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Eigenvalues of Quasisymmetric Automorphisms Determined by VMO Functions

Dedicated to Professor Eligiusz Złotkiewicz on the occasion of his 60th birthday

ABSTRACT. This article aims at proving the inclusion (0.3) and the identity (0.4). They provide information on quasisymmetric automorphisms of the unit circle and their eigenvalues.

0. Introduction. Let Γ be a Jordan arc or a Jordan curve in the extended complex plane $\hat{\mathbb{C}}$ and let $\mathrm{Hom}(\Gamma)$ be the family of all homeomorphic self-mappings of Γ. A homeomorphism ζ of a subarc I of the *unit circle* $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ onto Γ is said to be a *parametrization* of Γ. We call a homeomorphism $\gamma \in \mathrm{Hom}(\Gamma)$ to be *sense-preserving* and write $\gamma \in \mathrm{Hom}^+(\Gamma)$ if there exist $\sigma \in \mathrm{Hom}(\mathbb{T})$ and a parametrization $\zeta : I \to \Gamma$ such that $\gamma \circ \zeta = \zeta \circ \sigma$ on I and each continuous branch of $\arg \sigma(e^{it})$ is an increasing function of $t \in \mathbb{R}$.

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Assume now that Γ is a Jordan curve or arc which is locally rectifiable. We denote by $L^0(\Gamma)$ the class of all real-valued functions defined on Γ that are Lebesgue measurable with respect to the arc length measure $|\cdot|_1$. We adopt the standard notations $L^1(\Gamma)$ and $L^\infty(\Gamma)$ for the classes of all functions in $L^0(\Gamma)$ that are integrable (with respect to $|\cdot|_1$) and essentially bounded on Γ, respectively. The functional $\|\cdot\|_\infty$,

$$\|f\|_\infty := \operatorname*{ess\,sup}_{z\in\Gamma} |f(z)| \ , \quad f \in L^0(\Gamma) \ ,$$

is a pseudo-norm on the linear space $L^\infty(\Gamma)$. We say that a function $f : \Gamma \to \mathbb{R}$ is locally integrable on Γ and write $f \in L^1_{\rm loc}(\Gamma)$ if $f \in L^1(I)$ for every compact subarc $I \in \operatorname{Arc}^\infty(\Gamma)$, where $\operatorname{Arc}^\delta(\Gamma)$ stands for the set of all subarcs $I \subset \Gamma$ such that $0 < |I|_1 < \delta$, $0 < \delta \le \infty$. For every $f \in L^1_{\rm loc}(\Gamma)$ set

$$f_I := \frac{1}{|I|_1}\int_I f(z)|dz| \ , \quad I \in \operatorname{Arc}^\infty(\Gamma) \ ,$$

for the average of f over I and define

$$\|f\|_{*,\delta} := \sup\left\{\frac{1}{|I|_1}\int_I |f(z) - f_I||dz| : I \in \operatorname{Arc}^\delta(\Gamma)\right\} \ , \quad \delta > 0 \ .$$

The functional $\|\cdot\|_* := \|\cdot\|_{*,\infty}$ is a pseudo-norm on the spaces

$$\operatorname{BMO}(\Gamma) := \{f \in L^1_{\rm loc}(\Gamma) : \|f\|_* < \infty\}$$

and

$$\operatorname{VMO}(\Gamma) := \{f \in \operatorname{BMO}(\Gamma) : \lim_{\delta\to 0^+} \|f\|_{*,\delta} = 0\} \ ,$$

and for every $f \in \operatorname{BMO}(\Gamma)$, $\|f\|_* = 0$ iff f is a constant function almost everywhere (a.e.) on Γ. We recall that a function $f \in \operatorname{BMO}(\Gamma)$ ($f \in \operatorname{VMO}(\Gamma)$) is said to be of *bounded* (*vanishing*) *mean oscillation* on Γ. For a survey of the properties of the spaces $\operatorname{BMO}(\Gamma)$ and $\operatorname{VMO}(\Gamma)$ in cases $\Gamma = \mathbb{R}, \mathbb{T}$ we refer the reader to [G, Chapter VI]. We introduce the classes $\operatorname{HBMO}(\Gamma)$ and $\operatorname{HVMO}(\Gamma)$ of all $\gamma \in \operatorname{Hom}^+(\Gamma)$ absolutely continuous on Γ such that $\log|\gamma'| \in \operatorname{BMO}(\Gamma)$ and $\log|\gamma'| \in \operatorname{VMO}(\Gamma)$, respectively. Here and subsequently, $f'(z)$ denotes the derivative of a function $f : \Gamma \to \mathbb{R}$ at $z \in \Gamma$, i.e.,

$$f'(z) := \lim_{\Gamma\ni u\to z} \frac{f(u) - f(z)}{u - z}$$

provided the limit exists, while $f'(z) := 0$ otherwise. It is evident that the function $\rho_* : \operatorname{HBMO}(\Gamma) \times \operatorname{HBMO}(\Gamma) \to \mathbb{R}$ defined by

$$\rho_*(\gamma_1, \gamma_2) := \left\|\log\frac{|\gamma_1'|}{|\gamma_2'|}\right\|_* = \|\log|\gamma_1'| - \log|\gamma_2'|\|_* \ , \quad \gamma_1, \gamma_2 \in \operatorname{HBMO}(\Gamma) \ ,$$

is a pseudo-metric on HBMO(Γ). Since $L^\infty(\Gamma) \subset \mathrm{BMO}(\Gamma)$, we may consider the space

$$\mathrm{HBMO}^\infty(\Gamma) := \mathrm{cl}_{\rho_*}(\{\gamma \in \mathrm{HBMO}(\Gamma) : \log|\gamma'| \in L^\infty(\Gamma)\}) ,$$

where $\mathrm{cl}_\varrho(A)$ stands for the closure of $A \subset X$ in the pseudo-metric ϱ on the space X.

Following Beurling and Ahlfors [BA], for $M \geq 1$ we define the class $\mathrm{QS}(\Gamma; M)$ of all $\gamma \in \mathrm{Hom}^+(\Gamma)$ such that the inequality

$$|I''|_1 \leq M|I'|_1 \tag{0.1}$$

holds for all adjacent closed subarcs $I', I'' \subset \Gamma$ satisfying $0 < |I'|_1 = |I''|_1 < \infty$, where I' and I'' are said to be adjacent if the set $I' \cap I''$ consists of one or two points. A homeomorphism $\gamma \in \mathrm{QS}(\Gamma) := \bigcup_{M\geq 1} \mathrm{QS}(\Gamma; M)$ (resp. $\gamma \in \mathrm{QS}(\Gamma; M)$) is said to be a *quasisymmetric automorphism* (resp. M-*quasisymmetric automorphism*) of Γ.

For $K \geq 1$ let $\mathrm{Q}(\mathbb{T}; K)$ be the class of all $\gamma \in \mathrm{Hom}^+(\mathbb{T})$ such that γ has a K-quasiconformal extension to the *unit disk* $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and write $\mathrm{Q}(\mathbb{T}) := \bigcup_{K\geq 1} \mathrm{Q}(\mathbb{T}; K)$. By the properties of quasiconformal mappings (see [LV]), the functional τ,

$$\tau(\gamma_1, \gamma_2) := \inf\{\log K : \gamma_1 \circ \gamma_2^{-1} \in \mathrm{Q}(\mathbb{T}; K)\} , \quad \gamma_1, \gamma_2 \in \mathrm{Q}(\mathbb{T}) ,$$

is a pseudo-metric on $\mathrm{Q}(\mathbb{T})$ called the *Teichmüller pseudo-metric*. As shown by Krzyż in [K], $\mathrm{Q}(\mathbb{T}) = \mathrm{QS}(\mathbb{T})$. Moreover, modifying suitably the proof of [K, Thm. 2] (see [P3, p. 68]) and applying Lehtinen's estimate [L, Thm. 1] we obtain

$$\mathrm{QS}(\mathbb{T}; M) \subset \mathrm{Q}(\mathbb{T}; \min\{M^{3/2}, 2M - 1\}) \subset \mathrm{Q}(\mathbb{T}; M^2) . \tag{0.2}$$

Let $\mathrm{HA}(\mathbb{T})$ be the class of all $\gamma \in \mathrm{Hom}^+(\mathbb{T})$ such that γ has a conformal extension to an open annulus containing $\mathbb{T}$. We study the relationship between the classes $\mathrm{HA}(\mathbb{T})$, $\mathrm{HVMO}(\mathbb{T})$ and $\mathrm{cl}_\tau(\mathrm{HA}(\mathbb{T}))$. Our main aim is to prove the inclusion

$$\mathrm{HVMO}(\mathbb{T}) \subset \mathrm{cl}_\tau(\mathrm{HA}(\mathbb{T})) ; \tag{0.3}$$

see Theorem 2.4. Thus we provide the detailed proof of [P3, Thm. 3.4.7], which completes the discussion in [P3, Section 3.4]. As an application of (0.3), we obtain the identity

$$\Lambda_\gamma^* = \Lambda_\gamma , \quad \gamma \in \mathrm{HVMO}(\mathbb{T}) , \tag{0.4}$$

(see Corollary 2.5), where Λ_γ^* and Λ_γ are the sets of all *eigenvalues* and *spectral values* of γ, respectively, defined by means of the *generalized harmonic conjugation operator* A_γ introduced in [P1]. For the definitions of eigenvalues and spectral values of $\gamma \in \mathrm{Q}(\mathbb{T})$ the reader is referred to [P2, Definitions 1.1 and 1.2] or [P3, Definitions 3.2.2 and 3.2.1].

The considerations presented in this paper are based on an unpublished part of the author's Ph.D. thesis, whose one of the referees was Professor E. Złotkiewicz.

1. The class $\mathrm{HBMO}^\infty(\mathbb{R})$. In this section we study properties of the class $\mathrm{HBMO}^\infty(\mathbb{R})$ that turn out to be useful in the later discussion. In particular we establish the inclusion $\mathrm{HBMO}^\infty(\mathbb{R}) \subset \mathrm{QS}(\mathbb{R})$. Our considerations require the following John–Nirenberg theorem; cf. [JN], also see [G, p. 230].

Theorem 1.1. *There exist constants $C, c > 0$ such that for every function $f \in \mathrm{BMO}(\mathbb{R})$ and every interval $I \in \mathrm{Arc}^\infty(\mathbb{R})$ the inequality*

$$|\{t \in I : |f(t) - f_I| > \lambda\}|_1 \le C|I|_1 \exp\left(\frac{-c\lambda}{\|f\|_*}\right)$$

holds for all $\lambda > 0$.

Lemma 1.2. *Suppose that $f \in \mathrm{BMO}(\mathbb{R})$ and $I \in \mathrm{Arc}^\infty(\mathbb{R})$. If $\|f\|_* \le c/2$, then*

$$\begin{aligned}\frac{|I|_1}{C+1} \le \frac{|I|_1}{2Cc^{-1}\|f\|_* + 1} \le \int_I \exp(f(t) - f_I)dt \\ \le (2Cc^{-1}\|f\|_* + 1)|I|_1 \le (C+1)|I|_1\,,\end{aligned} \tag{1.1}$$

where C and c are the constants in Theorem 1.1.

Proof. For $\lambda > 0$ let $I_\lambda := \{t \in I : |f(t) - f_I| > \lambda\}$. Theorem 1.1 shows that

$$|I_\lambda|_1 \le C|I|_1 \exp\left(\frac{-c\lambda}{\|f\|_*}\right), \quad \lambda > 0\,.$$

Hence by the the assumption $\|f\|_* \le c/2$ we obtain

$$\begin{aligned}\int_I \exp|f(t) - f_I|dt = \int_I (\exp|f(t) - f_I| - 1)dt + |I|_1 \\ = \int_I \Big(\int_0^{|f(t)-f_I|} e^\lambda d\lambda\Big)dt + |I|_1 = \int_0^\infty e^\lambda |I_\lambda|_1 d\lambda + |I|_1\end{aligned} \tag{1.2}$$

$$\leq C|I|_1 \int_0^\infty \exp\left(1 - \frac{c}{\|f\|_*}\right) \lambda d\lambda + |I|_1 = \frac{C|I|_1\|f\|_*}{c - \|f\|_*} + |I|_1$$
$$\leq (2Cc^{-1}\|f\|_* + 1)|I|_1 \leq (C+1)|I|_1 .$$

If $g : I \to \mathbb{R}$ is a positive function then the Schwarz inequality shows that

$$|I|_1^2 = \left(\int_I \frac{1}{\sqrt{g(t)}} \sqrt{g(t)} dt\right)^2 \leq \int_I \frac{1}{g(t)} dt \int_I g(t) dt ,$$

and consequently

$$\int_I \frac{1}{g(t)} dt \geq |I|_1^2 \left(\int_I g(t) dt\right)^{-1} .$$

Setting $g(t) := \exp|f(t) - f_I|$, $t \in I$, we conclude from (1.2) that

$$\int_I \exp(-|f(t) - f_I|) dt \geq \frac{|I|_1}{2Cc^{-1}\|f\|_* + 1} \geq \frac{|I|_1}{C+1} .$$

Combining this with (1.2) we obtain (1.1). □

Theorem 1.3. *Given* $g \in \mathrm{BMO}(\mathbb{R})$ *and* $h \in L^\infty(\mathbb{R})$, *suppose that a homeomorphism* $\gamma \in \mathrm{Hom}^+(\mathbb{R})$ *is absolutely continuous on* $\mathbb{R}$ *and satisfies*

$$\text{(1.3)} \qquad \log \gamma'(t) = f(t) := g(t) + h(t) \quad \textit{for a.e. } t \in \mathbb{R} .$$

If $\|g\|_* \leq c/2$, *then*

$$\text{(1.4)} \qquad \gamma \in \mathrm{QS}\left(\mathbb{R}\, ; (2Cc^{-1}\|g\|_* + 1)^2 e^{4\|g\|_*} e^{2\|h\|_\infty}\right) ,$$

where C *and* c *are the constants in* Theorem 1.1. *In particular,* $\mathrm{HBMO}^\infty(\mathbb{R}) \subset \mathrm{QS}(\mathbb{R})$.

Proof. Since $\gamma \in \mathrm{Hom}^+(\mathbb{R})$ is absolutely continuous on $\mathbb{R}$,

$$|\gamma(I)|_1 = \int_I |\gamma'(t)| dt = \int_I e^{f(t)} dt , \quad I \in \mathrm{Arc}^\infty(\mathbb{R}) .$$

Given a pair of adjacent closed intervals I' and I'' with $0 < |I'|_1 = |I''|_1 < \infty$ let $I := I' \cup I''$. From (1.3) and Lemma 1.2 it follows that

$$|\gamma(I'')|_1 = \int_{I''} e^{g(t)} e^{h(t)} dt \leq \int_{I''} e^{\|h\|_\infty} e^{g(t)} dt$$
$$\leq e^{\|h\|_\infty} e^{g_{I''}} (2Cc^{-1}\|g\|_* + 1)|I''|_1$$

and

$$|\gamma(I')|_1 = \int_{I'} e^{g(t)} e^{h(t)} dt \geq \int_{I'} e^{-\|h\|_\infty} e^{g(t)} dt$$
$$\geq e^{-\|h\|_\infty} e^{g_{I'}} (2Cc^{-1}\|g\|_* + 1)^{-1} |I'|_1 .$$

Since $|g_{I'} - g_{I''}| \leq 4\|g\|_*$ (see [G, p. 223]), the above inequalities imply

$$\frac{|\gamma(I'')|_1}{|\gamma(I')|_1} \leq (2Cc^{-1}\|g\|_* + 1)^2 e^{g_{I''} - g_{I'}} e^{2\|h\|_\infty}$$
$$\leq M := (2Cc^{-1}\|g\|_* + 1)^2 e^{4\|g\|_*} e^{2\|h\|_\infty} ,$$

so that γ is an M–quasisymmetric automorphism, and consequently the inclusion (1.4) holds.

By definition, for every $\gamma \in \mathrm{HBMO}^\infty(\mathbb{R})$ there exist $h \in L^\infty(\mathbb{R})$ and $g \in \mathrm{BMO}(\mathbb{R})$ satisfying $\|g\|_* \leq c/2$ and (1.3). Then (1.4) yields the inclusion $\mathrm{HBMO}^\infty(\mathbb{R}) \subset \mathrm{QS}(\mathbb{R})$. □

Given $\gamma \in \mathrm{Hom}(\mathbb{R})$ assume that γ^{-1} is absolutely continuous on $\mathbb{R}$. Then for every measurable function $f : \mathbb{R} \to \mathbb{R}$ the composed function $f \circ \gamma$ is measurable on $\mathbb{R}$ as well and the mapping

(1.5) $$L^0(\mathbb{R}) \ni f \mapsto f \circ \gamma \in L^0(\mathbb{R})$$

is linear. If $\gamma \in \mathrm{HBMO}(\mathbb{R})$, then by definition, $|\log \gamma'(t)| < \infty$ for a.e. $t \in \mathbb{R}$, and hence $\gamma'(t) > 0$ for a.e. $t \in \mathbb{R}$. This means that the inverse homeomorphism γ^{-1} is absolutely continuous on $\mathbb{R}$ and (1.5) defines a self–mapping of $L^0(\mathbb{R})$. Moreover, the Jones result [J, Thm.] leads us to

Lemma 1.4. *If $\gamma \in \mathrm{HBMO}^\infty(\mathbb{R})$, then the mapping* (1.5) *is a linear homeomorphism of the space* BMO($\mathbb{R}$) *onto itself, i.e., there exists a positive constant c_γ such that*

(1.6) $$c_\gamma^{-1}\|f\|_* \leq \|f \circ \gamma\|_* \leq c_\gamma \|f\|_* , \quad f \in \mathrm{BMO}(\mathbb{R}) .$$

Proof. By definition, there exist $g \in \mathrm{BMO}(\mathbb{R})$ and $h \in L^\infty(\mathbb{R})$ satisfying (1.3) and $\|g\|_* \leq c/4$. Let $I \subset \mathbb{R}$, $0 < |I|_1 < \infty$, be a closed interval and let $E \subset I$ be a measurable set. Given a subset $A \subset \mathbb{R}$ we denote by χ_A the characteristic function of A, i.e., $\chi_A(t) := 1$ if $t \in A$ and $\chi_A(t) := 0$ if

$t \in \mathbb{R} \setminus A$. Combining Theorem 1.1 with the Schwarz inequality we obtain

$$\int_E (\exp|g(t) - g_I| - 1)dt = \int_E \left(\int_0^{|g(t)-g_I|} e^\lambda d\lambda \right) dt$$

$$= \int_0^\infty \left(e^\lambda \int_I \chi_{I_\lambda}(t) \chi_E(t) dt \right) d\lambda$$

$$\leq \int_0^\infty e^\lambda |I_\lambda|_1^{1/2} |E|_1^{1/2} d\lambda \leq \sqrt{C} |I|_1^{1/2} |E|_1^{1/2} \int_0^\infty \exp\left(\frac{2\|g\|_* - c}{2\|g\|_*} \lambda \right) d\lambda$$

$$= \frac{\sqrt{C}|I|_1^{1/2}|E|_1^{1/2} \cdot 2 \cdot \|g\|_*}{c - 2\|g\|_*} \leq \sqrt{C}|I|_1^{1/2}|E|_1^{1/2} ,$$

where C and c are the constants in Theorem 1.1 and

$$I_\lambda := \{t \in I : |g(t) - g_I| > \lambda\}.$$

Hence,

$$\int_E \exp(g(t))dt \leq (\sqrt{C} + 1)\exp(g_I)|I|_1^{1/2}|E|_1^{1/2} , \tag{1.7}$$

because $|E|_1 \leq |I|_1$. From Lemma 1.2 it follows that

$$\int_I \exp(g(t))dt \geq (C + 1)^{-1} \exp(g_I)|I|_1 . \tag{1.8}$$

Since $\gamma'(t) = \exp(g(t))\exp(h(t))$ for a.e. $t \in \mathbb{R}$, it follows that

$$\exp(-\|h\|_\infty)\exp(g(t)) \leq \gamma'(t) \leq \exp(\|h\|_\infty)\exp(g(t)) \quad \text{for a.e. } t \in \mathbb{R} .$$

Combining this with (1.7) and (1.8) we obtain

$$|\gamma(E)|_1 = \int_E \exp(g(t))\exp(h(t))dt$$

$$\leq \exp(\|h\|_\infty)(\sqrt{C} + 1)\exp(g_I)|I|_1^{1/2}|E|_1^{1/2}$$

and

$$|\gamma(I)|_1 = \int_I \exp(g(t))\exp(h(t))dt \geq \exp(-\|h\|_\infty)(C + 1)^{-1}\exp(g_I)|I|_1 .$$

Consequently,

$$\frac{|\gamma(E)|_1}{|\gamma(I)|_1} \leq \exp(2\|h\|_\infty)(\sqrt{C} + 1)(C + 1)\left(\frac{|E|_1}{|I|_1}\right)^{1/2} .$$

Thus γ induces the measure μ_γ, $\mu_\gamma(E) := |\gamma(E)|_1$ for every Borel set $E \subset \mathbb{R}$, which belongs to the so called Muckenhoupt class A_∞; cf. [G, p. 264] for the definition of A_∞. Applying [J, Thm.] and the Banach invertible operator theorem we obtain the assertion of the lemma. □

Theorem 1.5. *If* $\gamma \in \mathrm{HBMO}^{\infty}(\mathbb{R})$ *and if* $\eta \in \mathrm{HBMO}(\mathbb{R})$, *then* $\eta \circ \gamma \in \mathrm{HBMO}(\mathbb{R})$ *and* $\eta \circ \gamma^{-1} \in \mathrm{HBMO}(\mathbb{R})$. *Moreover, for every sequence* $\gamma_n \in \mathrm{HBMO}(\mathbb{R})$, $n \in \mathbb{N}$,

$$\rho_*(\gamma_n, \gamma) \to 0 \text{ as } n \to \infty \implies \rho_*(\gamma_n \circ \gamma^{-1}, \mathrm{id}_{\mathbb{R}}) \to 0 \text{ as } n \to \infty , \tag{1.9}$$

where id_X *stands for the identity operator on* X.

Proof. Fix $\gamma \in \mathrm{HBMO}^{\infty}(\mathbb{R})$ and $\eta \in \mathrm{HBMO}(\mathbb{R})$. Evidently, the composition $\eta \circ \gamma$ belongs to $\mathrm{Hom}^{+}(\mathbb{R})$ and is absolutely continuous on $\mathbb{R}$. Moreover, Lemma 1.4 implies

$$\log|(\eta \circ \gamma)'| = \log|\eta' \circ \gamma| + \log|\gamma'| = (\log|\eta'|) \circ \gamma + \log|\gamma'| \in \mathrm{BMO}(\mathbb{R}) ,$$

which means that

$$\eta \circ \gamma \in \mathrm{HBMO}(\mathbb{R}) . \tag{1.10}$$

As we noticed just before Lemma 1.4, the homeomorphism γ^{-1} belongs to $\mathrm{Hom}^{+}(\mathbb{R})$ and is absolutely continuous on $\mathbb{R}$. Applying Lemma 1.4 once again we see that

$$\log|(\eta \circ \gamma^{-1})'| = \log \frac{|\eta' \circ \gamma^{-1}|}{|\gamma' \circ \gamma^{-1}|} = (\log|\eta'| - \log|\gamma'|) \circ \gamma^{-1} \in \mathrm{BMO}(\mathbb{R}) ,$$

and consequently

$$\eta \circ \gamma^{-1} \in \mathrm{HBMO}(\mathbb{R}) . \tag{1.11}$$

Assume a sequence $\gamma_n \in \mathrm{HBMO}(\mathbb{R})$, $n \in \mathbb{N}$, satisfies $\rho_*(\gamma_n, \gamma) \to 0$ as $n \to \infty$. Combining (1.10) and (1.11) with (1.6) we obtain

$$\begin{aligned} \rho_*(\gamma_n \circ \gamma^{-1}, \mathrm{id}_{\mathbb{R}}) &= \big\|\log|(\gamma_n \circ \gamma^{-1})'|\big\|_* = \left\|\log \frac{|\gamma_n' \circ \gamma^{-1}|}{|\gamma' \circ \gamma^{-1}|}\right\|_* \\ &= \big\|(\log|\gamma_n'|) \circ \gamma^{-1} - (\log|\gamma'|) \circ \gamma^{-1}\big\|_* = \big\|(\log|\gamma_n'| - \log|\gamma'|) \circ \gamma^{-1}\big\|_* \\ &\le c_\gamma \big\|\log|\gamma_n'| - \log|\gamma'|\big\|_* = c_\gamma \rho_*(\gamma_n, \gamma) \to 0 , \quad n \to \infty , \end{aligned}$$

which proves (1.9). □

2. The class HVMO($\mathbb{T}$). In this section we establish our main results, that deal with the class HVMO($\mathbb{T}$); see Theorems 2.3, 2.4 and Corollary 2.5. For $z = x + iy \in \mathbb{C}_+ := \{w \in \mathbb{C} : \operatorname{Im} w > 0\}$ set

$$P_y(x) := -\frac{1}{\pi} \operatorname{Im} \frac{1}{z} = \frac{1}{\pi} \frac{y}{x^2 + y^2} \,.$$

The function $\mathbb{C}_+ \ni z \mapsto P_y(x) \in \mathbb{R}$ is the familiar Poisson kernel for the upper half plane $\mathbb{C}_+$. For every $f \in \mathrm{BMO}(\mathbb{R})$,

$$\int_{-\infty}^{+\infty} \frac{|f(t)|}{1+t^2} dt < \infty \,,$$

so that the function $t \mapsto P_y(x-t)f(t)$ belongs to $L^1(\mathbb{R})$ for all $x \in \mathbb{R}$ and $y > 0$, and we may define

$$P_y * f(x) := \int_{-\infty}^{+\infty} P_y(x-t)f(t)dt \,, \quad y > 0 \,, x \in \mathbb{R} \,.$$

To study the class HVMO($\mathbb{T}$) we need the following characterization of the space VMO($\mathbb{R}$); cf. [G, p. 250].

Theorem 2.1. *For every $f \in \mathrm{BMO}(\mathbb{R})$ the following conditions are equivalent:*

(i) $f \in \mathrm{VMO}(\mathbb{R})$;
(ii) $\|P_y * f - f\|_* \to 0$, as $y \to 0^+$;
(iii) *There exists a sequence $f_n \in \mathrm{BMO}(\mathbb{R})$, $n \in \mathbb{N}$ such that each function f_n is uniformly continuous on $\mathbb{R}$ and $\|f_n - f\|_* \to 0$ as $n \to \infty$.*

Each $\gamma \in \mathrm{Hom}^+(\mathbb{T})$ defines a unique $\hat{\gamma} \in \mathrm{Hom}^+(\mathbb{R})$ satisfying $0 \le \hat{\gamma}(0) < 2\pi$ and

$$\gamma(e^{it}) = e^{i\hat{\gamma}(t)} \,, \quad t \in \mathbb{R} \,, \tag{2.1}$$

called the *angular parametrization* or the *lifted mapping* of γ. By (2.1), $\hat{\gamma}$ satisfies

$$\hat{\gamma}(t + 2\pi) = \hat{\gamma}(t) + 2\pi \,, \quad t \in \mathbb{R} \,. \tag{2.2}$$

Lemma 2.2. *If $\gamma \in \mathrm{HBMO}(\mathbb{T})$, then $\hat{\gamma} \in \mathrm{HBMO}(\mathbb{R})$ and*

$$\rho_*(\eta,\gamma) \le \rho_*(\hat{\eta},\hat{\gamma}) \le 3\rho_*(\eta,\gamma)\,, \quad \eta,\gamma \in \mathrm{HBMO}(\mathbb{T})\,. \tag{2.3}$$

In particular, $\hat{\gamma} \in \mathrm{HBMO}^{\infty}(\mathbb{R}) \cap \mathrm{HVMO}(\mathbb{R})$ whenever $\gamma \in \mathrm{HVMO}(\mathbb{T})$.

Proof. For $f \in \mathrm{BMO}(\mathbb{T})$ let $\widetilde{f}(t) := f(e^{it})$, $t \in \mathbb{R}$. Fix $f \in \mathrm{BMO}(\mathbb{T})$ and assume

$$\int_0^{2\pi} f(e^{it})dt = 0\,. \tag{2.4}$$

Given a closed interval $I \subset \mathbb{R}$ assume that $2\pi < |I|_1 < \infty$. Then $I = I' \cup I''$, where I' and I'' are adjacent closed intervals such that $0 < |I''| \le 2\pi$ and $|I'| = 2n\pi$ for some $n \in \mathbb{N}$. It follows that

$$\begin{aligned}
&\frac{1}{|I|_1}\int_I |\widetilde{f}(t) - \widetilde{f}_I|dt \\
&\quad \le \frac{1}{|I|_1}\left(\int_{I'} |\widetilde{f}(t)|dt + \int_{I'} |\widetilde{f}_I|dt + \int_{I''} |\widetilde{f}(t) - \widetilde{f}_{I''}|dt + \int_{I''} |\widetilde{f}_{I''} - \widetilde{f}_I|dt\right) \\
&\quad \le \frac{|I'|_1}{|I|_1}\|f\|_* + \frac{|I''|_1}{|I|_1}\|f\|_* + \frac{1}{|I|_1}\left(|I'|_1|\widetilde{f}_I| + |I''|_1|\widetilde{f}_{I''} - \widetilde{f}_I|\right) \\
&\quad = \|f\|_* + \frac{1}{|I|_1}\left(\frac{|I'|_1|I''|_1}{|I|_1}|\widetilde{f}_{I''}| + |I''|_1\left(1 - \frac{|I''|_1}{|I|_1}\right)|\widetilde{f}_{I''}|\right) \\
&\quad = \|f\|_* + 2\frac{|I'|_1|I''|_1}{|I|_1|I|_1}|\widetilde{f}_{I''}| \\
&\quad \le \|f\|_* + \frac{2}{|I|_1}\left|\int_{I''} \widetilde{f}(t)dt\right| \le \|f\|_* + 2\cdot\frac{2\pi}{|I|_1}\cdot\frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|dt \le 3\|f\|_*\,.
\end{aligned}$$

Since $\|\widetilde{f}\|_{*,2\pi} = \|f\|_*$, it follows that

$$\|f\|_* \le \|\widetilde{f}\|_* \le 3\|f\|_*\,, \tag{2.5}$$

provided (2.4) holds. If f does not satisfy (2.4), then $f = (f - a) + a$ with $a := (2\pi)^{-1}\int_0^{2\pi} f(e^{it})dt$. Since $f - a \in \mathrm{BMO}(\mathbb{T})$ and (2.4) holds with f replaced by $f - a$, we conclude from (2.5) that $\|\widetilde{f}\|_* = \|\widetilde{f - a}\|_* \le 3\|f - a\|_* = 3\|f\|_*$. Therefore (2.5) holds for every $f \in \mathrm{BMO}(\mathbb{T})$.

If $\gamma \in \mathrm{HBMO}(\mathbb{T})$, then $f := \log|\gamma'| \in \mathrm{BMO}(\mathbb{T})$ satisfies (2.5). Therefore $\hat{\gamma} \in \mathrm{HBMO}(\mathbb{R})$ by the equality $\hat{\gamma}' = |\widetilde{\gamma'}|$. Given $\eta,\gamma \in \mathrm{HBMO}(\mathbb{T})$ set

$f := \log|\eta'| - \log|\gamma'|$. Since $f \in \mathrm{BMO}(\mathbb{T})$ and $\widetilde{f} = \log\hat{\eta}' - \log\hat{\gamma}'$, we deduce (2.3) from (2.5).

Assume now that $\gamma \in \mathrm{HVMO}(\mathbb{T})$. As shown above, $\hat{\gamma} \in \mathrm{HBMO}(\mathbb{R})$. Since for $0 < \delta < 2\pi$,

$$\|\log\hat{\gamma}'\|_{*,\delta} = \|\log|\gamma'|\|_{*,\delta} \to 0\ , \quad \text{as } \delta \to 0^+ ,$$

it follows that $\hat{\gamma} \in \mathrm{HVMO}(\mathbb{R})$. Moreover, by (2.2) the function $P_y * (\log\hat{\gamma}')$ is 2π–periodic and continuous on $\mathbb{R}$, and hence $P_y * (\log\hat{\gamma}') \in L^\infty(\mathbb{R})$ for each $y > 0$. Then Theorem 2.1 shows that $\hat{\gamma} \in \mathrm{HBMO}^\infty(\mathbb{R})$, which completes the proof. □

We are now in a position to prove our main results.

Theorem 2.3. *The inclusion* $\mathrm{HVMO}(\mathbb{T}) \subset \mathrm{QS}(\mathbb{T})$ *holds and the pseudo–metric* ρ_* *is stronger than the Teichmüller pseudo–metric* τ*, i.e. for all* $\gamma, \gamma_n \in \mathrm{HVMO}(\mathbb{T})$, $n \in \mathbb{N}$,

$$\rho_*(\gamma_n, \gamma) \to 0 \quad \text{as } n \to \infty \quad \Longrightarrow \quad \tau(\gamma_n, \gamma) \to 0 \quad \text{as } n \to \infty\ , \tag{2.6}$$

Proof. Let $\gamma \in \mathrm{HVMO}(\mathbb{T})$. By Lemma 2.2, $\hat{\gamma} \in \mathrm{HBMO}^\infty(\mathbb{R})$, and Theorem 1.3 gives $\hat{\gamma} \in \mathrm{QS}(\mathbb{R})$. Hence $\gamma \in \mathrm{QS}(\mathbb{T})$, which is clear from (2.1) and (0.1). Assume that a sequence $\gamma_n \in \mathrm{HVMO}(\mathbb{T})$, $n \in \mathbb{N}$, satisfies $\rho_*(\gamma_n, \gamma) \to 0$ as $n \to \infty$. By Lemma 2.2, $\hat{\gamma}_n \in \mathrm{HBMO}^\infty(\mathbb{T})$, $n \in \mathbb{N}$, and

$$\rho_*(\hat{\gamma}_n, \hat{\gamma}) \le 3\rho_*(\gamma_n, \gamma) \to 0 \quad \text{as } n \to \infty\ .$$

Theorem 1.5 now shows that

$$\|\log(\hat{\gamma}_n \circ \hat{\gamma}^{-1})'\|_* = \rho_*(\hat{\gamma}_n \circ \hat{\gamma}^{-1}, \mathrm{id}_\mathbb{R}) \to 0 \quad \text{as } n \to \infty\ .$$

Hence by Theorem 1.3 there exists a sequence $M_n \ge 1$, $n \in \mathbb{N}$, such that $\hat{\gamma}_n \circ \hat{\gamma}^{-1} \in \mathrm{QS}(\mathbb{R}; M_n)$, $n \in \mathbb{N}$, and $M_n \to 1$ as $n \to \infty$. Moreover, from (2.1) we see that for each $n \in \mathbb{N}$ the identity

$$\widehat{\gamma_n \circ \gamma^{-1}}(t) = \hat{\gamma}_n \circ \hat{\gamma}^{-1}(t) + 2k_n\pi\ , \quad t \in \mathbb{R}\ ,$$

holds with some integer k_n. Applying now (0.1) we obtain $\gamma_n \circ \gamma^{-1} \in \mathrm{QS}(\mathbb{T}; M_n)$, $n \in \mathbb{N}$. Then (0.2) implies that

$$\tau(\gamma_n, \gamma) \le \log M_n^2 \to 0 \quad \text{as } n \to \infty\ ,$$

which proves (2.6). □

Theorem 2.4. *) *The classes* HA($\mathbb{T}$), HVMO($\mathbb{T}$) *and* QS($\mathbb{T}$) *satisfy*

$$\text{(2.7)}\qquad \mathrm{HA}(\mathbb{T}) \subset \mathrm{cl}_{\rho_*}(\mathrm{HA}(\mathbb{T})) = \mathrm{HVMO}(\mathbb{T}) \subset \mathrm{cl}_{\tau}(\mathrm{HA}(\mathbb{T})) \subset \mathrm{QS}(\mathbb{T})\,.$$

Proof. By definition, each $\gamma \in \mathrm{HA}(\mathbb{T})$ has a conformal extension ω to an annulus $\Omega \supset \mathbb{T}$. Hence for every $z \in \mathbb{T}$, $|\gamma'(z)| = |\omega'(z)| > 0$, and so $\log|\gamma'| \in \mathrm{VMO}(\mathbb{T})$ as a continuous function. Thus $\gamma \in \mathrm{HVMO}(\mathbb{T})$, and the inclusion

$$\text{(2.8)}\qquad \mathrm{HA}(\mathbb{T}) \subset \mathrm{HVMO}(\mathbb{T})$$

holds. Fix $\gamma \in \mathrm{HVMO}(\mathbb{T})$. For every $n \in \mathrm{N}$, define

$$Q_n(z) := \frac{1}{\pi}\frac{n}{n^2z^2+1}\,, \quad z \in \mathbb{R}_{1/n}\,,$$

where $\mathbb{R}_\varepsilon := \{z \in \mathbb{C} : |\operatorname{Im} z| < \varepsilon\}$, $\varepsilon > 0$. By Lemma 2.2, the function

$$\mathbb{R} \ni t \mapsto f(t) := \log|\gamma'(e^{it})| = \log\hat{\gamma}'(t) \in \mathbb{R}$$

belongs to $\mathrm{BMO}(\mathbb{R})$. Then for all $n \in \mathrm{N}$ and $z \in \mathbb{R}_{1/n}$ the function

$$\mathbb{R} \ni t \mapsto Q_n(z-t)\log|\gamma'(e^{it})| \in \mathbb{C}$$

is integrable on $\mathbb{R}$ and we may define

$$Q_n * f(z) := \int_{-\infty}^{\infty} Q_n(z-t)f(t)dt = \int_{-\infty}^{\infty} Q_n(z-t)\log|\gamma'(e^{it})|dt,\ z \in \mathbb{R}_{1/n}.$$

Given $n \in \mathrm{N}$ the function $Q_n * f$ is analytic on the strip $\mathbb{R}_{1/n}$ and so is the function $\sigma_n : \mathbb{R}_{1/n} \to \mathbb{C}$,

$$\sigma_n(z) := c_n \int_0^z \exp(Q_n * f(w))dw\ , \quad z \in \mathbb{R}_{1/n}\,,$$

where the integral is taken along the line segment $[0,z]$ and $2\pi/c_n := \int_0^{2\pi}\exp(Q_n * f(t))dt$. Moreover, for all $z \in \mathbb{R}_{1/n}$,

$$Q_n*f(z+2\pi) = \int_{-\infty}^{\infty} Q_n(z+2\pi-t)f(t)dt = \int_{-\infty}^{\infty} Q_n(z-t)f(t+2\pi)dt = Q_n*f(z),$$

*)This theorem implies [P3, Thm. 3.4.7].

and consequently

(2.9)
$$\begin{aligned}\sigma_n(z+2\pi) &= c_n \int_0^{z+2\pi} \exp(Q_n * f(w))dw \\ &= c_n \int_0^{2\pi} \exp(Q_n * f(w))dw + c_n \int_{2\pi}^{z+2\pi} \exp(Q_n * f(w))dw \\ &= 2\pi + c_n \int_0^z \exp(Q_n * f(w+2\pi))dw = 2\pi + \sigma_n(z) .\end{aligned}$$

Since

(2.10) $$\sigma_n'(x) = c_n \exp(Q_n * f(x)) > 0 , \quad x \in \mathbb{R} ,$$

we conclude from (2.9) that there exists ε_n such that $0 < \varepsilon_n \leq 1/n$ and

$$\operatorname{Re} \sigma_n'(z) > 0 , \quad z \in \mathbb{R}_{\varepsilon_n} .$$

Therefore the mapping σ_n is conformal on the strip $\mathbb{R}_{\varepsilon_n}$ and by (2.9) so is the mapping ω_n on the annulus Ω_{ε_n}, where for each $n \in \mathbb{N}$,

$$\omega_n(z) := \exp(i\sigma_n(-i \log z)) \quad \text{and} \quad z \in \Omega_{\varepsilon_n} := \{z \in \mathbb{C} : |\log|z|| < \varepsilon_n\} .$$

Since $\omega_n(e^{it}) = e^{i\sigma_n(t)}$ for $t \in \mathbb{R}$ and $n \in \mathbb{N}$, we conclude from (2.10) that each function σ_n is increasing on $\mathbb{R}$, and so

(2.11) $$\gamma_n := \omega_{n|\mathbb{T}} \in \mathrm{HA}(\mathbb{T}) , \quad n \in \mathbb{N} .$$

Moreover, the identity

(2.12) $$|\gamma_n'(e^{it})| = \sigma_n'(t) , \quad t \in \mathbb{R} ,$$

holds for every $n \in \mathbb{N}$. By our assumption, $\log|\gamma'| \in \mathrm{VMO}(\mathbb{T})$ and Lemma 2.2 gives $f \in \mathrm{VMO}(\mathbb{R})$. Since $Q_n(x) = P_{1/n}(x)$ for $x \in \mathbb{R}$, we conclude from (2.10), (2.12), Lemma 2.2 and Theorem 2.1 that

$$\rho_*(\gamma_n, \gamma) \leq \rho_*(\hat\gamma_n, \hat\gamma) = \|\log \sigma_n' - \log \hat\gamma'\|_* = \|Q_n * f - f\|_* \to 0 , \quad n \to \infty .$$

Thus $\gamma \in \mathrm{cl}_{\rho_*}(\mathrm{HA}(\mathbb{T}))$ by (2.11), and so

(2.13) $$\mathrm{HVMO}(\mathbb{T}) \subset \mathrm{cl}_{\rho_*}(\mathrm{HA}(\mathbb{T})) .$$

Let now $\gamma \in \mathrm{cl}_{\rho_*}(\mathrm{HA}(\mathbb{T}))$. From (2.13) it follows that there exists a sequence $\gamma_n \in \mathrm{HA}(\mathbb{T})$, $n \in \mathrm{N}$, such that $\rho_*(\gamma_n, \gamma) \to 0$ as $n \to \infty$. Then Theorem 2.3 shows that $\tau(\gamma_n, \gamma) \to 0$ as $n \to \infty$, and so $\gamma \in \mathrm{cl}_\tau(\mathrm{HA}(\mathbb{T}))$. Thus

$$\mathrm{HVMO}(\mathbb{T}) \subset \mathrm{cl}_\tau(\mathrm{HA}(\mathbb{T})) \subset \mathrm{QS}(\mathbb{T})\,. \tag{2.14}$$

By (2.8) and by Lemma 2.2, $\hat{\gamma}, \hat{\gamma}_n \in \mathrm{HVMO}(\mathbb{R})$ for $n \in \mathrm{N}$ and

$$\| \log|\hat{\gamma}_n'| - \log|\hat{\gamma}'|\,\|_* = \rho_*(\hat{\gamma}_n, \hat{\gamma}) \le 3\rho_*(\gamma_n, \gamma) \to 0 \quad \text{as } n \to \infty\,.$$

Moreover, each function $\log|\hat{\gamma}_n'|$ is uniformly continuous on $\mathbb{R}$ being continuous and periodic. Theorem 2.1 now shows that $\log|\hat{\gamma}'| \in \mathrm{VMO}(\mathbb{R})$, and so $\gamma \in \mathrm{HVMO}(\mathbb{T})$. Therefore

$$\mathrm{cl}_{\rho_*}(\mathrm{HA}(\mathbb{T})) \subset \mathrm{HVMO}(\mathbb{T})\,. \tag{2.15}$$

Combining the inclusions (2.8) and (2.13)–(2.15) we obtain (2.7), which is our claim. □

Corollary 2.5. *If $\gamma \in \mathrm{HVMO}(\mathbb{T})$, then $\Lambda_\gamma^* = \Lambda_\gamma$. In particular, if $\gamma \in \mathrm{HVMO}(\mathbb{T}) \setminus \mathrm{Q}(\mathbb{T}; 1)$, then $\Lambda_\gamma^* \neq \emptyset$.*

Proof. The equality $\Lambda_\gamma^* = \Lambda_\gamma$ follows from the inclusion (2.14) and [P2, Thm. 2.1]; also cf. [P3, Corollary 3.4.5]. If $\gamma \in \mathrm{Q}(\mathbb{T}) \setminus \mathrm{Q}(\mathbb{T}; 1)$, then [P2, Thm. 1.4] (also see [P3, Corollary 3.2.7] and [KP, (3.6)]) shows that $\Lambda_\gamma \neq \emptyset$, which completes the proof. □

References

[BA] Beurling, A. and L. V. Ahlfors, *The boundary correspondence under quasiconformal mappings*, Acta Math. **96** (1956), 125–142.

[G] Garnett, J. B., *Bounded Analytic Functions*, Academic Press, New York, 1981.

[JN] John, F. and L. Nirenberg, *On functions of bounded mean oscilation*, Comm. Pure Appl. Math. **14** (1961), 415–426.

[J] Jones, P. W., *Homeomorphisms of the line which preserve BMO*, Arkiv för Mat. **21** (1983), 229–231.

[K] Krzyż, J. G., *Quasicircles and harmonic measure*, Ann. Acad. Sci. Fenn. Ser. A. I. Math. **12** (1987), 19–24.

[KP] ———, D. Partyka, *Generalized Neumann-Poincaré operator, chord-arc curves and Fredholm eigenvalues*, Complex Variables **21** (1993), 253-263.

[L] Lehtinen, M., *Remarks on the maximal dilatation of the Beurling–Ahlfors extension*, Ann. Acad. Sci. Fenn. Ser. A. I. Math. **9** (1984), 133–139.

[LV] Lehto, O. and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, Springer-Verlag, Berlin, 1973.

[P1] Partyka, D., *Generalized harmonic conjugation operator*, Ber. Univ. Jyväskylä Math. Inst. **55** (1993), 143-155 (Proc. of the Fourth Finnish-Polish Summer School in Complex Analysis at Jyväskylä, August 1992).

[P2] ______, *Spectral values and eigenvalues of a quasicircle*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **46** (1993), 81-98.

[P3] ______, *The generalized Neumann-Poincaré operator and its spectrum*, Dissertationes Math. No. 366, Institute of Mathematics, Polish Academy of Sciences, Warszawa, 1997.

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