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Integral Means of Harmonic Mappings

Dedicated to Professor Eligiusz Złotkiewicz on the occasion of his 60th birthday

ABSTRACT. In this paper we prove that if f is a univalent, sense-preserving, harmonic mapping of the unit disc onto the symmetric strip $|\operatorname{Im} w| < \pi/4$ such that $f(0) = 0 < f_z(0)$, then $f \in h^p$ for $0 < p < 1$. Moreover, we show that the harmonic Koebe function k_0 given by formula (1) is not in h^p if $p > 1/3$.

1. Introduction. Statement of results. Let Δ denote the open unit disc in the complex plane $\mathbb{C}$ and S_H denote the class of all complex valued, harmonic, sense-preserving univalent functions f in Δ, with the normalization $f(0) = 0 < f_z(0)$. Each $f \in S_H$ can be expressed as $f = h + \bar{g}$ where $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic in Δ. The subclasses of S_H consisting of harmonic mappings onto convex and close-to-convex regions will by denoted by K_H, C_H, respectively.

Let H^p (h^p), $0 < p < \infty$, be the standard Hardy space of analytic (harmonic) functions on Δ. In 1990 Y. Abu-Muhanna and A. Lyzzaik [AL] proved that if $f = h + \bar{g} \in S_H$, then $h, g \in H^p$ and $f \in h^p$ for every p, $p \in (0, (2A+2)^{-2})$, where $A = \sup\{|a_2|/f_z(0) : f \in S_H\}$. This result has been improved in [N] where the range of p was extended to $(0, A^{-2})$. There was also noticed that if $f = h + \bar{g} \in K_H$, then $g, h \in H^p$ and $f \in h^p$ for $0 < p < 1/2$. Moreover, an example of $f \in K_H$ such that $f \notin h^p$ for $p > 1/2$ was given.

In [AS] the authors showed that if $f \in K_H$ is such that $f(\Delta)$ is an unbounded domain which is neither a strip nor a half-plane then $f \in h^1$.

Let $\Omega = \{z \in \mathbb{C} : |\operatorname{Im} z| < \pi/4\}$ and $S_H(\Delta, \Omega) = \{f \in S_H : f(\Delta) = \Omega\}$. Here we prove the following

Theorem 1. *If $f \in S_H(\Delta, \Omega)$ then $f \in h^p$ for $0 < p < 1$.*

Futhermore, we give an example of $f \in S_H(\Delta, \Omega)$ such that $f \in h^1$ and $f \notin h^p$ for $p > 1$. For close-to-convex harmonic mappings the following theorem was proved in [N].

Theorem A. *If $f = h + \bar{g} \in C_H$ then $h, g \in H^p$ and $f \in h^p$ for $0 < p < \frac{1}{3}$.*

Let $k_0 : \Delta \to \mathbb{C} \setminus (\infty, -\frac{1}{6}]$ be the harmonic Koebe function given by the formula

(1)
$$k_0(z) = \frac{1}{6} \operatorname{Re}\left(\left(\frac{1+z}{1-z}\right)^3 - 1\right) + \frac{1}{4} i \operatorname{Im}\left(\frac{1+z}{1-z}\right)^2, \quad z \in \Delta \text{ (see [CS])}.$$

The function k_0 is in $h^{1/3}$ (see [N]). In this paper we prove

Theorem 2. *$k_0 \notin h^p$ for $p > 1/3$.*

2. Proof of Theorem 1. In the proof of Theorem 1 we will need the Baernstein star-function. If u is a real valued integrable function on $[-\pi, \pi]$ then

$$u^*(\theta) = \sup_{|E|=2\theta} \int_E u(t)\,dt, \quad 0 \leq \theta \leq \pi.$$

The following properties of star-functions are well known [D2]:

(i) $(u+v)^*(\theta) \leq u^*(\theta) + v^*(\theta), \quad 0 \leq \theta \leq \pi$.
Equality occurs if both u and v are symmetrically decreasing.

(ii) If $u(re^{i\theta})$, $v(re^{i\theta})$ are subharmonic in Δ and u is subordinate to v then for each r, $0 \leq r < 1$, $u^*(re^{i\theta}) \leq v^*(re^{i\theta})$.

For $p > 0$ and f harmonic on Δ set

$$M_p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p\,dt, \quad 0 \leq r < 1.$$

The next result we need is the so called dual of the Hardy – Littlewood inequality and is due to T. Flett (see [F1], [F2]).

Theorem B. *Let $0 < p \leq 2$ and h be an analytic function on Δ. If*

$$\int_0^1 (1-r)^{p-1} M_p(r, h')dr < \infty ,$$

then $h \in H^p$.

Proof of Theorem 1. First of all notice that for $f \in \mathcal{S}_H(\Delta, \Omega)$ and suitably chosen $t > 0$ and $\alpha \in \mathbb{R}$ the affine transformation

$$w \mapsto T(w) = te^{-i\alpha}(f_z(0)w - f_{\bar{z}}(0)\bar{w})$$

maps Ω onto itself univalently [GS, Example 1.] and the composition $T \circ f \circ \lambda_\alpha$, where $\lambda_\alpha : z \mapsto e^{i\alpha}z$ belongs to

$$S_H^0(\Delta, \Omega) = \{f \in S_H(\Delta, \Omega) \; : \; f_{\bar{z}}(0) = 0\}.$$

Hence we may restrict our attention to the subclass $S_H^0(\Delta, \Omega)$. W. Hengartner and G. Schober proved [HS] that $f \in \overline{S_H^0(\Delta, \Omega)}$ if and only if

$$f(z) = \mathrm{Re}\left\{\int_0^z \frac{p(\zeta)d\zeta}{1-\zeta^2}\right\} + \frac{i}{2}\arg\left(\frac{1+z}{1-z}\right), \tag{2}$$

where p is an analytic function in Δ such that $\mathrm{Re}\, p > 0$ and $p(0) = 1$.

It is clear that $\mathrm{Im}\, f \in h^\infty$. Writing

$$F(z) = \int_0^z \frac{p(\zeta)d\zeta}{1-\zeta^2}, \quad z \in \Delta$$

we have

$$\log|F'(z)| = \log\left|\frac{1}{1-z^2}\right| + \log|p(z)|, \quad z \in \Delta . \tag{3}$$

Since the function $z \mapsto 1/(1-z^2)$ maps Δ onto the half-plane $\{z \in \mathbb{C} : \mathrm{Re}\, z > 1/2\}$ and carries 0 to 1, we conclude that it is subordinate to $z \mapsto (1+z)/(1-z)$ and so is p. Hence in view of the above cited properties of the Baernstein star-function we get, for $0 \leq \theta \leq \pi$,

$$\left(\log\left|\frac{1}{1-r^2e^{2i\theta}}\right|\right)^* \leq \left(\log\left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right|\right)^*$$

and

$$\left(\log\left|p(re^{i\theta})\right|\right)^* \leq \left(\log\left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right|\right)^* .$$

It is easy to see that the function $\theta \mapsto \log\left|(1+re^{i\theta})/(1-re^{i\theta})\right|$ is symmetrically decreasing for each fixed $r \in (0,1)$. So, by (i)

$$(\log|F'(re^{i\theta})|)^* \leq \left(2\log\left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right|\right)^* = \left(\log\left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right|^2\right)^*, \quad 0 \leq \theta \leq \pi\,.$$

Applying Lemma 5 in [D2, p. 218] with $\phi : x \mapsto e^{px}$, $p > 0$, we conclude that

$$\int_{-\pi}^{\pi} |F'(re^{i\theta})|^p d\theta \leq \int_{-\pi}^{\pi} \left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right|^{2p} d\theta.$$

Moreover, by a Lemma in [D1, p. 65] there exists $C_p > 0$ such that for $p > 1/2$

$$\int_0^{2\pi} |F'(re^{i\theta})|^p d\vartheta \leq \frac{C_p}{(1-r)^{2p-1}} \quad \text{for } 0 < r < 1\,.$$

Thus

$$\int_0^1 (1-r)^{p-1} M_p(r, F')dr \leq C_p \int_0^1 (1-r)^{-p} dr < +\infty$$

for $p \in (1/2, 1)$. By Theorem B, the function F is in H^p, $0 < p < 1$, and the desired result follows.

<u>**Remark.**</u> It follows from the proof of Theorem 1 that all functions from $S_H^0(\Delta, \Omega)$ are in h^p for $0 < p < 1$. This improves the result contained in Lemma 2.2 in [CL].

3. Proof of Theorem 2. We start with the following, easily verifiable,

Lemma 1. *If the functions*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n\,, \quad g(x) = \sum_{n=0}^{\infty} b_n x^n\,,$$

where $a_k\,, b_k > 0\,,\ k = 0, 1, 2, \ldots$ *defined in the interval* $(-1, 1)$ *are such that*

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} g(x) = +\infty \quad \text{and} \quad \lim_{n\to\infty} \frac{a_n}{b_n} = A \in (0, \infty)\,,$$

then

$$\lim_{x\to 1^-} \frac{f(x)}{g(x)} = A\,.$$

Proof of Theorem 2. We can assume that $p \in \left(\frac{1}{3}, \frac{1}{2}\right)$. Since $\operatorname{Im} k_0 \in h^p$, $0 < p < 1/2$, it is enough to prove that

$$\lim_{r \to 1^-} \int_0^{2\pi} \left| \operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)^3 \right|^p d\theta = +\infty . \tag{4}$$

Using the inequality $|a+b|^p \geq |a|^p - |b|^p$, $0 < p < 1$, we obtain

$$\begin{aligned} \int_0^{2\pi} \left| \operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)^3 \right|^p d\theta &\geq 12^p r^{2p}(1-r^2)^p \int_0^{2\pi} \frac{\sin^{2p}\theta d\theta}{(1+r^2-2r\cos\theta)^{3p}} \\ &- (1-r^2)^{3p} \int_0^{2\pi} \frac{d\theta}{(1+r^2-2r\cos\theta)^{3p}} . \end{aligned} \tag{5}$$

Let

$$\begin{aligned} I_1(r) &= 12^p r^{2p}(1-r^2)^p \int_0^{2\pi} \frac{\sin^{2p}\theta d\theta}{(1+r^2-2r\cos\theta)^{3p}} \\ &= \frac{2\cdot 12^p r^{2p}(1-r^2)^p}{(1+r^2)^{3p}} \int_0^{\pi} \frac{\sin^{2p}\theta d\theta}{(1-c\cos\theta)^{3p}} , \end{aligned} \tag{6}$$

where $c = 2r/(1+r^2)$. Making the substitution $t = \cos\theta$ gives

$$\begin{aligned} \int_0^{\pi} \frac{\sin^{2p}\theta d\theta}{(1-c\cos\theta)^{3p}} &= \int_{-1}^{1} \frac{(1-t^2)^{p-\frac{1}{2}} dt}{(1-ct)^{3p}} \\ &> \int_0^1 \frac{(1-t^2)^{p-\frac{1}{2}} dt}{(1-ct)^{3p}} \geq 2^{p-\frac{1}{2}} \int_0^1 \frac{(1-t)^{p-\frac{1}{2}} dt}{(1-ct)^{3p}} . \end{aligned}$$

Expanding the function $t \mapsto 1/(1-ct)^{3p}$ into a power series and integrating term by term we obtain

$$\int_0^1 \frac{(1-t)^{p-\frac{1}{2}} dt}{(1-ct)^{3p}} = \frac{1}{p+1/2} + \sum_{n=1}^{\infty} a_n c^n =: F_1(c) , \tag{7}$$

where

$$a_n = \frac{3p(3p+1)\cdots(3p+n-1)}{(p+1/2)(p+3/2)\cdots(p+1/2+n)} , \quad n = 1, 2, \ldots$$

Now, if

$$G_1(c) = \frac{1}{(1-c)^{2p-1/2}} = \sum_{n=0}^{\infty} a'_n c^n ,$$

then by Gauss's formula (e.g. [Co], p. 174) we have

$$\lim_{n\to\infty}\frac{a_n'}{n^{2p-3/2}}=\frac{1}{\Gamma(2p-1/2)}$$

and

$$\lim_{n\to\infty}\frac{a_n}{a_n'}=\frac{\Gamma(2p-1/2)\Gamma(p+1/2)}{\Gamma(3p)}=B(2p-1/2,p+1/2).$$

Hence by Lemma 1

$$\lim_{c\to 1^-}\frac{F_1(c)}{G_1(c)}=B(2p-1/2,p+1/2).$$

This means that if we take any $\varepsilon>0$ then there exists $\delta>0$ such that for $c>1-\delta$

(8) $$F_1(c)\geq(B(2p-1/2,p+1/2)-\varepsilon)G_1(c).$$

It follows from (6)–(8) that

(9)

$$\begin{aligned}I_1(r)&\geq\frac{2\cdot 12^p r^{2p}(1-r^2)^p}{(1+r^2)^{3p}}2^{p-1/2}\left(B(2p-1/2,p+1/2)-\varepsilon\right)G_1(c)\\&=\frac{2^{p+1/2}\cdot 12^p r^{2p}(1+r)^p}{(1+r^2)^{p+1/2}}\left(B(2p-1/2,p+1/2)-\varepsilon\right)\cdot\frac{1}{(1-r)^{3p-1}}.\end{aligned}$$

Let now

$$\begin{aligned}I_2(r)&=(1-r^2)^{3p}\int_0^{2\pi}\frac{d\theta}{(1+r^2-2r\cos\theta)^{3p}}\\&=\frac{2\cdot(1-r^2)^{3p}}{(1+r^2)^{3p}}\int_0^{\pi}\frac{d\theta}{(1-c\cos\theta)^{3p}},\end{aligned}$$

where again $c=\frac{2r}{1+r^2}$. Using the same technique as above we will estimate $I_2(r)$. We have

$$\int_0^{\pi}\frac{d\theta}{(1-c\cos\theta)^{3p}}=\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}(1-ct)^{3p}}.$$

Note that

$$\int_{-1}^{0}\frac{dt}{\sqrt{1-t^2}(1-ct)^{3p}}\leq\int_0^1\frac{dt}{\sqrt{1-t}}=2,$$

and

$$\int_0^1 \frac{dt}{\sqrt{1-t^2}(1-ct)^{3p}} \le \int_0^1 \frac{dt}{\sqrt{1-t}(1-ct)^{3p}} = 2+\sum_{n=1}^{\infty} b_n c^n =: F_2(c),$$

where

$$b_n = \frac{3p(3p+1)\cdots(3p+n-1)}{(1/2)(1/2+1)\cdots(1/2+n)}, \quad n=1,2,\ldots$$

Now we compare the behaviour of F_2 and the function

$$G_2(c) = \frac{1}{(1-c)^{3p-1/2}} = \sum_{n=0}^{\infty} b'_n c^n.$$

In this case we get

$$\lim_{n\to\infty} \frac{b'_n}{n^{3p-3/2}} = \frac{1}{\Gamma(3p-1/2)}$$

and

$$\lim_{n\to\infty} \frac{b_n}{b'_n} = \frac{\Gamma(1/2)\Gamma(3p-1/2)}{\Gamma(3p)} = B(1/2,3p-1/2).$$

Therefore

$$\lim_{c\to 1^-} \frac{F_2(c)}{G_2(c)} = B(1/2,3p-1/2).$$

This implies that for $\varepsilon > 0$ there exists $\delta_1 > 0$ such that

$$F_2(c) \le (B(1/2,3p-1/2)+\varepsilon)G_2(c) \quad \text{for } c > 1-\delta_1.$$

It follows from the above estimates that

(10)

$$I_2(r) \le \frac{2\cdot(1-r^2)^{3p}}{(1+r^2)^{3p}}(2+G_2(c)(B(1/2,3p-1/2)+\varepsilon))$$

$$= \frac{4\cdot(1-r^2)^{3p}}{(1+r^2)^{3p}} + \frac{2\cdot(1+r)^{3p}}{(1+r^2)^{1/2}}(B(1/2,3p-1/2)+\varepsilon)\frac{1}{(1-r)^{3p-1}}.$$

Now, combining inequalities (5), (9) and (10), we obtain

$$\int_0^{2\pi} \left|\operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)^3\right|^p d\theta \ge -\frac{4\cdot(1-r^2)^{3p}}{(1+r^2)^{3p}} + \frac{2(1+r)^p}{(1+r^2)^{1/2}}\cdot\frac{1}{(1-r)^{3p}}$$

$$\times\left[\frac{2^{p-\frac{1}{2}}12^p r^{2p}}{(1+r^2)^p}\left(B\Big(p+\frac{1}{2},2p-\frac{1}{2}\Big)-\varepsilon\right)-(1+r)^{2p}\left(B\Big(\frac{1}{2},3p-\frac{1}{2}\Big)+\varepsilon\right)\right].$$

Let $U(p,r,\varepsilon)$ denote the expression in the square brackets in the last inequality. Then

$$\lim_{r\to 1^-} U(p,r,\varepsilon) = \frac{12^p}{\sqrt{2}} B\Big(p+\frac{1}{2}, 2p-\frac{1}{2}\Big) - 2^{2p} B\Big(\frac{1}{2}, 3p-\frac{1}{2}\Big) - \varepsilon\left(\frac{12^p}{\sqrt{2}} + 2^{2p}\right).$$

Consider now the function

$$\alpha(p) = \frac{12^p}{\sqrt{2}} B(p+1/2, 2p-1/2) - 2^{2p} B(1/2, 3p-1/2) \quad \text{for} \quad p \in \left(\frac{1}{3}, \frac{1}{2}\right).$$

In view of the formula

$$B(a, 1-a) = \frac{\pi}{\sin a\pi}, \quad 0 < a < 1,$$

we have $\alpha(1/3) = \pi(12^{1/3}2^{1/2} - 2^{2/3}) =: 2\alpha_0 > 0$. The continuity of $\alpha(p)$ implies $\alpha(p) \geq \alpha_0 > 0$ for $p \in (1/3, 1/3+\gamma)$, $\gamma > 0$. Taking $\varepsilon > 0$ small enough we get

$$\lim_{r\to 1^-} U(p,r,\varepsilon) > 0.$$

Note that $\lim_{r\to 1^-} -4(1-r^2)^{3p}/(1+r^2)^{3p} = 0$ for $p \in (1/3, 1/2)$.

Consequently, there exists a positive constant $A > 0$ such that

$$\int_0^{2\pi} \left| \operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)^3 \right|^p d\theta \geq \frac{A}{(1-r)^{3p-1}} \xrightarrow[r\to 1^-]{} \infty,$$

which completes the proof.

4. Example. Let f be defined by

$$f(z) = \frac{1}{2}\operatorname{Re}\left(\frac{z}{1+z} + \frac{z}{1-z}\right) + \frac{i}{2}\arg\left(\frac{1+z}{1-z}\right), \quad z \in \Delta.$$

It follows from Theorem 2.9 in [HS] that $f \in S_H^0(\Delta, \Omega)$. We claim that $f \in h^1$ and that $f \notin h^p$ if $p > 1$. It is enough to show that $\operatorname{Re} f \in h^1$.

For $0 < r < 1$ we have

$$\int_0^{2\pi} \left| \operatorname{Re} \frac{re^{i\theta}}{1-re^{i\theta}} \right| d\theta = 2r \int_0^{\pi} \frac{|\cos\theta - r| d\theta}{1+r^2-2r\cos\theta}$$

$$= 2r \int_0^{\arccos r} \frac{(\cos\theta - r)d\theta}{1+r^2-2r\cos\theta} + 2r \int_{\arccos r}^{\pi/2} \frac{(r-\cos\theta)d\theta}{1+r^2-2r\cos\theta}$$

$$+ 2r \int_{\pi/2}^{\pi} \frac{(r-\cos\theta)d\theta}{1+r^2-2r\cos\theta}$$

$$\leq 2r \int_0^{\arccos r} \frac{(\cos\theta - r)d\theta}{1+r^2-2r\cos\theta} + 2r(2+\pi).$$

The substitution $t = \cos\theta$ gives

$$\int_0^{\arccos r} \frac{(\cos\theta - r)d\theta}{1 + r^2 - 2r\cos\theta} = \frac{1}{1+r^2}\int_r^1 \frac{(t-r)dt}{(1-ct)\sqrt{1-t^2}}$$
$$\leq \frac{1-r}{1+r^2}\int_0^1 \frac{dt}{(1-ct)\sqrt{1-t}},$$

where $c = 2r/(1+r^2)$. Proceeding as in the proof of Theorem 2 we see that there exists a positive constant C such that

$$\int_0^1 \frac{dt}{(1-ct)\sqrt{1-t}} \leq \frac{C}{(1-c)^{1/2}}.$$

This means that

$$\int_0^{\arccos r} \frac{(\cos\theta - r)d\theta}{1 + r^2 - 2r\cos\theta} \leq \frac{C}{\sqrt{1+r^2}} \leq C.$$

Since

$$\int_0^{2\pi} \left|\operatorname{Re}\frac{re^{i\theta}}{1 - re^{i\theta}}\right| d\theta = \int_0^{2\pi} \left|\operatorname{Re}\frac{re^{i\theta}}{1 + re^{i\theta}}\right| d\theta,$$

the desired statement follows.

Assume now that $p \in (1,2)$. For $z = re^{i\theta}$ we have

$$\operatorname{Re} f(z) = \operatorname{Re}\frac{z}{1-z^2} = \frac{r\cos\theta(1-r^2)}{1 - 2r^2\cos 2\theta + r^4}.$$

Hence

$$\int_0^{2\pi} |\operatorname{Re} f(re^{i\theta})|^p d\theta = 2r^p(1-r^2)^p \int_0^{\pi} \frac{|\cos\theta|^p d\theta}{(1 - 2r^2\cos 2\theta + r^4)^p}$$
$$\geq 2r^p(1-r^2)^p \int_0^{\pi/4} \frac{\cos^p\theta d\theta}{(1 - 2r^2\cos 2\theta + r^4)^p}$$
$$> 2r^p(1-r^2)^p \int_0^{\pi/4} \frac{\cos^p(2\theta) d\theta}{(1 - 2r^2\cos 2\theta + r^4)^p}.$$

In a similar way we obtain

$$\int_0^{2\pi} |\operatorname{Re} f(re^{i\theta})|^p d\theta > C\cdot\frac{r^p}{(1-r^2)^{p-1}} \xrightarrow[r\to 1^-]{} \infty.$$

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