ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA

VOL. LII.1, 13 SECTIO A 1998

MARIA NOWAK

Coefficient Multipliers of Spaces of Analytic Functions

Dedicated to Professor Eligiusz Złotkiewicz on the occasion of his 60th birthday

ABSTRACT. In this paper we compare some known results on coefficient multipliers for Hardy and Bergman spaces. We also use them to prove an extension of the Bergman space version of the Hausdorff-Young Theorem (Sect. 3) and to answer in the negative open Problem 10 posed in the survey paper [CL] (Remark 3). Moreover, the multipliers of the Bergman space B^p, $0 < p \leq 2$, into BMOA and the multipliers of BMOA into the Bloch space are characterized.

1. Introduction. Notations. Let X and Y be two sequence spaces. A sequence of complex numbers $\lambda = \{\lambda_n\}$ is called a multiplier from X to Y if $\{\lambda_n x_n\} \in Y$ whenever $\{x_n\} \in X$. The set of all multipliers from X to Y will be denoted by (X, Y).

While dealing with spaces of analytic functions in the unit disc $\mathbb{D}$ we associate with each function $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ the corresponding sequence of Taylor coefficients $\{\hat{f}(n)\}$. On the other hand, any sequence $\lambda = \{\lambda_n\}$ such that $\limsup_{n\to\infty} \sqrt[n]{|\lambda_n|} \leq 1$ may be considered as a holomorphic function in $\mathbb{D}$, defined by $\lambda(z) = \sum_{n=0}^{\infty} \lambda_n z^n$.

If $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$ and $g(z)=\sum_{n=0}^{\infty}\hat{g}(n)z^n$ then we write

$$f\star g(z)=\sum_{n=0}^{\infty}\hat{f}(n)\hat{g}(n)z^n .$$

For a function f holomorphic in $\mathbb{D}$ and $0\le r<1$ we set

$$M_p(f,r)=\left(\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\right)^{1/p},\quad 0<p<\infty ,$$
$$M_\infty(f,r)=\max_{0\le\theta\le 2\pi}|f(re^{i\theta}|.$$

The Hardy space H^p, $0<p\le\infty$, is the space of those f for which $M_p(f,r)$ remains bounded as $r\to 1^-$.

Let $d\sigma$ denote the normalized area measure on $\mathbb{D}$. The Bergman space B^p, $0<p<\infty$ consists, of all holomorphic functions on $\mathbb{D}$ such that

$$\|f\|_{B^p}=\left(\int_{\mathbb{D}}|f(z)|^p d\sigma(z)\right)^{1/p}<\infty .$$

Moreover, let $l(p,q)$, $0<p,q\le\infty$, denote the space of sequences $\{\lambda_n\}\in\mathbb{C}$ such that

$$\|\{\lambda_n\}\|_{(p,q)}=\left(\sum_{n=0}^{\infty}\left(\sum_{k\in I_n}|\lambda_k|^p\right)^{q/p}\right)^{1/q}<\infty ,$$

where $I_0=\{0\}$ and $I_n=\{k\in\mathrm{N}:2^{n-1}\le k<2^n\}$, $n\ge 1$. In the cases p or $q=\infty$ we replace the corresponding sum by a supremum. The following lemma, proved in [K], characterizes the multiplier space $(l(r,s),l(p,q))$.

Lemma. *Let $0<r,s,p,q\le\infty$ and define α and β by*

$$\frac{1}{\alpha}=\frac{1}{p}-\frac{1}{r},\quad \text{if } r>p,\quad \alpha=\infty,\quad \text{if } r\le p,$$
$$\frac{1}{\beta}=\frac{1}{q}-\frac{1}{s},\quad \text{if } s>q,\quad \beta=\infty,\quad \text{if } s\le q.$$

Then $(l(r,s),l(p,q))=l(\alpha,\beta)$.

We will also deal with the Bloch space $\mathcal{B}$ consisting of those functions f analytic on $\mathbb{D}$ for which $\|f\|_{\mathcal{B}}=\sup_{z\in\mathbb{D}}(1-|z|)|f'(z)|<\infty$ and with BMOA space of those functions f in H^1 for which boundary functions (that will be also denoted by f) are of bounded mean oscillation, i.e.

$$\|f\|_{BMO} = \sup_I \frac{1}{|I|} \int_I |f(\theta) - f_I| d\theta < \infty ,$$

where I is an interval contained in $\partial\mathbb{D}$ and $f_I = (1/|I|) \int_I f(\theta) d\theta$.

Throughout this paper all functions f will be holomorphic on the unit disc. We will denote by p' the conjugate exponent of p, i.e. $1/p + 1/p' = 1$, and by C a general constant that may vary from line to line.

The authoress is grateful to Professor Peter L. Duren for informing her about the papers [Hor] and [V].

2. Remarks on multipliers (H^p, H^q) **and** (B^p, B^q). In 1941 Hardy and Littlewood [HL] proved the following:

Theorem HL. *If* $1 \le p \le 2 \le q < \infty$, $1/s = (1/q) - (1/p) + 1$ *and*

$$M_s(f', r) \le \frac{C}{1 - r}, \quad 0 < r < 1, \tag{2.1}$$

then $f \in (H^p, H^q)$.

Thirty years later C.N. Kellogg ([K]) obtained another necessary condition for a membership in the multiplier space (H^p, H^q). Namely, he proved

Theorem K1. *If* $1 \le p \le 2 \le q < \infty$ *and* $1/t = (1/p) - (1/q)$ *then*

$$l(t, \infty) \subset (H^p, H^q). \tag{2.2}$$

Note that $t = s'$, where s is defined in Theorem HL.

It seems interesting to decide which of these two theorems is stronger. J.H. Hedlund [H] proved that for $q = 2$ Theorem K1 does include Theorem HL. The next statement shows that this is not true for all $q \ge 2$.

Proposition 1. *Assume that* $1 \le p \le 2 \le q < \infty$ *and* $1/s = (1/q) - (1/p) + 1$. *If* $s \le 2$ *(or equivalently* $p \ge 2q/(q+2)$*) then*

$$\{f : M_s(f', r) = O((1 - r)^{-1})\} \subset l(s', \infty).$$

If $s \ge 2$ *(or equivalently* $p \le 2q/(q+2)$*) then*

$$l(s', \infty) \subset \{f : M_s(f', r) = O((1 - r)^{-1})\}.$$

Remark 1. The foregoing propositon is a generalization of Lemma 8 in [AS], which states that

$$\{f:\ M_2(f',r) = O\left((1-r)^{-1}\right)\} = l(2,\infty) \tag{2.3}$$

and to show it we apply a similar reasoning to that in the proof of this result.

Proof of Proposition 1. Note that always $s \geq 1$. Assume first that $s \leq 2$ and $M_s(f',r) \leq C/(1-r)$. Let $g(z) = zf'(z)$ and $g_r(z) = g(rz)$ for $0 \leq r < 1$ and $z \in \mathbb{D}$. Then the Hausdorff-Young theorem (cf. [D, p.94]) implies

$$\sum_{n=1}^{N} n^{s'}|\hat{f}(n)|^{s'} r^{Ns'} \leq \sum_{n=1}^{\infty} (n|\hat{f}(n)|r^n)^{s'} \leq \|g_r\|_{H^s}^{s'} \leq M_s^{s'}(f',r) \leq \frac{C}{(1-r)^{s'}} .$$

Taking $r = 1 - 1/N$ we obtain $\sum_{n=1}^{N} n^{s'}|\hat{f}(n)|^{s'} = O(N^{s'})$ which is equivalent to the defining condition for $l(s',\infty)$.

Assume now that $s \geq 2$ and $\{\hat{f}(n)\} \in l(s',\infty)$. This implies (see [AS, p.262])

$$\sum_{n=jm+1}^{(j+1)m} |\hat{f}(n)|^{s'} \leq C, \quad m,j = 1,2,\ldots . \tag{2.4}$$

Moreover, the dual part of the Hausdorff-Young theorem gives

$$M_s(f',r) \leq C\left(\sum_{n=1}^{\infty} (n|\hat{f}(n)|r^n)^{s'}\right)^{1/s'} . \tag{2.5}$$

In view of (2.4) we have, for $1 - 1/(N-1) < r \leq 1 - 1/N$,

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{s'}|\hat{f}(n)|^{s'} r^{ns'} &\leq \left(\sum_{n=1}^{N} + \sum_{n=N+1}^{2N} + \sum_{n=2N+1}^{3N} + \ldots\right) n^{s'}|\hat{f}(n)|^{s'}(1-1/N)^{s'n} \\
&\leq CN^{s'} + C(2N)^{s'}(1-1/N)^{Ns'} + C(3N)^{s'}(1-1/N)^{2Ns'} + \ldots \\
&\leq CN^{s'}\left(1 + 2^{s'}(1-1/N)^{Ns'} + 3^{s'}(1-1/N)^{2Ns'} + \ldots\right) \\
&\leq CN^{s'} \sum_{k=0}^{\infty} (k+1)^{s'} 2^{-ks'} \leq CN^{s'} \leq 2C(N-1)^{s'} \leq \frac{C}{(1-r)^{s'}} .
\end{aligned}$$

This, together with (2.5), gives $M_s(f',r) \le C/(1-r)$.

□

It is well known that many results for H^p spaces have their analogues in the theory of Bergman spaces. The next two results due to J. Xiao and O. Blasco are such analogues of Theorems HL and K1, respectively.

Theorem X ([X]). *Suppose that* $1 < p \le q < \infty$ *and* $1/s = (1/q) - (1/p) + 1$. *If* $M_s(f',r) = O\left((1-r)^{-1/s}\right)$, *then* $f \in (B^p, B^q)$.

Theorem B ([B]). *Let* $1 \le p \le 2 \le q < \infty$ *and* $1/t = (1/p) - (1/q)$. *If* $\{n^{1/t}\lambda_n\} \in l(t,\infty)$, *then* $\{\lambda_n\} \in (B^p, B^q)$.

The next statement provides us with a comparison of Theorems X and B for $1 < p \le 2 \le q < \infty$.

Proposition 2. *If* $1 \le s \le 2$, *then*

$$M_s(f',r) = O\left((1-r)^{-1/s}\right) \Longrightarrow \left\{n^{1/s'} \hat{f}(n)\right\} \in l(s',\infty).$$

If $s \ge 2$ *then*

$$\left\{n^{1/s'}\lambda_n\right\} \in l(s',\infty) \Longrightarrow M_s(\lambda',r) = O\left((1-r)^{-1/s}\right),$$

where λ *is a function defined by* $\lambda(z) = \sum_{n=0}^{\infty} \lambda_n z^n$, $z \in \mathbb{D}$.

Proof. Assume first that $1 \le s \le 2$.
As in the proof of Proposition 1, we get $\sum_{n=1}^{N} n^{s'} |\hat{f}(n)|^{s'} = O(N^{s'-1})$ and this means that $\left\{n^{1/s'}\hat{f}(n)\right\} \in l(s',\infty)$. If $s \ge 2$ and $\left\{n^{1/s'}\lambda_n\right\} \in l(s',\infty)$, then a reasoning similar to that used in the second part of the proof of Proposition 1 gives

$$\sum_{n=1}^{\infty} n^{s'} |\lambda_n|^{s'} r^{ns'} \le \sum_{n=1}^{\infty} n^{s'-1} n |\lambda_n|^{s'} r^{ns'} \le \frac{C}{(1-r)^{s'-1}}.$$

Thus, by (2.5) $M_s(\lambda',r) \le C/(1-r)^{1/s}$.

□

Remark 2. Applying the equalities (cf. [D], [HL], [S])

$$(2.6) \qquad (H^1, H^2) = \left\{f : M_2(f',r) = O\left((1-r)^{-1}\right)\right\} = l(2,\infty)$$

we can easily prove the Paley result $H^1 \subset l(\infty, 2)$. In fact, (2.6) implies $H^1 \subset (l(2,\infty), H^2) = (l(2,\infty), l(2,2)) = l(\infty, 2)$. Moreover, by Theorem 3.1 in [X] (see also [V, p. 347]) and Proposition 2

$$(B^1, B^2) = \left\{f:\ M_2(f', r) = O\left((1-r)^{-1/2}\right)\right\}$$
$$= \left\{f:\ \{n^{1/2}\hat{f}(n)\} \in l(2,\infty)\right\}.$$

Thus, an analogous reasoning gives $B^1 \subset \{\{\lambda_n\}:\ \{n\lambda_n\} \in l(\infty, 2)\}$.

Remark 3. Using relation (2.3) we can answer in the negative open Problem 10 in [CL, p. 108] and show that the inclusion $H^1 \subset l(2,\infty)$ is not true. To this end consider the functions $k_p(z) = 1/(1-z)^p$, $z \in \mathbb{D}, 0 < p < 1$. All these functions are in H^1. However, if $1/2 < p < 1$, then

$$M_2(k_p', r) = \left(\int_0^{2\pi} \frac{p^2\, d\theta}{|1 - re^{i\theta}|^{2p+2}}\right)^{1/2}$$

behaves like $(1-r)^{-p-1/2}$, i.e. the ratio $M_2(k_p', r)/(1-r)^{p+1/2}$ has a positive limit as $r \to 1^-$. So, by (2.3), $k_p \notin l(2,\infty)$.

3. Extension of the Hausdorff-Young Theorem for Bergman spaces.

In [K] the following extension of the Hausdorff-Young Theorem for H^p spaces was obtained

Theorem K2. *If $1 \le p \le 2$ and $f \in H^p$, then $\{\hat{f}(n)\} \in l(p', 2)$ and*

$$\|\hat{f}(n)\|_{(p',2)} \le C\|f\|_{H^p}.$$

If $1 < p \le 2$ and $\{\lambda_n\} \in l(p, 2)$ then $\lambda \in H^{p'}$ and $\|\lambda\|_{H^{p'}} \le C\|\{\lambda_n\}\|_{(p,2)}$.

For the Bergman space the following analogue of the Hausdorff-Young Theorem was proved by Ch. Horowitz [Hor].

Theorem H. *Assume that $1 < p \le 2$. If $f \in B^p$, then*

$$\sum_{n=1}^{\infty} n^{1-p'} |\hat{f}(n)|^{p'} < \infty.$$

Conversely, if $\sum_{n=1}^{\infty} n^{1-p} |\lambda_n)|^p < \infty$, then $\lambda \in B^{p'}$.

Here, using Theorem B we prove

Theorem 1. *If* $f \in B^p$, $1 \le p \le 2$, *then* $\{n^{-1/p}\hat{f}(n)\} \in l(p',2)$ *and*

$$\|\{n^{-1/p}\hat{f}(n)\}\|_{(p',2)} \le C\|f\|_{B^p}. \tag{3.1}$$

If $\{n^{-1/p'}\lambda_n\} \in l(p,2)$, $1 < p \le 2$, *then* $\lambda \in B^{p'}$ *and*

$$\|\lambda\|_{B^{p'}} \le C\|\{n^{-1/p'}\lambda_n\}\|_{(p,2)}. \tag{3.2}$$

Proof. Assume first that $f \in B^p$, $1 \le p \le 2$. Let $1/t = (1/p) - (1/2)$ and define $\Lambda^t = \{\{\lambda_n\} : \{n^{1/t}\lambda_n\} \in l(t,\infty)\}$. Then by Theorem B $\Lambda^t \subset (B^p, B^2)$. This implies

$$B^p \subset (\Lambda^t, B^2). \tag{3.3}$$

Now, applying the facts

(1) $f \in B^2 \iff \{n^{-1/2}\hat{f}(n)\} \in l^2 = l(2,2)$,
(2) $(l(t,\infty), l(2,2)) = l(p',2)$

we get $(\Lambda^t, B^2) = \{\{a_n\} : \{n^{-1/p}a_n\} \in l(p',2)\}$. This and (3.3) give the inclusion $B^p \subset \{\{a_n\} : \{n^{-1/p}a_n\} \in l(p',2)\}$. Moreover, the closed graph theorem implies (3.1).

Now suppose that $\{\lambda_n\}$ is a sequence of complex numbers such that $\{n^{-1/p'}\lambda_n\} \in l(p,2)$. For $z \in \mathbb{D}$, let $\lambda_N(z) = \lambda_1 z + \lambda_2 z^2 + \ldots + \lambda_{2^N} z^{2^N}$. Then, for any $g \in B^p$,

$$\left|\int_{\mathbb{D}} \bar{\lambda}_N g\, d\sigma\right| = \left|\sum_{k=1}^{2^N} \frac{1}{2k+1}\bar{\lambda}_k \hat{g}(k)\right| \le \sum_{n=1}^{N} \sum_{k\in I_n} |k^{-1}\lambda_k \bar{\hat{g}}(k)|$$

$$\le \sum_{n=1}^{N}\left(\sum_{k\in I_n} |k^{-1/p'}\lambda_k|^p\right)^{1/p}\left(\sum_{k\in I_n}|k^{-1/p}\hat{g}(k)|^{p'}\right)^{1/p'}$$

$$\le \left(\sum_{n=1}^{N}\left(\sum_{k\in I_n}|k^{-1/p'}\lambda_k|^p\right)^{2/p}\right)^{1/2}\left(\sum_{n=1}^{N}\left(\sum_{k\in I_n}|k^{-1/p}\hat{g}(k)|^{p'}\right)^{2/p'}\right)^{1/2}$$

By (3.1)

$$\left(\sum_{n=1}^{N}\left(\sum_{k\in I_n}|k^{-1/p}\hat{g}(k)|^{p'}\right)^{2/p'}\right)^{1/2} \le \|\{n^{-1/p}\hat{g}(k)\}\|_{(p',2)} \le C\|g\|_{B^p}.$$

Thus we get
$$\left|\int_{\mathbb{D}} \bar{\lambda}_N g d\sigma\right| \le C\|\{n^{-1/p'}\lambda_n\}\|_{(p,2)}\|g\|_{B^p}\,.$$

Since for $1 < p < \infty$ the dual of B^p can be identified with $B^{p'}$ (e.g. [A, p.7]), taking supremum over all $g \in B^p$ such that $\|g\|_{B^p} \le 1$ we obtain

$$\|\lambda_N\|_{B^{p'}} \le C|\{n^{-1/p'}\lambda_n\}\|_{(p,2)}\,. \tag{3.4}$$

Using similar arguments one can show that $\{\lambda_N\}$ is a Cauchy sequence in $B^{p'}$, so it converges to $\lambda(z) = \sum_{n=1}^{\infty} \lambda_n z^n$ in $B^{p'}$. Taking $N \to \infty$ in (3.4) we get (3.2).

□

4. (B^p, BMOA), $0 < p \le 2$. It was shown in [S] and [SZ] that for $p = 1$ condition (2.1) in Theorem HL is also necessary i.e. if $2 \le q < \infty$, then

$$f \in (H^1, H^q) \iff M_{q'}(f', r) \le \frac{C}{1-r}, \quad 0 < r < 1\,. \tag{4.1}$$

Using this fact and a duality argument M. Mateljevic and M. Pavlovic [MP] proved that for $1 < p \le 2$
$$(H^p, BMOA) = \left\{f :\ M_{p'}(f', r) = O\left((1-r)^{-1}\right), \quad 0 < r < 1\right\}.$$

They also showed that $(H^1, BMOA) = (H^1, \mathcal{B}) = \mathcal{B}$. The last result was extended in [N] where the following characterization of the space $(H^p, \mathcal{B})$, $0 < p < \infty$, was obtained

(i) if $1 \le p < \infty$, then $(H^p, \mathcal{B}) = \left\{f :\ M_{p'}(f', r) = O\left((1-r)^{-1}\right)\right\}$,

(ii) if $0 < p < 1$ and n is an integer such that $1/p < (n+1)$, then
$$\begin{aligned}(H^p, H^\infty) &= (H^p, BMOA) = (H^p, \mathcal{B})\\ &= \left\{g :\ M_\infty(g^{(n)}, r) = O\left((1-r)^{-n-1+(1/p)}\right)\right\} = A_n\end{aligned}$$

The proof of (ii) was based on the equality $(H^p, H^\infty) = A_n$ (due to Duren and Shields [DS]).

Recently J. Xiao [X] proved the following:

if $1 \le p < \infty$ then

$$(B^p, \mathcal{B}) = \left\{f :\ M_{p'}(f'', r) = O\left((1-r)^{-1-1/p'}\right)\right\}, \tag{4.2}$$

if $0 < p \le 1$ and n is an integer such that $2/(n+1) < p \le 2/n$, then

$$(B^p, H^\infty) = \left\{g :\ M_\infty(g^{(n)}, r) = O\left((1-r)^{-n-1+\frac{2}{p}}\right)\right\} = B_n\,. \tag{4.3}$$

We can now prove the main result of this section

Theorem 2. *If* $1 < p \leq 2$, *then*

$$(B^p, BMOA) = \left\{ f:\ M_{p'}(f', r) = O\left((1-r)^{-1/p'}\right)\right\}. \tag{4.4}$$

If $0 < p \leq 1$, *then*

$$(B^p, H^\infty) = (B^p, BMOA) = (B^p, \mathcal{B}) = B_n. \tag{4.5}$$

Proof. Assume first that $1 < p \leq 2$. Let g be a function such that $M_{p'}(g, r') = O\left((1-r)^{-1/p'}\right)$ and $f \in B^p$. We have to show that $h = f \star g$ is in BMOA. Let the multiplier transformation $D^s g$ of g, $g(z) = \sum_{n=0}^\infty \hat{g}(n) z^n$, be defined by $D^s g(z) = \sum_{n=0}^\infty (n+1)^s \hat{g}(n) z^n$, s being any real number. We have

$$|D^2 h(r^2 e^{it})| = \left| \frac{1}{2\pi} \int_0^{2\pi} D^1 f(re^{i\theta}) D^1 g(re^{i(t-\theta)}) d\theta \right|$$
$$\leq M_{p'}(D^1 g, r) M_p(D^1 f, r) \leq \frac{c}{(1-r)^{1-\frac{1}{p}}} M_p(r, D^1 f)$$

Hence

$$|h''(r^2 e^{it})|^p (1-r)^{2p-1} \leq c \int_0^{2\pi} |D^1 f(re^{i\theta})|^p (1-r)^p d\theta.$$

Integrating both sides of this inequality with respect to r we get

$$\int_0^1 M_\infty^p(h'', r)(1-r)^{2p-1} dr \leq c \int_0^1 \int_0^{2\pi} |D^1 f(re^{i\theta})|^p (1-r)^p d\theta dr.$$

Since $f \in B^p$, by Theorem 9 in [HL] the integral on the right-hand side of the last inequality is finite. We will show that for any p, $0 < p \leq 2$,

$$\int_0^1 M_\infty^p(h'', r)(1-r)^{2p-1} dr < \infty \Longrightarrow h \in BMOA. \tag{4.6}$$

By the Hardy-Littlewood Lemma (see e.g. [MP, p.75])

(4.7)
$$\int_0^1 M_\infty^p(h'', r)(1-r)^{2p-1} dr < \infty \Longrightarrow \int_0^1 M_\infty^p(h', r)(1-r)^{p-1} dr < \infty.$$

Furthermore, it has been noticed by Stroethoff [Str, p.417] that if for some $p, 0 < p \le 2$,

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{p-1} \frac{(1-|a|^2)}{|1-\bar{a}z|^2} dv(z) < \infty$$

then $f \in$ BMOA. It follows from the above that

$$\begin{aligned} &\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |h'(z)|^p (1-|z|)^{p-1} \frac{(1-|a|^2)}{|1-\bar{a}z|^2} dv(z) \\ &\le \sup_{a\in U} \int_0^1 M_\infty^p(h',r)(1-r)^{p-1} \int_0^{2\pi} \frac{(1-|a|^2)}{|1-\bar{a}re^{it}|^2} dt dr \\ &\le \int_0^1 M_\infty^p(h',r)(1-r)^{p-1} dr < \infty \end{aligned}$$

which implies $h \in$BMOA.

So we have proved the inclusion

$$\left\{ f : \ M_{p'}(f',r) = O\left((1-r)^{-1/p'}\right)\right\} \subset (B^p, \mathrm{BMOA}).$$

Equality (4.4) follows from (4.2) and the inclusion $(B^p, BMOA) \subset (L^p_a, \mathcal{B})$. In case $p = 1$ (4.2) and (4.3) imply that $(B^p, H^\infty) = (B^p, \mathcal{B})$ and the inclusions $H^\infty \subset$ BMOA $\subset \mathcal{B}$ give (4.5). To prove (4.5) for $0 < p < 1$ it is enough to apply arguments similar to that in the proof of Theorem 2 in [N].

□

5. (BMOA, $\mathcal{B}$). Since BMOA is in some sense the natural limit of H^p as $p \to \infty$, the next result can be treated as an extension of result (i) from the preceding section.

Theorem 3.

(5.1) $$(BMOA, \mathcal{B}) = (VMOA, \mathcal{B}) = \left\{ g : M_1(g',r) = O\left((1-r)^{-1}\right)\right\}.$$

Proof. If $f \in$ BMOA then there exists a bounded function b on the unit circle such that $f = Pb$, where P denotes the Szegö projection and norms $\|f\|_{BMOA} = |f(0)| + \|f\|_{BMO}$ and $\|b\|_\infty$ are equivalent. If g is a function satisfying the condition (5.1) and $0 < r < 1$, then we have

$$\begin{aligned} |(f\star g)'(re^{i\varphi})re^{i\varphi}| &= \frac{1}{2\pi}\left|\int_0^{2\pi} f(e^{i\theta}) g'(re^{i(\varphi-\theta)}) e^{i(\varphi-\theta)} d\theta\right| \\ &= \frac{1}{2\pi}\left|\int_0^{2\pi} Pb(e^{i\theta}) g'(re^{i(\varphi-\theta)}) e^{i(\varphi-\theta)} d\theta\right| \end{aligned}$$

$$= \frac{1}{2\pi}\left|\int_0^{2\pi} b(e^{i\theta})g'(re^{i(\varphi-\theta)})e^{i(\varphi-\theta)}d\theta\right|$$

$$\le C\|b\|_\infty M_1(g',r) \le C/1-r,$$

which means that $f\star g \in \mathcal{B}$.

Now assume that g is an analytic function on $\mathbb{D}$ such that $f\star g\in\mathcal{B}$ for each $f\in$ BMOA. Then by the closed graph theorem

$$\|f\star g\|_{\mathcal{B}} \le C\|f\|_{BMOA}. \tag{5.2}$$

Let for $0\le r<1$, $g_r(z)=g(rz)$, $z\in\overline{\mathbb{D}}$. Since the Hardy space H^1 is equivalent to the dual of VMOA (equipped with the BMOA norm, see e.g.[Zhu]) we have

$$M_1(g',r)=\|g'_r\|_{H^1}\le C \sup_{\substack{f\in VMOA\\ \|f\|_{BMOA}\le 1}} \frac{1}{2\pi}\left|\int_0^{2\pi} f(e^{-i\theta})g'(re^{i\theta})d\theta\right| \le$$

$$\le C \sup_{\substack{f\in BMOA\\ \|f\|_{BMOA}\le 1}} \frac{1}{2\pi}\left|\int_0^{2\pi} f(e^{-i\theta})g'(re^{i\theta})d\theta\right|$$

$$\le C|(f\star g)'(r)|\le C\frac{1}{1-r}.$$

□

Remark 4. It was shown in [MZ] that the functions given by the formula

$$g_{a,\lambda}(z)=\sum_{n=0}^{\infty}\frac{\Gamma(a+n+2)}{\Gamma(a+i\lambda+n+2)}z^n, \quad a>0,\ \lambda\in\mathbb{R},$$

are multipliers from BMOA into $\mathcal{B}$. In view of Theorem 3 this implies that $M_1(g'_{a,\lambda},r)=O\left((1-r)^{-1}\right)$.

To prove the next theorem we will need the Abel dual. The Abel dual of a sequence space A, denoted by A^a, is defined to be the space of sequences $\{\lambda_n\}$ such that $\lim_{r\to 1}\sum_{n=1}^{\infty}\lambda_n a_n r^n$ exists for all $\{a_n\}\in A$. Let $\mathcal{B}_0$ denotes the little Bloch space, i.e. the space of analytic functions in $\mathbb{D}$ such that $(1-|z|^2)|f'(z)|\to 0$ as $|z|\to 1$. It follows from [A, pp.14-20] that

$$(B^1)^a=\left\{f:\ \sum_{n=1}^{\infty} n\hat{f}(n)z^n\in\mathcal{B}\right\} \tag{5.3}$$

and

$$(\mathcal{B}_0)^a = \left\{ f : \sum_{n=0}^{\infty} n\hat{f}(n)z^n \in B^1 \right\} \tag{5.4}$$

Moreover, by Theorem 3.1 in [X]

$$(B^1, B^1) = \left\{ g : M_1(g', r) = O\left((1-r)^{-1}\right)\right\} .$$

Using these facts and a duality argument, we can prove

Theorem 4. $(\mathcal{B}, \mathcal{B}) = \left\{ g : M_1(g', r) = O\left((1-r)^{-1}\right)\right\}$.

Proof. Since for sequence spaces A, B $(A, B) \subset (B^a, A^a)$ and $A \subset A^{aa}$ (see [AS]) we get $(B^1, B^1) \subset (\mathcal{B}, \mathcal{B}) \subset (\mathcal{B}_0, \mathcal{B}) \subset ((\mathcal{B})^a, (\mathcal{B}_0)^a) \subset (B^1, B^1)$. This implies $(B^1, B^1) = (\mathcal{B}, \mathcal{B})$. □

Similar arguments yield

Theorem 5. *For* $1 \le p \le \infty$, $(l^p, \mathcal{B}) = l(p,' \infty)$.

Proof. It has been shown in [B, p.55] and in [DS] that for $1 \le q \le \infty$

$$(B^1, l^q) = \{\{\lambda_n\} : \{n\lambda_n\} \in l(q, \infty)\}. \tag{5.5}$$

In view of the inclusions $(B^1, l^q) \subset (l^{q'}, (B^1)^a) \subset ((B^1)^{aa}, l^q) \subset (B^1, l^q)$ we get $(B^1, l^q) = (l^{q'}, (B^1)^a)$. This, combined with (5.5) and (5.3), gives us the desired result.

□

Remark 5. Note that, by [MP, p. 82], for $1 \le p \le 2$ $(l^p, \mathcal{B}) = (l^p, \mathrm{BMOA})$.

References

[AS] Anderson, J. M. and A. L. Shields, *Coefficient multipliers on Bloch functions*, Trans. Amer. Math. Soc. **224** (1976), 256-269.

[A] Axler, S., *Bergman spaces and their operators. Surveys of some recent results on operator theory*, Pitman Res. Notes Math. Ser. No **171** (1988), 1-50.

[B] Blasco, O., *Multipliers on spaces of analytic functions*, Canad. J. Math. **47** (1) (1995), 44-64.

[CL] Campbell, D. M. and R. J. Leach, *A survey of H^p Multipliers as Related to Classical Function Theory*, Complex Variables Theory Appl. **3** (1984), 85-111.

[D] Duren, P. L., *Theory of H^p spaces*, Academic Press, New York 1970.

[DS] Duren, P. L. and A. Shields, *Coefficient multipliers of H^p and B^p spaces*, Pacific J. Math. **32** (1970), 69-78.

[HL] Hardy, G. H. and J. E. Littlewood, *Theorems concerning mean values of analytic or harmonic functions*, Quart. J. Math., Oxford Ser., **8** (1941), 221-256.

[H] Hedlund, J. M., *Multipliers of H^p spaces*, J. Math. Mech. **18** (1969), 1067-1074.

[Hor] Horowitz, C. A., *Zeros of functions in the Bergman spaces*, Ph.D. Thesis, University of Michigan, 1974.

[K] Kellogg, C. N., *An extension of the Hausdorff-Young Theorem*, Michigan Math. J. **18** (1971), 121-127.

[MZ] MacGregor, T. and K. Zhu, *Coefficient multipliers between Hardy and Bergman spaces*, Mathematika **42** (1994), 523-533.

[MP] M.Mateljevic, M.Pavlovic, *Multipliers of H^p and $BMOA$*, Pacific J. Math. **146** (1990), 71-84.

[N] Nowak, M., *A note on coefficient multipliers $(H^p, \mathcal{B})$ and $(H^p, BMOA)$*, Topics in Complex Analysis, Banach Center Publications, vol. 31, Warszawa 1995, 299-302.

[Sle] Sledd, W. T., *On multipliers of H^p spaces*, Indiana Univ. Math. J. **27** (1978), 797-803.

[SZ] Stein, E. M. and A. Zygmund, *Boundedness of translation invariant operators on Hölder spaces and L^p spaces*, Ann. of Math. **85** (1967), 337-349.

[Str] Stroethoff, K., *Besov-type characterizations for the Bloch space*, Bull. Austral. Math. Soc. **39** (1989), 405–420.

[V] Vukotić, D., *On the coefficient multipliers of Bergman spaces*, J. London Math. Soc. **(2)50** (1994), 341-348.

[X] J.Xiao, J., *On multipliers of some analytic spaces*, Math. Japonica **41 No.2** (1995), 285-296.

[Zhu] K.Zhu, K., *Operator theory in function spaces*, Marcel Dekker, Inc., New York, 1990.

Instytut Matematyki UMCS
pl. Marii Curie-Skłodowskiej 1
20-031 Lublin , Poland
e-mail: nowakm@golem.umcs.lublin.pl

received December 20, 1997