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Fuchsian Groups Isomorphism, Conformally Natural Quasiconformal Extension and Harmonic Mappings

To Eli on the occasion of 60-th birthday

ABSTRACT. This paper is a slightly extended version of a talk given by the author at the XVII-th Nevanlinna Colloquium (EPFL Lausanne, August 14-20, 1997). The characterization of conformally natural quasiconformal extension of a quasisymmetric automorphism is given and its role in retrieving the isomorphism of special Fuchsian groups is presented. Moreover, another method for solution of this problem based on harmonic maps is proposed.

1. Introduction. According to the usual definition a Fuchsian group is a discontinuous group G of Möbius transformations with an invariant disk D, i.e., $g(D) = D$ for any $g \in G$. However, the theory of Riemann surfaces deals with more special Fuchsian groups.

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A Riemann surface W is defined as a topological surface S endowed with a conformal structure. By the uniformization theorem (cf. e.g. [7]) W is conformally equivalent to the quotient surface Ω/G, where $\Omega \subset \widehat{\mathbb{C}}$ is the universal covering surface of S and G is the covering group of conformal self-mappings of Ω. We may take as Ω one of the following standard domains: the unit disk Δ, the finite plane $\mathbb{C}$, or the extended plane $\widehat{\mathbb{C}}$. The case $\Omega = \Delta$ is most important and then G is a discontinuous, fixed-point free subgroup of the group $\mathfrak{M}$ of all Möbius self-mappings of Δ.

Any $g \in \mathfrak{M}$ has the form $g(z) = e^{i\alpha}(z-a)/(1-\overline{a}z)$, $0 \leq \alpha < 2\pi$, $|a| < 1$, and we may distinguish three cases:

(p) g has one fixed point $\zeta \in \mathbb{T} = \partial\Delta$ (parabolic case characterized by $|a| = \sin\alpha/2 > 0$);

(h) g has two different fixed points $\zeta_1, \zeta_2 \in \mathbb{T}$ (hyperbolic case characterized by $|a| > \sin\alpha/2$);

(e) g has one fixed point $\zeta \in \Delta$ (elliptic case characterized by $|a| < \sin\alpha/2$).

The subclasses of $\mathfrak{M}$ consisting of g satisfying one of the above given conditions may be denoted by $\mathfrak{M}_p$, $\mathfrak{M}_h$ and $\mathfrak{M}_e$, respectively. Obviously any fixed-point free Fuchsian group G satisfies $G \subset \{\mathrm{id}\} \cup \mathfrak{M}_p \cup \mathfrak{M}_h$, where id is the identity mapping. If $g \in \mathfrak{M}_h$ then the circular arc joining in Δ fixed points ζ_1, ζ_2 of g and orthogonal to $\mathbb{T}$ is said to be *the axis of* g.

In what follows we deal with isomorphisms of special Fuchsian groups G of the first kind, so G is supposed to satisfy the following conditions:

(i) G is *fixed-point free*, i.e., $g(\zeta) = \zeta$ and $g \neq \mathrm{id}$ implies $|\zeta| = 1$;

(ii) G is *discontinuous*, i.e., each $z \in \Delta$ has a neighbourhood N_z which does not contain any pair of points equivalent under G;

(iii) G is *of the first kind*, i.e. fixed points of $g \in G$ are dense on $\mathbb{T}$.

Isomorphism of Fuchsian groups was investigated by several authors. A. Marden [9] dealt with finitely generated Fuchsian groups with elliptic elements admitted. We follow here Lehto [6] and Tukia [12] who treated the isomorphism of special Fuchsian groups.

Then we have

Theorem A. *Suppose θ is an isomorphism between Fuchsian groups G, $\widetilde{G}$ which satisfy* (i) - (iii). *Suppose $\theta(g) \in \widetilde{G}$ is parabolic if and only if $g \in G$ is. Then θ generates a mapping from the set X of fixed points of G onto the set $\widetilde{X}$ of fixed points of $\widetilde{G}$. This mapping can be extended to a homeomorphism γ of $\mathbb{T}$ if and only if the following axis condition is satisfied: $g_1, g_2 \in G$ have intersecting axes if and only if $\theta(g_1), \theta(g_2)$ do. The homeomorphism $\gamma : \mathbb{T} \to \mathbb{T}$ is said to be the boundary homeomorphism of the isomorphism θ. It satisfies the condition*

$$\gamma \circ g = \theta(g) \circ \gamma \quad \text{on } \mathbb{T} \text{ for any } g \in G. \tag{1.1}$$

Corollary. *We may assume that γ is sense-preserving.*

In fact, γ is generated by the correspondence between fixed points ζ_1, ζ_2 of $g \in G$ and fixed points τ_1, τ_2 of $\tilde{g} \in \theta(g)$. If $\gamma(\zeta_1) = \tau_1$ then by (1.1) $\gamma \circ g(\zeta_1) = \gamma(\zeta_1) = \tau_1 = \tilde{g}(\tau_1) = \tilde{g} \circ \gamma(\zeta_1)$. Assume now that $\gamma^*(\zeta_1) = \tau_2$, $\gamma^*(\zeta_2) = \tau_1$. This corresponds to changing the orientation of γ but leaving the sets X, $\widetilde{X}$ unchanged. Consequently, we obtain another boundary homeomorphism γ^* satisfying

$$\gamma^* \circ g(\zeta_1) = g(\zeta_1) = \gamma^*(\zeta_1) = \tau_2 = \tilde{g}(\tau_2) = \tilde{g} \circ \gamma^*(\zeta_1),$$

i.e., $\gamma^* \circ g = \tilde{g} \circ \gamma^*$ which also satisfies (1.1). If γ is sense-reversing then γ^* is sense-preserving.

An automorphism (i.e., a sense-preserving homeomorphism) γ of $\mathbb{T}$ is said to be *compatible with the group* $G \subset \mathfrak{M}$ iff for any $g \in G$ there exists $\tilde{g} \in \mathfrak{M}$ such that

$$\tilde{g}|\,\mathbb{T} = \gamma \circ g \circ \gamma^{-1}. \tag{1.2}$$

If γ is quasisymmetric on $\mathbb{T}$ and compatible with a group G then an isomorphic group $\widetilde{G}$ can be determined explicitly in terms of a conformally natural quasiconformal extension of γ to the unit disk. This notion due to Tukia [13], as well as its application, will be treated in the next section. In the last section another solution of (1.2), without quasisymmetry assumption, will be presented.

2. Conformally natural quasiconformal extension.

An automorphism of $\mathbb{T}$ given by the equation $\gamma(e^{it}) = \exp i\varphi(t)$, $t \in \mathbb{R}$, is said to be M-quasisymmetric on $\mathbb{T}$ if the function $\varphi(t) - t = \sigma(t)$ has a continuous, 2π-periodic extension on $\mathbb{R}$, where $\varphi(t) = t + \sigma(t)$ is strictly increasing and satisfies the familiar M-condition of Beurling-Ahlfors:

$$M^{-1} \le [\varphi(t+h) - \varphi(t)][\varphi(t) - \varphi(t-h)]^{-1} \le M, \quad h, t \in \mathbb{R},\ h \ne 0, \tag{2.1}$$

cf. [1], [4]. Then we write $\gamma \in QS(M)$ and set $QS = \bigcup_{M \ge 1} QS(M)$. Note that with this definition of quasisymmetry no point on $\mathbb{T}$ is distinguished. The condition $\gamma \in QS$ is necessary and sufficient for γ to have a quasiconformal (qc. for short) extension to Δ, cf. [1], [4], [8].

Definition. A qc. automorphism w of Δ is said to be a conformally natural qc. extension to Δ (CNQE for short) of its boundary values γ

iff for any $\lambda, \nu \in \mathfrak{M}$ the qc. automorphism $\lambda \circ w \circ \nu$ of Δ has boundary values $\lambda \circ \gamma \circ \nu$.

In what follows we show that any $\gamma \in QS$ has a CNQE and give a geometrical construction covering all CNQE-s.

Suppose that Γ is a quasicircle in the finite plane and $D, D^* \ni \infty$, are components of $\hat{\mathbb{C}} \setminus \Gamma$. Moreover, suppose that f^{-1}, F map D, D^* conformally onto Δ, $\Delta^* = \hat{\mathbb{C}} \setminus \overline{\Delta}$, respectively. It is well-known that f and F can be extended as homeomorphisms to the closures of corresponding domains and $F \circ f = \gamma \in QS$. According to the sewing theorem for conformal mappings, cf. [8], [10], the converse statement is also true: For any $\gamma \in QS$ there exists a quasicircle Γ in the finite plane and conformal mappings f, F as above, such that $\gamma = F \circ f$. The quasicircle Γ and conformal mappings f, F may be called as associated with $\gamma \in QS$.

In this notation we have

Proposition 1. *Suppose that for $\gamma \in QS$ the quasicircle Γ and the conformal mappings f, F are associated with γ. If $S(z) = 1/\bar{z}$ and J is an arbitrary qc. reflection in Γ then*

$$w = S \circ F \circ J \circ f \tag{2.2}$$

is a CNQE of γ.

Proof. Obviously w, as given by formula (2.2), is a qc. self-mapping of Δ with boundary values $F \circ f = \gamma$. Consider now conformal mappings $f_1 = f \circ \nu$, $F_1 = \lambda \circ F$, where $\lambda, \nu \in \mathfrak{M}$. Obviously f_1 and F_1^{-1} map Δ and Δ^* conformally onto D and D^*, resp. Hence F_1, f_1, as well as Γ, are associated with $\gamma_1 \in QS$, where

$$\gamma_1 = F_1 \circ f_1 = \lambda \circ F \circ f \circ \nu = \lambda \circ \gamma \circ \nu. \tag{2.3}$$

According to (2.2) the qc. extension w_1 of γ_1 to Δ generated by J has the form

$$\begin{aligned} w_1 &= S \circ F_1 \circ J \circ f_1 = S \circ \lambda \circ (F \circ J \circ f) \circ \nu \\ &= \lambda \circ (S \circ F \circ J \circ f) \circ \nu = \lambda \circ w \circ \nu , \end{aligned}$$

due to the identity $\lambda \circ S = S \circ \lambda$. This ends the proof.

The converse statement is also true and it may be expressed as

Proposition 2. *If w is a CNQE of $\gamma \in QS$ then there exist a quasicircle Γ and a qc. reflection J in Γ such that w satisfies (2.2), where f, F are conformal mappings associated with γ.*

Proof. Given w being a CNQE of γ find a quasicircle Γ and conformal mappings f, F associated with γ. Then J can be evaluated from (2.2) as a mapping

$$D \to D^* : J = F^{-1} \circ S \circ w \circ f^{-1}.$$

Inserting this value into (2.2) we obtain the desired representation of w.

Note that CNQE of a given γ is not unique. As shown by Kühnau [5], there are infinitely many extremal qc. reflections for some pretty regular quasicircles. This holds e.g. for γ being the image curve of the circle $\{z : |z-1| = 1\}$ under the mapping $z \to z^\alpha$, $0 < \alpha < 1$.

Following Tukia [13] we now present a nice application of CNQE of $\gamma \in QS$ to isomorphisms of Fuchsian groups.

Proposition 3. *Let G be a Fuchsian group satisfying the assumptions of* Theorem A *and let the boundary homeomorphism $\gamma \in QS$ be compatible with G. If w is a* CNQE *of γ then*

$$\theta(g) = w \circ g \circ w^{-1}, \; g \in G, \tag{2.4}$$

represents an isomorphism of G.

Proof. Suppose that (1.1) holds on $\mathbb{T}$. Then we have

$$\theta(g) \circ \gamma \circ g^{-1} = \gamma \quad \text{on } \mathbb{T} \tag{2.5}$$

with $\theta(g), g^{-1} \in \mathfrak{M}$. If w is a CNQE of γ then, by Definition of CNQE (with $\lambda = \theta(g)$, $g^{-1} = \nu$), we obtain

$$\theta(g) \circ w \circ g^{-1} = w \quad \text{on } \Delta \tag{2.6}$$

and hence (2.4) readily follows.

Tukia was first to realize the importance of CNQE, however, his construction was fairly complicated, cf. [7; p. 194]. Another, more simple and explicit construction of a CNQE was proposed by Douady and Earle [3]. As observed by D. Partyka [11], the inverse function $F(\gamma, z)$ to the Douady-Earle extension can be expressed in a relatively simple way more suitable for numerical treatment. To this end, set for $z \in \Delta$, $h_z(\zeta) = (\zeta - z)/(1 - \overline{z}\zeta)$. Evidently, for a fixed $\gamma \in QS$ also $h_z \circ \gamma \in QS$ and by Radó-Kneser-Choquet theorem (cf. e.g. [2]) the Poisson extension of $h_z \circ \gamma$ to Δ, i.e.,

$w \to P[h_z \circ \gamma](w)$ is a univalent, harmonic self-mapping of Δ. Consequently, $P[h_z \circ \gamma](w) = 0$ has a unique solution $w = \Phi(z) := F(\gamma, z)$. Since in (2.4) both w and w^{-1} appear, $F(\gamma, z)$ may be as useful as CNQE of γ in some cases.

3. Boundary homeomorphism without quasisymmetry. Suppose that G is a special Fuchsian group satisfying the assumptions of Theorem A and the boundary homeomorphism γ of $\mathbb{T}$ is compatible with G. This means that for any $g \in G$ there exists $\tilde{g} \in \mathfrak{M}$ such that

$$\tilde{g}|\mathbb{T} = \gamma \circ g \circ \gamma^{-1}. \tag{3.1}$$

It follows from (3.1) that $\gamma \circ g \circ \gamma^{-1}$ are boundary values of a function $\tilde{g}$ harmonic in Δ. Hence $\tilde{g}$ may be retrieved from its boundary values in a unique manner by the Poisson extension, i.e.,

$$\tilde{g}(z) = P[\gamma \circ g \circ \gamma^{-1}](z), \quad z \in \Delta, \ g \in G. \tag{3.2}$$

Formula (3.2) determines $\tilde{g}$ uniquely, contrary to (2.4). It seems plausible that there might be several isomorphic groups $\tilde{G}$ due to different CNQE of the boundary homeomorphism γ.

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