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Two-point Comparisons Between Hyperbolic and Euclidean Geometry on Plane Regions

Dedicated to Professor Eligiusz Złotkiewicz on the occasion of his 60th birthday

ABSTRACT. Sharp two-point comparison theorems between hyperbolic and euclidean geometry on various classes of simply connected regions are known. We consider the problem of determining all hyperbolic regions in the complex plane for which similar comparisons exist. Let $\lambda_\Omega(w)|dw|$ denote the hyperbolic metric and $d_\Omega(A,B)$ the associated distance function on a hyperbolic region Ω. Two-point comparison theorems give upper and lower bounds on the euclidean distance $|A-B|$ of two points in Ω in terms of the hyperbolic distance $d_\Omega(A,B)$ and the values, $\lambda_\Omega(A)$ and $\lambda_\Omega(B)$, of the hyperbolic density at A and B. We demonstrate that two-point comparisons of a certain type exist if and only if the region is uniformly perfect. A region Ω is uniformly perfect if there is a constant $c=c(\Omega)>0$ such that $\lambda_\Omega \geq c/\delta_\Omega$, where $\delta_\Omega(w)$ denotes the euclidean distance from w to $\partial\Omega$.

1. Introduction. In the following Ω always denotes a hyperbolic region in the complex plane $\mathbb{C}$; that is, $\mathbb{C}\backslash\Omega$ contains at least two points.

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Let $\lambda_\Omega(w)|dw|$ denote the hyperbolic metric on Ω. It is defined by

$$\lambda_\Omega(f(z)) = \frac{1}{|D_1 f(z)|},$$

where $f : \mathbb{D} \to \Omega$ is any holomorphic universal covering projection ($\mathbb{D}$ denotes the unit disk) and $D_1 f(z) = (1-|z|^2)f'(z)$. This is independent of the choice of covering projection of $\mathbb{D}$ onto Ω. Note that f is a conformal mapping when Ω is simply connected. The hyperbolic distance on Ω is defined by

$$d_\Omega(A,B) = \inf \int_\gamma \lambda_\Omega(w)|dw|,$$

where the infimum is taken over all paths γ in Ω joining A and B. A path δ connecting A and B is called a hyperbolic geodesic arc if

$$d_\Omega(A,B) = \int_\delta \lambda_\Omega(w)|dw|.$$

Hyperbolic geodesic arcs always exist, but need not be unique unless Ω is simply connected. For the unit disk

$$d_\mathbb{D}(a,b) = \operatorname{artanh}\left|\frac{a-b}{1-\bar{a}b}\right|$$

and the unique hyperbolic geodesic arcs are parts of circles that are orthogonal to the unit circle. For hyperbolic regions hyperbolic geodesic arcs are the images of geodesic arcs in $\mathbb{D}$ under covering projections. For any covering projection $f : \mathbb{D} \to \Omega$, $d_\Omega(f(a), f(b)) \le d_\mathbb{D}(a,b)$. If Ω is simply connected, then $d_\Omega(f(a), f(b)) = d_\mathbb{D}(a,b)$, or f is an isometry. The quantity $\lambda_\Omega(w)$ is called the density of the hyperbolic metric. It is best viewed as the quotient of the hyperbolic metric $\lambda_\Omega(w)|dw|$ and the euclidean metric $1|dw|$. The density $\lambda_\Omega(w)$ means the infinitesimal distortion at w between hyperbolic and euclidean geometry at w.

Blatter [B] introduced a type of two-point distortion theorem for univalent functions; it is interesting that his distortion theorem characterized univalent functions. The work of Blatter was extended by Kim and Minda [KM] and then put into definitive form by Jenkins [J]. The result asserts that if f is univalent in $\mathbb{D}$, then for all $a, b \in \mathbb{D}$ and $p \ge 1$

$$\text{(1)} \quad \frac{\sinh(2d_\mathbb{D}(a,b))}{2[2\cosh(2pd_\mathbb{D}(a,b))]^{1/p}}\,[|D_1 f(a)|^p + |D_1 f(b)|^p]^{1/p} \le |f(a) - f(b)|.$$

Equality holds if and only if $f = S \circ K \circ T$, where S is any conformal automorphism of $\mathbb{C}$, $K(z) = z/(1-z)^2$ is the Koebe function and T is a conformal automorphism of $\mathbb{D}$ with $T(a), T(b) \in (-1,1)$. Conversely, if f is holomorphic on $\mathbb{D}$ and satisfies inequality (1) for some $p \geq 1$, then f is either univalent or constant. Jenkins' proof shows that (1) is not valid for univalent functions when $0 < p < 1$.

Kim and Minda [KM] observed that two-point distortion theorems yield comparison theorems between hyperbolic and euclidean geometry. Inequality (1) implies that if Ω is simply connected, then for $A, B \in \Omega$ and $p \geq 1$

$$(2) \qquad \frac{\sinh(2d_\Omega(A,B))}{2[2\cosh(2pd_\Omega(A,B))]^{1/p}}\left[\frac{1}{\lambda_\Omega^p(A)}+\frac{1}{\lambda_\Omega^p(B)}\right]^{1/p} \leq |A-B|.$$

Equality holds if and only if Ω is a slit plane and A, B lie on the extensions of the slit into Ω. Because inequality (1) characterizes univalent functions, it is plausible that inequality (2) would characterize simply connected regions. However, Kim and Minda showed that narrow annuli satisfy inequality (2) for all $p \geq 1$. At the same time they obtained two-point distortion theorems analogous to (1) which characterize convex univalent functions. For convex regions Ω this result gives

$$(3) \qquad \frac{\sinh(d_\Omega(A,B))}{[2\cosh(pd_\Omega(A,B))]^{1/p}}\left[\frac{1}{\lambda_\Omega^p(A)}+\frac{1}{\lambda_\Omega^p(B)}\right]^{1/p} \leq |A-B|$$

for all $A, B \in \Omega$ and $p \geq 1$. Equality holds if and only if Ω is a half-plane and the line through A, B is perpendicular to the edge of the half-plane. It is interesting that inequality (3) characterizes convex regions. Precisely, if a hyperbolic region Ω satisfies (3) for some $p \geq 1$, then Ω is convex. In inequalities (1), (2) and (3) the lower bound is a decreasing function of p and the limiting cases $p = \infty$ are invariant versions of classical growth theorems.

Upper bounds analogous to (2) and (3) follow from associated two-point distortion theorems ([MM$_3$], [MM$_6$]). If Ω is simply connected, then

$$(4) \qquad |A-B| \leq \frac{[2\cosh(2pd_\Omega(A,B))]^{1/p}\sinh(2d_\Omega(A,B))}{2[\lambda_\Omega^p(A)+\lambda_\Omega^p(B)]^{1/p}}$$

for $A, B \in \Omega$ and $p \geq 1$ with equality as in (2). If Ω is convex, then

$$(5) \qquad |A-B| \leq \frac{[2\cosh(pd_\Omega(A,B))]^{1/p}\sinh(d_\Omega(A,B))}{[\lambda_\Omega^p(A)+\lambda_\Omega^p(B)]^{1/p}}$$

for $A, B \in \Omega$ and $p \geq 1$ with equality as in (3). In (4) and (5) the upper bounds are increasing functions of p and the limiting cases $p = \infty$ are invariant versions of classical growth theorems. There are thin annuli that satisfy (4). The weakest inequality in (5), the case $p = \infty$, does characterize convex regions.

Jenkins [J] has given a two-point distortion theorem for univalent functions which gives rise to an inequality different from (4). For two-point distortion theorem in other contexts, see [FG], [MM_4] and [MM_5].

The preceding results suggest the problem of determining comparisons between hyperbolic and euclidean geometry analogous to (2), (3), (4) and (5) for other regions in $\mathbb{C}$. Except for annuli no results analogous to (2) - (5) are known for nonsimply connected regions.

To be precise, we say a region Ω satisfies a two-point comparison between hyperbolic and euclidean geometry if there exist constants $n > 0$ and $p \geq 1$ such that either

$$\text{(6)} \qquad \frac{\sinh(nd_\Omega(A,B))}{n[2\cosh(pnd_\Omega(A,B))]^{1/p}}\left[\frac{1}{\lambda_\Omega^p(A)}+\frac{1}{\lambda_\Omega^p(B)}\right]^{1/p} \leq |A-B|$$

or

$$\text{(7)} \qquad |A-B| \leq \frac{[2\cosh(pnd_\Omega(A,B))]^{1/p}\sinh(nd_\Omega(A,B))}{n[\lambda_\Omega^p(A)+\lambda_\Omega^p(B)]^{1/p}}$$

for all $A, B \in \Omega$. Our main result is that the two-point comparisons (6) or (7) characterize uniformly perfect regions; see the theorem in Section 3 for a precise statement. Recall that Ω is called uniformly perfect if

$$c(\Omega) = \inf\{\lambda_\Omega(w)\delta_\Omega(w) : w \in \Omega\} > 0,$$

where $\delta_\Omega(w) = \operatorname{dist}(w, \partial\Omega)$ is the euclidean distance from w to $\partial\Omega$. Uniformly perfect regions play an important role in function theory. The concept can be extended to Riemann surfaces; Sugawa [S] gives a nice, unified presentation of a number of different characterizations of this useful concept.

2. Preliminaries. In addition to the differential operator D_1, we will use the two differential operators D_2 and D_3 given by

$$\begin{aligned} D_2 f(z) &= (1-|z|^2)^2 f''(z) - 2\bar{z}(1-|z|^2)f'(z), \\ D_3 f(z) &= (1-|z|^2)^3 f'''(z) - 6\bar{z}(1-|z|^2)^2 f''(z) \\ &\quad + 6\bar{z}^2(1-|z|^2)f'(z). \end{aligned}$$

These differential operators satisfy the important invariance property that $|D_j(S \circ f \circ T)| = |D_j f| \circ T \ (j = 1, 2, 3)$ whenever S is a euclidean motion of $\mathbb{C}$ and T is a conformal automorphism of $\mathbb{D}$ [MM$_1$]. It is convenient to introduce for locally univalent holomorphic functions f the abbreviation

$$Q_f(z) = \frac{D_2 f(z)}{D_1 f(z)} = (1-|z|^2)\,\frac{f''(z)}{f'(z)} - 2\bar{z}.$$

We note that

$$\frac{D_3 f(z)}{D_1 f(z)} - \frac{3}{2}\left(\frac{D_2 f(z)}{D_1 f(z)}\right)^2 = (1-|z|^2)^2 S_f(z),$$

where

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is the Schwarzian derivative of f.

As we noted earlier there are very many ways to characterize uniformly perfect regions. We need one of these alternate characterizations. Consider the domain constant

$$\eta(\Omega) = \sup_{w\in\Omega} \frac{\left|\frac{\partial}{\partial w}\log\lambda_\Omega(w)\right|}{\lambda_\Omega(w)}.$$

The domain constants $c(\Omega)$ and $\eta(\Omega)$ are related by

$$\frac{1}{2c(\Omega)} \le \eta(\Omega) \le \frac{1}{c(\Omega)} - c(\Omega);$$

see [O] for the lower bound and [MM$_2$] for the upper bound. Thus, Ω is uniformly perfect if and only if $\eta(\Omega) < \infty$. The quantity $\eta(\Omega)$ can be expressed conveniently in terms of any covering projection $f : \mathbb{D} \to \Omega$; in fact [HM],

$$\eta(\Omega) = \frac{1}{2}\sup_{z\in\mathbb{D}} |Q_f(z)|.$$

The right-hand side of this equality is the linearly invariant order of f in the sense of Pommerenke [P].

The following result (or some equivalent version) has been used in the proofs of most two-point distortion theorems; see [MM$_3$] for a proof. med-skip

Lemma 1. *Suppose* $v \in C^2[-L, L], v > 0, |v'| \le kv, v'' \le k^2 v$ *and* $p \ge 1$. *Then*

(i) $$\frac{2[v(L)^p + v(-L)^p]^{1/p}\sinh(kL)}{k[2\cosh(pkL)]^{1/p}} \le \int_{-L}^{L} v(s)ds$$

and

(ii) $$\int_{-L}^{L} \frac{ds}{v(s)} \le \frac{2[2\cosh(pkL)]^{1/p}\sinh(kL)}{k[v(L)^p + v(-L)^p]^{1/p}}.$$

Equality holds in (i) *and* (ii) *if* $v(s) = Ae^{\pm ks}$, *where* $A > 0$.

3. Main result. We now show that two-point comparisons between hyperbolic and euclidean geometry characterize uniformly perfect regions.

Theorem 1. *Let* Ω *be a hyperbolic plane region.*

(i) *If* Ω *is uniformly perfect, then the two-point comparisons* (6) *(with* $n = \sqrt{2\eta^2(\Omega) - \frac{1}{2}}$*) and* (7) *(with* $n = \sqrt{\frac{1}{2} + \eta^2(\Omega)}$*) hold for all* $p \ge 1$.

(ii) *If either* (6) *or* (7) *holds for some* $n > 0$ *and some* $p \ge 1$, *then* Ω *is uniformly perfect with* $\eta(\Omega) \le 2n - \frac{1}{2n}$ *when* (6) *holds and*

$$\eta(\Omega) \le \sqrt{2n^2 + \frac{2(n^2-1)}{3p}} \le \sqrt{\frac{8n^2-2}{3}} < 1.633n.$$

when (7) *holds.*

Proof. (i) We first establish (7) for all $p \ge 1$ with $n = \sqrt{\frac{1}{2} + \eta^2(\Omega)}$. Fix $A, B \in \Omega$ and let Γ be a hyperbolic geodesic arc joining A to B. Let $f : \mathbb{D} \to \Omega$ be a holomorphic covering projection. Then there exist $a, b \in \mathbb{D}$ and a hyperbolic geodesic arc γ from a to b with $f(a) = A$, $f(b) = B$ and $f \circ \gamma = \Gamma$. Note that $d_{\mathbb{D}}(a,b) = d_\Omega(A,B)$. Let $\gamma : z = z(s)$, $-L \le s \le L$, be a hyperbolic arclength parametrization of γ. This means that $2L$ is the hyperbolic length of γ which is $d_{\mathbb{D}}(a,b)$ and $z'(s) = (1-|z(s)|^2)e^{i\theta(s)}$, where $e^{i\theta(s)}$ is a unit tangent vector for γ at $z(s)$. Set $v(s) = |D_1 f(z(s))|^{-1}$. Then as in [MM$_3$] $v'(s) = -v(s)\,\mathrm{Re}\{e^{i\theta(s)}Q_f(z(s))\}$ and

$$\begin{aligned} v''(s) &= v(s)\left[\frac{1}{2}|Q_f(z(s))|^2 - \mathrm{Re}\{e^{2i\theta(s)}(1-|z(s)|^2))^2 S_f(z(s))\} + 2\right] \\ &\le v(s)\left[\frac{1}{2}|Q_f(z(s))|^2 + (1-|z(s)|^2)^2|S_f(z(s))| + 2\right]. \end{aligned}$$

Set $\eta = \eta(\Omega)$. Since Ω is uniformly perfect,

(8) $$|Q_f(z)| \le 2\eta$$

so that $|v'| \le 2\eta v$.

In [H] it is shown that (8) implies

$$
\begin{aligned}
\frac{1}{2}|Q_f(z)|^2 + (1-|z|^2)^2|S_f(z)| &\le 2\left[\eta^2 + \left(\eta + \frac{1}{2\eta}\right)\sqrt{\eta^2-1}\right] \\
&= 2\left[\eta^2 + \left(\eta^2 + \frac{1}{2}\right)\sqrt{1-\frac{1}{\eta^2}}\right] \\
&< 2\left[\eta^2 + \left(\eta^2 + \frac{1}{2}\right)\left(1-\frac{1}{2\eta^2}\right)\right] \\
&= 2\left(2\eta^2 - \frac{1}{4\eta^2}\right) < 4\eta^2.
\end{aligned} \tag{9}
$$

Therefore, $v'' \le 4\left(\frac{1}{2}+\eta^2\right)v$. This shows that for $k = 2\sqrt{\frac{1}{2}+\eta^2} = 2n$ the hypotheses of the lemma are satisfied. Then part (ii) of the lemma gives

$$
\begin{aligned}
|A-B| = |f(a)-f(b)| &\le \int_\Gamma |dw| = \int_\gamma |f'(z)||dz| \\
&= \int_{-L}^{L} |f'(z(s))|(1-|z(s)|^2)ds = \int_{-L}^{L} |D_1 f(z(s))|ds \\
&= \int_{-L}^{L} \frac{ds}{v(s)} \le \frac{2[2\cosh(2pnL)]^{1/p}\sinh(2nL)}{2n\left[\frac{1}{|D_1 f(a)|^p} + \frac{1}{|D_1 f(b)|^p}\right]^{1/p}} \\
&= \frac{[2\cosh(pnd_\Omega(A,B))]^{1/p}\sinh(nd_\Omega(A,B))}{n[\lambda_\Omega^p(a) + \lambda_\Omega(B)]^{1/p}}.
\end{aligned}
$$

Next, we show that (6) holds for all $p \ge 1$ with $n = \sqrt{2}\eta(\Omega)$. Initially, we suppose $A, B \in \Omega$ and the closed euclidean line segment $\Gamma = [A,B]$ is contained in Ω. Let $f : \mathbb{D} \to \Omega$ be a holomorphic covering projection. Then there is a path γ in $\mathbb{D}$, a lift of Γ, with $f \circ \gamma = \Gamma$. If a is the initial point of γ and b the terminal point, then $f(a) = A$, $f(b) = B$.

Let $\gamma : z = z(s)$, $-L \le s \le L$, be a hyperbolic arclength parametrization of γ. Since $2L$ is the hyperbolic length of γ, $2L \ge d_{\mathbb{D}}(a,b) \ge d_\Omega(A,B)$. In this case we choose $v(s) = |D_1 f(z(s))|$. Just as in [MM$_3$] we obtain $v'(s) = v(s)\operatorname{Re}\{e^{i\theta(s)}Q_f(z(s))\}$ and

$$
\begin{aligned}
v''(s) = v(s)\Bigg[&|Q_f(z(s))|^2 \\
&+ \operatorname{Re}\left\{e^{2i\theta(s)}\left[\frac{D_3 f(z(s))}{D_1 f(z(s))} - \left(\frac{D_2 f(z(s))}{D_1 f(z(s))}\right)^2\right]\right\} - 2\Bigg] \\
\le v(s)&\left[\frac{3}{2}|Q_f(z(s))|^2 + (1-|z(s)|^2)^2|S_f(z(s))| - 2\right].
\end{aligned}
$$

From $|Q_f(z)| \leq 2\eta$, where $\eta = \eta(\Omega)$, we obtain $|v'| \leq 2\eta v$. By making use of (8) and (9) we find $v'' \leq (8\eta^2 - 2)v$. In this case the hypotheses of the lemma are satisfied with $k = 2\sqrt{2\eta^2 - \frac{1}{2}} = 2n$ and part (i) of the lemma produces

$$\begin{aligned}
|A - B| = |f(a) - f(b)| &= \int_\Gamma |dw| = \int_\gamma |f'(z)||dz| \\
&= \int_{-L}^{L} |D_1 f(z(s))| ds = \int_{-L}^{L} v(s) ds \\
&\geq \frac{2\sinh(2nL)}{2n[2\cosh(2pnL)]^{1/p}} [|D_1 f(a)|^p + |D_1 f(b)|^p]^{1/p} \\
&= \frac{\sinh(2nL)}{n[2\cosh(2pnL)]^{1/p}} \left[\frac{1}{\lambda_\Omega^p(A)} + \frac{1}{\lambda_\Omega^p(B)}\right]^{1/p}.
\end{aligned}$$

Since $2L \geq d_\Omega(A, B)$ and the function $h(t) = \sinh(t)/[2\cosh(pt)]^{1/p}$ is increasing for $t > 0$ and $p \geq 1$, we conclude that (6) holds when $[A, B] \subset \Omega$.

Now, we show that (6) holds when $[A, B]$ is not contained in Ω. In this circumstance we can find $\alpha, \beta \in \partial\Omega$ such that the half-open segments $[A, \alpha), (\beta, B] \subset \Omega$ and $[A, \alpha) \cup (\beta, B] \subset [A, B]$. Choose $C \in [A, \alpha)$. Then $[A, C] \subset \Omega$ and the validity of (6) in this case gives

$$|A - C| \geq \frac{\sinh(nd_\Omega(A, C))}{n[2\cosh(pnd_\Omega(A, C)]^{1/p}} \left[\frac{1}{\lambda_\Omega^p(A)} + \frac{1}{\lambda_\Omega^p(C)}\right]^{1/p}.$$

As $C \to \alpha$ along $[A, \alpha)$, $\lambda_\Omega(C) \to \infty$ and $d_\Omega(A, C) \to \infty$ because the hyperbolic distance is complete. Since $h(t) \to 1/2$ as $t \to \infty$, we find that

$$(10) \qquad |A - \alpha| \geq \frac{1}{2n\lambda_\Omega(A)}$$

by letting $C \to \alpha$. Similarly,

$$|\beta - B| \geq \frac{1}{2n\lambda_\Omega(B)},$$

so

$$|A - B| \geq |A - \alpha| + |\beta - B| \geq \frac{1}{2n}\left[\frac{1}{\lambda_\Omega(A)} + \frac{1}{\lambda_\Omega(B)}\right].$$

Since $h(t)$ increases to $1/2$ as t tends to infinity,

$$|A - B| \geq \frac{\sinh(nd_\Omega(A, B))}{n[2\cosh(pnd_\Omega(A, B))]^{1/p}} \left[\frac{1}{\lambda_\Omega(A)} + \frac{1}{\lambda_\Omega(B)}\right].$$

As

$$\left[\frac{1}{\lambda_\Omega(A)}+\frac{1}{\lambda_\Omega(B)}\right]^p \geq \frac{1}{\lambda_\Omega^p(A)}+\frac{1}{\lambda_\Omega^p(B)}$$

for $p \geq 1$, this establishes (6) when $[A, B]$ is not contained in Ω.

(ii) Initially, we suppose (6) holds for some $n > 0$ and $p \geq 1$. Fix $A \in \Omega$. Choose $\alpha \in \partial\Omega$ with $|A-\alpha| = \delta_\Omega(A)$. By using the same method that led to (10), we deduce that

$$\delta_\Omega(A) = |A-\alpha| \geq \frac{1}{2n\lambda_\Omega(A)},$$

so

$$\lambda_\Omega(A)\delta_\Omega(A) \geq \frac{1}{2n}.$$

Hence, $c(\Omega) \geq 1/(2n) > 0$ and Ω is uniformly perfect. We remark that $c(\Omega) \leq 1/2$ ([Hi], [HM]) which implies that we must actually have $n \geq 1$ in any two-point comparison inequality (6). Since $\eta(\Omega) \leq \frac{1}{c(\Omega)} - c(\Omega)$, we have $\eta(\Omega) \leq 2n - \frac{1}{2n}$.

All that remains is to prove that if (7) holds for some $n > 0$ and some $p \geq 1$, then Ω is uniformly perfect. Fix $A \in \Omega$. In order to simplify subsequent calculation, we assume $A = 0$. This is no problem since (7) is invariant under translations. Let $f : \mathbb{D} \to \Omega$ be the holomorphic covering projection with $f(0) = 0$ and $f'(0) > 0$. Since inequality (7) is invariant under stretchings ($z \mapsto rz$, $r > 0$) of Ω, we may suppose $f'(0) = 1$. Because $d_\mathbb{D}(0, z) = d_\Omega(0, f(z))$ for all z sufficiently near 0, inequality (7) yields

$$(11) \qquad |f(z)| \leq \frac{[2\cosh(pnd_\mathbb{D}(0,z)]^{1/p}\sinh(nd_\mathbb{D}(0,z))}{n\left[1+\frac{1}{(1-|z|^2)^p|f'(z)|^p}\right]^{1/p}}$$

for all z near the origin. We use this inequality to obtain information about the coefficients in the Taylor series of f about the origin. If

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots,$$

then we obtain

$$|f(z)| = |z|[1 + \mathrm{Re}\{a_2z\} + \frac{1}{4}|a_2|^2|z|^2 + \mathrm{Re}\{(a_3 - a_2^2/4)z^2\} + o(|z|^2).$$

Also,

$$[2\cosh(pnd_\mathbb{D}(0,z))]^{1/p} = 2^{1/p}\left[1+\frac{n^2p}{2}|z|^2 + o(|z|^3)\right],$$

$$\sinh(nd_\mathbb{D}(0,z)) = n\left[|z| + \left(\frac{1}{3}+\frac{n^2}{6}\right)|z|^3 + o(|z|^3)\right],$$

$$\frac{1}{\left[1+\frac{1}{(1-|z|^2)^p|f'(z)|^p}\right]^{1/p}} = 2^{-1/p}\Bigg[1+\mathrm{Re}\{a_2z\}+\left(\frac{1-p}{4}|a_2|^2-\frac{1}{2}\right)|z|^2 + \mathrm{Re}\left\{\left(\frac{3}{2}a_3-\frac{p+3}{4}a_2^2\right)z^2\right\}+o(|z|^2)\Bigg]$$

so that

$$\frac{2\cosh(pnd_{\mathbb{D}}(0,z))]^{1/p}\sinh(nd_{\mathbb{D}}(0,z))}{n\left[1+\frac{1}{(1-|z|^2)^p|f'(z)|^p}\right]^{1/p}} = |z|\Bigg[1+\mathrm{Re}\{a_2z\} + \left(\frac{n^2p}{2}+\frac{n^2-1}{6}+\frac{1-p}{4}|a_2|^2\right)|z|^2 + \mathrm{Re}\left\{\left(\frac{3}{2}a_3-\frac{p+3}{4}a_2^2\right)z^2\right\}+o(|z|^2)\Bigg].$$

Thus, (11) yields

$$\mathrm{Re}\left\{\left(-\frac{1}{2}a_3+\frac{p+2}{4}a_2^2\right)z^2\right\} \le \left(\frac{n^2p}{2}+\frac{n^2-1}{6}-\frac{p}{4}|a_2|^2\right)|z|^2+o(|z|^2).$$

From this we may conclude that

$$\left|a_3-\frac{p+2}{2}a_2^2\right|+\frac{p}{2}|a_2|^2 \le n^2p+\frac{n^2-1}{3}.$$

In particular, as $p \ge 1$, $\frac{1}{4}|Q_f(0)|^2 = |a_2|^2 \le 2n^2+\frac{2(n^2-1)}{3p}$, or

$$\frac{1}{2}|Q_f(0)| \le \sqrt{2n^2+\frac{2(n^2-1)}{3p}}.$$

This is equivalent to

$$\frac{\left|\frac{\partial \log \lambda_\Omega}{\partial w}(A)\right|}{\lambda_\Omega(A)} \le \sqrt{2n^2+\frac{2(n^2-1)}{3p}}.$$

Because $A \in \Omega$ is arbitrary, we deduce $\eta(\Omega) \le \sqrt{2n^2+\frac{2(n^2-1)}{3p}}$. This proves that Ω is uniformly perfect.

4. Concluding remarks. Are there classes of uniformly perfect regions for which the best possible constant n in (6) and (7) can be determined? There are several classes of regions for which the answer is known. For convex regions, $n = 1$ ([KM], [MM3]). More generally, $n = 1 + \alpha$ if Ω is strongly close-to-convex of order $\alpha \in (0,1]$ [MM3]. For simply connected regions $n = 2$ is optimal ([B], [KM], [J], [MM6]). In each of these cases n is the maximal value of $\eta(\Omega)$ over the class of regions. Are there other classes of regions for which a similar result holds?

In the special case ($p = \infty$) of formulas (6) and (7) we can determine the optimal value of n. For $p = \infty$ formulas (6) and (7) become

$$\frac{1 - \exp(-2nd_\Omega(A,B))}{2n \min\{\lambda_\Omega(A), \lambda_\Omega(B)\}} \le |A - B| \tag{12}$$

and

$$|A - B| \le \frac{\exp(2nd_\Omega(A,B)) - 1}{2n \max\{\lambda_\Omega(A), \lambda_\Omega(B)\}}, \tag{13}$$

respectively. Below we show that the best choice of n in both cases is $\eta(\Omega)$.

Theorem 2. *Let Ω be a hyperbolic plane region.*

(i) If Ω is uniformly perfect, then the two-point comparisons (12) *and* (13) *both hold with $n = \eta(\Omega)$.*

(ii) If either (12) *or* (13) *holds, then $n \ge \eta(\Omega)$.*

Proof. (i) We first establish (13) with $n = \eta(\Omega)$. Fix $A \in \Omega$. Let $f : \mathbb{D} \to \Omega$ be the holomorphic covering projection with $f(0) = A$ and $f'(0) > 0$. Then $f'(0) = 1/\lambda_\Omega(A)$, $g(z) = [f(z) - A]/f'(0)$ is a normalized ($g(0) = 0, g'(0) = 1$) locally univalent function on $\mathbb{D}$ and $|Q_f(z)| = |Q_g(z)| \le 2\eta(\Omega)$. In fact, g is linearly invariant with order $\eta = \eta(\Omega)$. The growth theorem for linearly invariant functions [P] gives

$$|g(z)| \le \frac{1}{2\eta}\left[\left(\frac{1+|z|}{1-|z|}\right)^\eta - 1\right],$$

or

$$|f(z) - A| \le \frac{\exp(2\eta d_\mathbb{D}(0,z)) - 1}{2\eta\lambda_\Omega(A)}.$$

Given $B \in \Omega$ we can determine $b \in \mathbb{D}$ with $f(b) = B$ and $d_\mathbb{D}(0,b) = d_\Omega(A,B)$. For $z = b$ we obtain

$$|A - B| \le \frac{\exp(2\eta d_\Omega(A,B)) - 1}{2\eta\lambda_\Omega(A)}.$$

This holds for all $A, B \in \Omega$. If we interchange the roles of A and B, then we get

$$|A-B| \leq \frac{\exp(2\eta d_\Omega(A,B))-1}{2\eta\lambda_\Omega(B)}.$$

The two preceding inequalities prove that (13) holds with $n = \eta(\Omega)$.

Next, we establish (12) with $n = \eta(\Omega)$. We first establish (12) under the assumption that the line segment $[A, B]$ is contained in Ω. We proceed as above, so let f and g be as there and choose $b \in \mathbb{D}$ with $d_{\mathbb{D}}(0,b) = d_\Omega(A,B)$. Then Satz 1.1 of [P] yields

$$\frac{1}{2\eta}\left[1-\left(\frac{1+|b|}{1-|b|}\right)^{\eta}\right] \leq |g(b)|$$

since the euclidean segment from $g(0) = 0$ to $g(b)$ lies in the Riemann image surface of g. This results in

$$\frac{1-\exp(-2\eta d_\Omega(A,B))}{2\eta\lambda_\Omega(A)} \leq |A-B|.$$

Since the similar inequality holds when the roles of A and B are reversed, we have proved (12) in the special case that $[A, B] \subset \Omega$.

Now we turn to the case when $[A, B]$ is not contained in Ω. Then we may select $\alpha, \beta \in \partial\Omega$ such that $[A, \alpha) \cup (\beta, B] \subset [A, B] \cap \Omega$. Then

$$|A-\alpha| \geq \delta_\Omega(A) \geq \frac{1}{2\eta\lambda_\Omega(A)},$$

and

$$|B-\alpha| \geq \delta_\Omega(B) \geq \frac{1}{2\eta\lambda_\Omega(B)}$$

since $c(\Omega) \geq 1/(2\eta(\Omega))$[0]. This produces

$$\begin{aligned}|A-B| \geq |A-\alpha| + |\beta - B| &\geq \frac{1}{2\eta}\left[\frac{1}{\lambda_\Omega(A)}+\frac{1}{\lambda_\Omega(B)}\right] \\ &\geq \frac{1-\exp(-2\eta d_\Omega(A,B))}{2\eta\max\{\lambda_\Omega(A),\lambda_\Omega(B)\}},\end{aligned}$$

so (12) holds in general for $n = \eta(\Omega)$.

(ii) We must show that if either (12) or (13) holds, then $n \geq \eta(\Omega)$. Fix $A \in \Omega$; there is no harm in assuming $A = 0$. Let $f : \mathbb{D} \to \Omega$ be the covering projection with $f(0) = A$ and $f'(0) > 0$. As in the proof of part (ii) of

Theorem 1, we may even assume that $f'(0) = 1$. Say $f(z) = z + a_2 z^2 + \cdots$, so

$$|f(z)| = |z|[1 + \mathrm{Re}\{a_2 z\} + o(|z|)]$$

and $\lambda_\Omega(A) = 1$. Also, for all z sufficiently near the origin, $d_{\mathbb{D}}(0,z) = d_\Omega(A, f(z))$. For z near the origin, inequality (12) gives

$$|f(z)| \geq \frac{1}{2n}[1 - \exp(-2nd_{\mathbb{D}}(0,z))] = \frac{1}{2n}\left[1 - \left(\frac{1-|z|}{1+|z|}\right)^n\right]$$
$$= |z|[1 - n|z| + o(|z|)]$$

Then $1 + \mathrm{Re}\{a_2 z\} + o(|z|) \geq 1 - n|z| + o(|z|)$ so $\mathrm{Re}\{a_2 e^{i\theta}\} + o(1) \geq -n$ for all $\theta \in \mathbb{R}$. This yields $-|a_2| \geq -n$, or $|a_2| \leq n$. Thus,

$$\frac{1}{2}|Q_f(0)| = |a_2| \leq n$$

so $|Q_f(z)| \leq 2n$. This implies that $\eta(\Omega) \leq \eta$. Similarly, (13) yields

$$|f(z)| \leq \frac{1}{2n}[\exp(2nd_{\mathbb{D}}(0, f(z))) - 1] = \frac{1}{2n}\left[\left(\frac{1+|z|}{1-|z|}\right)^n - 1\right]$$
$$= |z|[1 + n|z| + o(|z|)].$$

Then

$$1 + \mathrm{Re}\{a_2 z\} + o(|z|) \leq 1 + n|z| + o(|z|),$$

which gives $|a_2| \leq n$. This shows that $|Q_f(z)| \leq 2n$, so $\eta(\Omega) \leq n$.

We can obtain general lower bounds on n from Theorem 1(ii). If (6) holds, then $\eta(\Omega) \leq 2n - \frac{1}{2n}$ which gives

$$\frac{1}{4}\left[\eta^2(\Omega) + \sqrt{\eta^2(\Omega) + 4}\right] \leq n.$$

Similarly, if (7) holds, then

$$\sqrt{\frac{3\eta^2(\Omega) + 2}{8}} \leq n.$$

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