need only to show that

$$
\begin{equation*}
T=\sum_{n=2}^{\infty}(n+\lambda-1)\left|B_{n}\right| \leq \lambda \tag{3.2}
\end{equation*}
$$

In the sequel we will use the triangle inequality for $(a, n)$ :

$$
\begin{equation*}
|(a, n)|=|a(a+1)(a+2) \cdots(a+n-1)| \leq(|a|, n) \tag{3.3}
\end{equation*}
$$

Then, we have

$$
\begin{aligned}
T & =\sum_{n=2}^{\infty}(n-1+\lambda) \frac{|(a, n-1)(b, n-1)|}{(c, n-1)(1, n-1)}\left|a_{n}\right| \\
& \leq 2(1-\beta) \sum_{n=2}^{\infty}(n-1+\lambda) \frac{(|a|, n-1)(|b|, n-1)}{(c, n-1)(1, n-1) n}(\text { by }(3.1) \text { and }(3.3)) \\
& =2(1-\beta) \sum_{n=1}^{\infty}[(n+1)+(\lambda-1)] \frac{(|a|, n)(|b|, n)}{(c, n)(1, n)(n+1)} \\
& =2(1-\beta)\left[\sum_{n=1}^{\infty} \frac{(|a|, n)(|b|, n)}{(c, n)(1, n)}+(\lambda-1) \sum_{n=1}^{\infty} \frac{(|a|, n)(|b|, n)}{(c, n)(1, n+1)}\right]:=T_{1}
\end{aligned}
$$

Using the formula (1.3) and Lemma 3.1 (iii), we find that

$$
\begin{aligned}
T_{1} & =2(1-\beta)\left[\left(\frac{\Gamma(c-|a|-|b|) \Gamma(c)}{\Gamma(c-|a|) \Gamma(c-|b|)}-1\right)\right. \\
& \left.+(\lambda-1)\left\{\frac{1}{(|a|-1)(|b|-1)}\left(\frac{\Gamma(c+1-|a|-|b|) \Gamma(c)}{\Gamma(c-|a|) \Gamma(c-|b|)}-(c-1)\right)-1\right\}\right] \\
& =2(1-\beta)\left[\frac{\Gamma(c-|a|-|b|) \Gamma(c)}{\Gamma(c-|a|) \Gamma(c-|b|)}\left\{1+\frac{(\lambda-1)(c-|a|-|b|)}{(|a|-1)(|b|-1)}\right\}\right. \\
& \left.-\frac{(\lambda-1)(c-1)}{(|a|-1)(|b|-1)}-\lambda\right] \leq \lambda, \text { by }(2.1)
\end{aligned}
$$

Thus, we have $T \leq T_{1} \leq \lambda$ and therefore (3.2) holds. From this, we conclude that the function $z F(a, b ; c ; z) * f(z)$ belongs to $S_{\lambda}^{*}$, which completes the proof.

Proof of Theorem 2.2. The proof follows the same pattern as the proof of Theorem 2.1 until we reach the definition of the number $T_{1}$ which now takes the form

$$
T_{1}:=2(1-\beta)\left[\sum_{n=1}^{\infty} \frac{(|b|, n)}{(c, n)}+(\lambda-1) \sum_{n=1}^{\infty} \frac{(|b|, n)}{(c, n)(n+1)}\right] .
$$

Using (1.3) and Lemma 3.1 (iv) we get

$$
\begin{aligned}
T_{1} & =2(1-\beta)\left[\frac{\Gamma(c-|b|-1) \Gamma(c)}{\Gamma(c-1) \Gamma(c-|b|)}+(\lambda-1) \frac{c-1}{b-1}(\psi(c-1)-\psi(c-b))-\lambda\right] \\
& =2(1-\beta)\left[\frac{c-1}{c-|b|-1}+(\lambda-1) \frac{c-1}{b-1}(\psi(c-1)-\psi(c-b))-\lambda\right] \\
& <2(1-\beta)\left[\lambda\left(1+\frac{1}{2(1-\beta)}\right)-\lambda\right]=\lambda
\end{aligned}
$$

by assumption.
Proof of Proposition 2.5. In order to prove that $L_{\Pi_{b, c}}\left(h_{1}\right) \geq 0$, it suffices to prove that

$$
H_{b, c}(y)=\int_{0}^{1} t \mathrm{II}_{b, c}(t) \frac{1-2 t-t^{2}}{\left(1+t^{2}-2 y t\right)(1+t)^{2}} d t \geq 0
$$

The explanation of this can be found in [5], so we will not repeat that here. The proof is based on direct computation, and we only show the $b=3$ case. (The $b=2$ case is proved in [5].) When $b=3$ and $c=8$ we get $\Pi_{3,8}(t)=78(1-t)^{5}(1+5 t) / 8$ and hence

$$
\begin{aligned}
H_{3,8}(y) & =\frac{7}{2} \int_{0}^{\sqrt{2}-1} \frac{t(1-t)^{5}(1+5 t)\left(1-2 t-t^{2}\right)}{\left(1+t^{2}-2 y t\right)(1+t)^{2}} d t \\
& -\frac{7}{2} \int_{\sqrt{2}-1}^{1} \frac{t(1-t)^{5}(1+5 t)\left(t^{2}+2 t-1\right)}{\left(1+t^{2}-2 y t\right)(1+t)^{2}} d t
\end{aligned}
$$

$$
\begin{aligned}
& \geq \frac{7}{2} \int_{0}^{\sqrt{2}-1} \frac{t(1-t)^{5}(1+5 t)\left(1-2 t-t^{2}\right)}{(1+t)^{4}} d t \\
& -\frac{7}{2} \int_{\sqrt{2}-1}^{1} \frac{t(1-t)^{3}(1+5 t)\left(t^{2}+2 t-1\right)}{(1+t)^{2}} d t \\
& =\frac{510587}{60}-\frac{13132 \sqrt{2}}{3}-3346 \log 2>0 .
\end{aligned}
$$

Proof of Theorem 2.6. Let $a \in \mathbb{C} \backslash\{0\}, c>2+|a|+|b|$ and $\lambda \in(0,1]$. Assume that (2.3) holds. Let $f(z)=z+\sum_{n=2}^{\infty} a_{n} z^{n} \in \mathcal{S}$. Consider $z F(a, b ; c ; z) * f(z)=z+\sum_{n=2}^{\infty} B_{n} z^{n}$ where $B_{1}=1$ and for $n \geq 1$

$$
B_{n}=\frac{(a, n-1)(b, n-1)}{(c, n-1)(1, n-1)} a_{n} .
$$

Therefore, according to (1.1), we need only to show that

$$
T=\sum_{n=2}^{\infty}(n+\lambda-1)\left|B_{n}\right| \leq \lambda .
$$

Since $f \in \mathcal{S}$, we have $\left|a_{n}\right| \leq n$, and therefore

$$
\begin{aligned}
T & \leq \sum_{n=2}^{\infty} n(n-1+\lambda) \frac{|(a, n-1)(b, n-1)|}{(c, n-1)(1, n-1)} \\
& \leq(\lambda-1) \sum_{n=1}^{\infty} \frac{(n+1)(|a|, n)(|b|, n)}{(c, n)(1, n)}+\sum_{n=1}^{\infty} \frac{(n+1)^{2}(|a|, n)(|b|, n)}{(c, n)(1, n)}:=T_{2} .
\end{aligned}
$$

From Lemma 3.1 (i) and (ii) and (2.3), we see that the inequality $T_{2} \leq \lambda$ holds, which implies $T \leq \lambda$. Therefore, the operator $\mathcal{I}_{a, b ; c}(f)$ maps $S$ into $\mathcal{S}_{\lambda}^{*}$, and the proof is complete.
4. Concluding remarks. Clearly, the condition on the parameters $a, b$ and $c$ for the univalence of functions $z F(a, b ; c ; z)$ and $z F\left(a, b ; c ; z^{2}\right)$ are different. In [7, Theorem 4.1] Ponnusamy and Vuorinen determined conditions on $a, b, c>0$ for the function $z F\left(a, b ; c ; z^{2}\right)$ to be close-to-convex with respect to the convex function $(1 / 2) \log ((1+z) /(1-z))$. On the other hand, not much is known about the starlikeness of an odd function $z F\left(a, b ; c ; z^{2}\right)$. One of the known results in this direction is the following [7, Theorem 6.8]:

Theorem 4.1. Let $-1 \leq a<0,0 \leq b, b \neq 1$ and for $0 \leq \beta<1$, let

$$
c \geq 1+\max \left\{1-\beta+|a+b+1+2 \beta|, 1-a b-\frac{(a+1)(b+1)(1+\beta)}{1-\beta}\right\}
$$

Then the odd hypergeometric function $z F\left(a, b ; c ; z^{2}\right)$ is in $\mathcal{S}^{*}(\beta)$.

Finally we state a result giving simple conditions for $z F\left(a, b ; c ; z^{2}\right)$ to belong to $\mathcal{S}_{\lambda}^{*}$. The exact range of the parameters for which $F\left(a, b ; c ; z^{2}\right)$ belongs to $\mathcal{S}$ or $\mathcal{S}^{*}$ remains an open problem.

Theorem 4.2. Let $a, b \in \mathbb{C} \backslash\{0\}, c>1+|a|+|b|$ and $\lambda \in(0,1]$. If $c$ satisfies the condition

$$
\frac{\Gamma(c-|a|-|b|) \Gamma(c)}{\Gamma(c-|a|) \Gamma(c-|b|)}\left[\frac{2|a b|}{c-1-|a|-|b|}+\lambda\right] \leq 2 \lambda
$$

then the odd hypergeometric function $z F\left(a, b ; c ; z^{2}\right)$ is in $\mathcal{S}_{\lambda}^{*}$.

The proof of this result is similar to the proof of Theorem 2.1.

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