ANNALES
UNIVERSITATIS MARIAE CURIE – SKŁODOWSKA
LUBLIN – POLONIA
VOL. LII.1, 7 SECTIO A 1998

STANISŁAWA KANAS and JAN STANKIEWICZ

Univalence of Confluent Hypergeometric Functions

Dedicated to Professor Eligiusz Złotkiewicz

ABSTRACT. Conditions of univalence and convexity for a confluent hypergeometric function Φ with complex coefficients are obtained. Condition of starlikeness of $z\Phi(z)$ is given.

1. Introduction. Let H denote the class of functions with the classical normalization which are analytic in the open unit disk U and let S be the class of functions in H which are univalent in the disk U. Subclasses of the class S being the collection of *starlike* and *convex* functions are denoted by S^* and S^c, respectively.

By a *confluent hypergeometric* function, also known as a *Kummer function*, we mean the function

$$\Phi(a;c;z) = {}_1F_1(a;c;z) = 1 + \frac{a}{c}\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} + \dots, \tag{1.1}$$

defined for $a, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \cdots$. The series (1.1) is convergent throughout the z-plane and represents therefore an integral function.

1991 *Mathematics Subject Classification*. Primary 30C45; Secondary 33E05.
Key words and phrases. Hypergeometric functions, confluent hypergeometric functions, convex functions, univalent functions, starlike functions.

A particular example is the exponential function: $\Phi(a;a;z) = \exp\ z$.

The function $\Phi(a;c;z)$ satisfies the confluent hypergeometric differential equation of Kummer

$$zw''(z) + (c-z)w'(z) - aw(z) = 0. \tag{1.2}$$

Moreover, it is easy to verify that for Re $c >$ Re $a > 0$, (see, e.g. [3]),

$$\Phi(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}e^{tz}dt = \int_0^1 e^{tz}d\mu(t), \tag{1.3}$$

where $\mu(t)$ is a probability measure on $[0,1]$.

If f and g are analytic in U and $g \in S$, we say that f is subordinate to g if $f(0) = g(0)$ and $f(U) \subset g(U)$. Then we write $f \prec g$.

Lemma 1.1 ([1]). *Let Ω be a set in the complex plane and let the function $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ satisfy the condition: $\psi(is,t,u+iv;z) \notin \Omega$ for $z \in U$, $s,t,u,v \in \mathbb{R}$,*

$$t \le -\frac{1+s^2}{2} \quad \text{and} \quad t+u \le 0.$$

If $p(z)$ is analytic in U with $p(0) = 1$ and $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$ for $z \in U$, then Re $p(z) > 0$ *in U.*

Properties of the hypergeometric functions was the subject of numerous investigations. A new method of obtaining different results concerning hypergeometric functions was proposed in [2]. Applying the theory of differential subordination the authors studied various properties of the function Φ with real coefficients. Among others they proved

Lemma 1.2 ([2]). *If a and c are real and satisfy one of the conditions:*

(i) $a > 0$ *and* $c \ge a$,

(ii) $a \le 0$ *and* $c \ge 1 + \sqrt{1+a^2}$,

then Re $\Phi(a;c;z) > 0$, $z \in U$, *where $\Phi(a;c;z)$ is a confluent hypergeometric function.*

In this paper we determine the conditions on the complex-valued parameters a and c for which the function Φ is univalent, convex and $z\Phi(z)$ is starlike in U.

2. Main results.

Theorem 2.1. *If complex numbers* $a = a_1 + ia_2$ *and* c *satisfy one of the following conditions:*

(i) $\operatorname{Re} c > \operatorname{Re} a > 0$,

(ii) $\operatorname{Re} a < 0$ *and* $\operatorname{Re} c > 2 + \sqrt{a_1^2 + (1 + a_2)^2}$,

(iii) $\operatorname{Re} a = 0$ *and* $\operatorname{Re} c > 2 + |\operatorname{Im} a|$,

then $\operatorname{Re} \Phi(a; c; z) > 0$ *for* $z \in U$.

Proof. If the condition (i) holds then the result follows from the equality (1.3), so we may consider only the cases (ii) and (iii).

Write $p(z) = \Phi(a; c; z)$. Then, obviously $p(0) = 1$. Since the function p satisfies Kummer hypergeometric differential equation (1.2), we have

$$z^2 p''(z) + (c - z)zp'(z) - azp(z) = 0. \tag{2.1}$$

Let $\psi(r_1, r_2, r_3; z) = r_3 + (c - z)r_2 - azr_1$ *and* $\Omega = \{0\}$. Then (2.1) can be rewritten in the form

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega, \quad z \in U. \tag{2.2}$$

In order to prove $\operatorname{Re} p(z) > 0$ we will use Lemma 1.1. Then for $z = x + iy$, $c = c_1 + ic_2$, $a = a_1 + ia_2$ we have

$$\operatorname{Re} \psi(is, t, u + iv; x + iy) = u + t + t(c_1 - 1 - x) + s(a_2 x + a_1 y).$$

Hence for $u + t \leq 0$, and $t \leq -(1 + s^2)/2$

$$\operatorname{Re} \psi(is, t, u + iv; x + iy) \leq -\frac{s^2 + 1}{2}(c_1 - 1 - x) + s(a_2 x + a_1 y) := Q(s).$$

We have to prove that $Q(s) < 0$.

Suppose that the condition (ii) holds. Since $c_1 > 2$, $Q(s)$ is a quadratic polynomial in s which attains its maximum Q_0 at s_0. If $s_0 = 0$ then $Q_0 = -\frac{1}{2}(c_1 - 1 - x) < 0$. If $s_0 \neq 0$ then $s_0 = \dfrac{a_2 x + a_1 y}{c_1 - 1 - x}$, $(Q'(s_0) = 0)$ and

$$Q_0 = -\frac{1}{2}(c_1 - 1 - x)(s_0^2 - 2s_0 + 1) = -\frac{1}{2}(c_1 - 1 - x)(s_0 - 1)^2.$$

Unless $s_0 = 1$, we have $Q_0 < 0$.

Suppose that $s_0 = 1$, or equivalently $c_1 - 1 = (a_2 + 1)x + a_1 y$. Hence

$$\begin{aligned} c_1 - 1 = |(a_2 + 1)x + a_1 y| &\leq \sqrt{x^2 + y^2}\sqrt{a_1^2 + (1 + a_2)^2} \\ &\leq \sqrt{a_1^2 + (1 + a_2)^2} < c_1 - 2, \end{aligned}$$

i.e. $c_1 - 1 < c_1 - 2$, a contradiction.

If $a_1 = 0$, (the condition (iii)), an analogous procedure gives the proof of $Q(s) < 0$ under the assumption $c_1 > 2 + |a_2|$. Then $s_0 = a_2 x/(c_1 - 1 - x)$ and

$$Q_0 = -\frac{1}{2}(c_1 - 1 - x)(s_0 - 1)^2.$$

The condition $s_0 = 0$ implies $Q_0 < 0$, whereas $s_0 = 1$ gives $a_2 x = c_1 - 1 - x$ i.e. $(a_2 + 1)x = c_1 - 1$ and this contradicts the assumption $|a_2| + 1 < c_1 - 1$.

Thus, in the case (ii) and (iii) all assumptions of Lemma 1.1 are satisfied, and so we have $\operatorname{Re} p(z) > 0$ in U which is equivalent to $\operatorname{Re}\Phi(a;c;z) > 0$. This completes the proof.

Remark 2.1 In the case of real a, c, Theorem 2.1 reduces to Theorem 1.1, due to Miller and Mocanu, [2].

Taking into account the relation $c\,\Phi'(a;c;z) = a\,\Phi(a+1;c+1;z)$ we obtain as a direct consequence of Theorem 2.1 the following condition of univalence of the Kummer function.

Theorem 2.2. *If a, c are complex numbers, $a \neq 0$, which satisfy one of the following conditions:*

(i) $\operatorname{Re} c > \operatorname{Re} a > -1$,

(ii) $\operatorname{Re} a < -1$ *and* $\operatorname{Re} c > 1 + \sqrt{(1 + \operatorname{Re} a)^2 + (1 + \operatorname{Im} a)^2}$,

(iii) $\operatorname{Re} a = -1$ *and* $\operatorname{Re} c > 1 + |\operatorname{Im} a|$,

then $\operatorname{Re}(c/a)\Phi'(a;c;z) > 0$ *in* U, *which means that* $\Phi(a;c;z)$ *is univalent in* U.

Theorem 2.3. *If $a = a_1 + ia_2$ and $c = c_1 + ic_2$ with $a \neq 0$, $a_1 \geq -1$ are complex numbers which satisfy the following conditions*

$$c_1 > \begin{cases} |a_1| + \dfrac{1}{2} + \left|a_2 - \dfrac{c_2}{3}\right| + \dfrac{1}{6}c_2^2 & \text{if } |a_1| \geq \dfrac{1}{3}, \\[2ex] \dfrac{3}{2}a_1^2 + \dfrac{2}{3} + \left|a_2 - \dfrac{c_2}{3}\right| + \dfrac{1}{6}c_2^2 & \text{if } |a_1| \leq \dfrac{1}{3}, \end{cases} \tag{2.4}$$

then $\Phi(a;c;z) \in S^c$, $z \in U$.

Proof. Let the complex numbers a, c, satisfy the assumptions of theorem. Then also conditions (i) - (iii) of Theorem 2.2 are satisfied, so $\Phi'(a;c;z) \neq 0$ in U. Hence the function

$$q(z) = 1 + \frac{z\Phi''(z)}{\Phi'(z)} \tag{2.5}$$

is analytic in U, with $q(0)=1$. The function $\Phi(a;c;z)$ satisfies the differential equality (1.2), and so the equality

$$zq'(z)+q^2(z)+(c-2-z)q(z)-az-c+1=0,$$

where q is defined in (2.5).

Reasoning along the same lines as in the proof of Theorem 2.1, put

$$\psi(r_1,r_2;z)=r_2+r_1^2+(c-2-z)r_1-az-c+1, \text{ and } \Omega=\{0\}.$$

Then, setting $z=x+iy$ and applying $t\le-(1+s^2)/2$, we have

$$\begin{aligned}\text{Re } \psi(is,t,z) &= t-s^2-s(c_2-y)-(a_1x-a_2y)-c_1+1\\ &\le -\frac{3}{2}s^2-s(c_2-y)-(a_1x-a_2y)-c_1+\frac{1}{2}:=Q(s).\end{aligned}$$

We next show that $Q(s)<0$ for all real s and $x^2+y^2<1$. The discriminant

$$\begin{aligned}\Delta &= (c_2-y)^2+6a_2y-6a_1x-6c_1+3\\ &< -x^2-6a_1x+y(6a_2-2c_2)+c_2^2-6c_1+4:=h(x),\end{aligned}$$

and $h'(x)=-2x-6a_1=0$, when $x_0=-3a_1$. Then for $|a_1|\ge 1/3$, $h'(x)$ is either negative or positive in the interval $(-1,1)$. Thus we have $h(x)<h(-1)$ or $h(x)<h(1)$, respectively. Taking into account the first condition in (2.4) we deduce

$$h(\pm1)=6|a_1|+c_2^2-6c_1+3+y(6a_2-2c_2)\le-|6a_2-2c_2|+y(6a_2-2c_2)<0,$$

for all $y\in(-1,1)$. If $|a_1|\le 1/3$ we have $h(x)\le h(x_0)$. But, from the second condition in (2.4)

$$h(x_0)=9a_1^2+c_2^2-6c_1+4+y(6a_2-2c_2)\le-|6a_2-2c_2|+y(6a_2-2c_2)<0.$$

Thus, in both cases the discriminant is negative and so is $Q(s)$ for all real s and $x^2+y^2<1$.

By Lemma 1.1 we conclude that Re $p(z)=$ Re $[1+z\Phi''(z)/\Phi'(z)]>0$ which means that the function $\Phi(a;c;z)$ is convex in U.

By the relation $(a-1)z\Phi(a;c;z)=(c-1)z\Phi'(a-1;c-1;z)$ we arrive at the following result:

Theorem 2.4. *If $a = a_1 + ia_2$ and $c = c_1 + ic_2$ with $a \neq 0$ are complex numbers, which satisfy the following conditions*

$$(2.6) \qquad c_1 > \begin{cases} |a_1 - 1| + \dfrac{3}{2} + \left|a_2 - \dfrac{c_2}{3}\right| + \dfrac{1}{6}c_2^2 & \text{if } |a_1 - 1| \geq \dfrac{1}{3}, \\ \dfrac{3}{2}(a_1 - 1)^2 + \dfrac{5}{3} + \left|a_2 - \dfrac{c_2}{3}\right| + \dfrac{1}{6}c_2^2 & \text{if } |a_1 - 1| \leq \dfrac{1}{3}, \end{cases}$$

then $z\Phi(a; c; z) \in S^*$, $z \in U$.

Acknowledgments. The authors wish to thank the referee for helpful suggestions on an earlier draft of this paper.

References

[1] Miller, S. S. and P.T. Mocanu, *Differential subordinations and inequalities in the complex plane*, J. Differential Equations **67, 2** (1987), 199-211.

[2] ———, *Univalence of Gaussian and confluent hypergeometric functions*, Proc. Amer. Math. Soc. **110(2)** (1990), 333-342.

[3] Sansone, G. and J. Gerretsen, *Lectures on the theory of functions of a complex variable*, Vol. II, Wolters-Noordhoof Publishing, Groningen, The Netherlands, 1969..

Department of Mathematics
Technical University of Rzeszów
ul. W. Pola 2, 35-959 Rzeszów, Poland
e-mail: skanas@prz.rzeszow.pl
e-mail: jstan@prz.rzeszow.pl

received February 23, 1998