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Introduction to Almost Hyperbolic Pseudodistances via Intermediate Dimensional-Invariant Measures

Dedicated to Professor Eligiusz Złotkiewicz

ABSTRACT. In 1989 two of us (P.D. and J.Ł.) introduced a Dirichlet integral-type biholomorphic-invariant pseudodistance connected with bordered holomorphic chains whose regular part was treated as a Riemann surface [4]. The condition for a complex manifold that the pseudodistance on it was a distance defined a class of hyperbolic-like manifolds which had an important property of extendability of holomorphic mappings, analogous to the hyperbolic manifolds, Stein spaces, and complex spaces with a Stein covering. Further results in this direction were published in 1996 by G. Boryczka and L.M. Tovar [1]. The present research introduces a modified approach exploring, in addition, the intermediate one- and two-dimensional measures due to D.Eisenman (now Pelles) [5].

1. Introduction. The importance of the subject is motivated by a number of results by A. Andreotti, W. Stoll, and K. Kobayashi, referred to in [4]. The authors believe that this introduction to a new approach will open new possibilities in continuing those lines, in particular in the aspect of interrelations between the complex dynamics and hyperbolic geometry.

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2. An analogue of the hyperbolic pseudodistance related to intermediate measures. Let X be a complex manifold of complex dimension n. Consider a compact connected C^1-cycle γ [3] of (real) dimension one on X. Suppose that Γ is an irreducible complex analytic subvariety of complex dimension one of $V = X \setminus \operatorname{spt}\gamma$, with support spt Γ relatively compact on X. Let Γ represent an elementary bordered holomorphic chain [4].

A *bordered holomorphic chain* passing through distinct points z_0, $z \in X$ is defined as a finite sum $\sum_{j\in I}\Gamma_j$ of elementary chains Γ_j such that each elementary chain Γ_j passing through distinct points z_{j-1}, z_j of $\mathcal{U}$, $j = 1,\dots,p$, is such that z_0 is the first given point, while z_p is the last one: $z_p = z$. Let γ_j denote the border of Γ_j.

For each elementary chain Γ'_j passing through the points z_{j-1}, z_j with Γ'_j contained in a fixed elementary chain Γ_j, we have a holomorphic mapping

$$\phi_j : \sum_j \to \Gamma_j \subset X \setminus \operatorname{spt}\gamma_j$$

such that, for a discrete set $E_j \subset \Gamma_j$, the set Reg $\Gamma_j = \Gamma_j \setminus E_j$, called the *regular part* of Γ_j, is the image of a connected Riemann surface S under a biholomorphic mapping $f_j = \phi_j|S$. Let γ'_j be the border of Γ'_j.

Assume that X is (k,m)-hyperbolic for $k = 1$ or 2, and a fixed $m \geq n$, in the sense of Eisenman-Kobayashi [5-7]. Set $\alpha = 1 - n/m$. For a fixed elementary chain Γ_j, let

$$(1) \qquad \mu^{\alpha}_{\Gamma_j}[u] := \inf_{\Gamma'_j\subset\Gamma_j}\left\{[\mu_1(\gamma'_j, f_j^{-1})_m/\mu_2(\Gamma'_j, f_j^{-1})_m]\left|\int_{\Gamma'_j} du\wedge d^c u\right|\right\},$$

where $\mu = \mu_1$ and $\mu = \mu_2$ are the intermediate one- and two-dimensional measures [6], u belongs to an admissible family $F[\mathcal{U}] = \operatorname{adm}(X,\mathcal{U})$ of plurisubharmonic functions, defined in the usual way [4] for a given locally finite open covering $\mathcal{U}$ of X, and the infimum in (1) is taken over all compact connected C^1-cycles of dimension one within Γ_j.

We have

Lemma 1. *The expression* (1) *is well defined.*

Thus with any bordered holomorphic chain passing through the points z_0, z of X, such that $\mu_1(\gamma'_j, f_j^{-1})_m$ is uniformly bounded in Γ, we may associate the expression $\mu^{\alpha}_{\Gamma}(z_0,z)[u] := \sum_{j\in J}\mu_{\Gamma_j}(z_{j-1},z_j)[u]$. Using this expression we set

$$\mu^{\alpha}_{X}(z_0,z)[u,\mathcal{U}] := \inf\{\mu_{\Gamma}(z_0,z)[u] : \Gamma \text{ passing through } z_0, z\}.$$

Finally, we define an *almost hyperbolic pseudodistance:*

(2) $$\rho_X^\alpha(z_0,z)[\mathcal{U}] := \sup\{\mu_X^\alpha(z_0,z)[u,\mathcal{U}] \colon u \in F[\mathcal{U}]\}.$$

We have to prove its finiteness and that it is indeed a pseudodistance.

Lemma 2. *Let z_0, z_1 and z_2 be points on a (k,m)-hyperbolic n-dimensional complex manifold X for $k = 1$ and 2, and a fixed $m \geq n$. Set $\alpha = 1 - n/m$. Then, for any locally finite open covering $\mathcal{U}$ of X, we have*

$$\rho_X^\alpha(z_0,z_2)[\mathcal{U}] \leq \rho_X^\alpha(z_0,z_1)[\mathcal{U}] + \rho_X^\alpha(z_1,z_2)[\mathcal{U}].$$

Proof. Let Γ_0, Γ_1, and Γ_2 be bordered holomorphic chains passing through z_0, z_1; z_1, z_2; z_0, z_2, respectively. Then $\Gamma_1 + \Gamma_2$ is also a bordered holomorphic chain passing through z_0, z_2 and everywhere in $F[\mathcal{U}]$ we have

$$\mu_{\Gamma_1+\Gamma_2}^\alpha(z_0,z_2)[u] \leq \mu_{\Gamma_1}^\alpha(z_0,z_1) + \mu_{\Gamma_2}^\alpha(z_0,z_2).$$

Hence, for any u and $\mathcal{U}$,

$$\begin{aligned}\mu_X^\alpha(z_0,z_2) &= \inf_\Gamma \mu_\Gamma^\alpha(z_0,z_2) \leq \inf_{\Gamma_1+\Gamma_2} \mu_{\Gamma_1+\Gamma_2}^\alpha(z_0,z_2)\\ &\leq \inf_{\Gamma_1,\Gamma_2} [\mu_{\Gamma_1}^\alpha(z_0,z_1) + \mu_{\Gamma_2}^\alpha(z_1,z_2)]\\ &\leq \inf_{\Gamma_1} \mu_{\Gamma_1}^\alpha(z_0,z_1) + \inf_{\Gamma_2} \mu_{\Gamma_2}^\alpha(z_1,z_2) = \mu_X^\alpha(z_0,z_1) + \mu_X^\alpha(z_1,z_2),\end{aligned}$$

where the infima are taken with respect to bordered holomorphic chains passing through the points indicated in the brackets. Consequently,

$$\begin{aligned}\rho_X^\alpha(z_0,z_2)[\mathcal{U}] &= \sup_u \mu_X^\alpha(z_0,z_2)[u,\mathcal{U}]\\ &\leq \sup_u\{\mu_X^\alpha(z_0,z_1)[u,\mathcal{U}] + \mu_X^\alpha(z_1,z_2)[u,\mathcal{U}]\}\\ &\leq \sup_u \mu_X^\alpha(z_0,z_1)[u,\mathcal{U}] + \sup_u \mu_X^\alpha(z_1,z_2)[u,\mathcal{U}]\\ &= \mu_X^\alpha(z_0,z_2)[\mathcal{U}] + \mu_X^\alpha(z_1,z_2)[\mathcal{U}],\end{aligned}$$

where the suprema are taken with respect to u ranging over $F[\mathcal{U}]$.

Lemma 3. *Let z_0, z be points on a (k,m)-hyperbolic n-dimensional complex manifold X for $k = 1$ and 2, and a fixed $m \geq n$. Set $\alpha = 1 - n/m$. Then, for any locally finite open covering $\mathcal{U}$ of X, we have*

$$\rho_X^\alpha(z_0,z)[\mathcal{U}] \leq +\infty.$$

Proof. Since $\mu_X^\alpha(z_0, z)[u, \mathcal{U}]$ is defined as the infimum of all the expressions $\mu_\Gamma^\alpha(z_0, z)[u]$ with respect to bordered holomorphic chains Γ passing through z_0, z, without any loss of generality we may suppose that Γ is an elementary chain passing through z_0, z_1. Moreover, since the closure $\mathrm{cl}_X \mathrm{spt}\Gamma$ is compact, we may suppose that it is contained in a connected Riemann surface $S \subset U_j$, U_j being a member of $\mathcal{U}$, and that S is biholomorphically equivalent to the unit disc. Since, as it is well known [2],

$$\sup\left\{\left|\int_\Gamma du \wedge d^c u\right| : u \in F[U]\right\}$$

is bounded, this proves the lemma.

Remark 1. Under the hypotheses of Lemma 1, $\rho_X^\alpha(z_0, z)[\mathcal{U}] \geq 0$ and $\rho_X^\alpha(z, z_0)[\mathcal{U}] = \rho_X^\alpha(z_0, z)[\mathcal{U}]$. If $z = z_0$, then the length of $\phi^{-1}[\gamma']$ can be as small as we desire, so $\rho_X^\alpha(z_0, z)[\mathcal{U}] = 0$.

From Lemma 2, by Remark 1, we infer

Proposition 1. *Let X be a (k, m)-hyperbolic n-dimensio- nal complex manifold for $k = 1, 2$ and a fixed $m \geq n$. Set $\alpha = 1 - n/m$. Then, for any locally finite open covering $\mathcal{U}$ of X, the corresponding expression ρ_X^α given by (2) is a continuous pseudodistance.*

By Proposition 1, we trivially get (for the proof, cf. [1]):

Proposition 2. *Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be two (k, m)-hyperbolic n-dimensional complex manifolds with k, m, α as in Proposition 1, locally finite open coverings $\mathcal{U}$ and $\mathcal{V}$, and admissible families $F[U]$ and $F[\mathcal{V}]$ of pluriharmonic functions. Let $f : X \to Y$ be a biholomorphic mapping such that $f[\mathcal{U}] = \mathcal{V}$. Then*

$$\rho_X^\alpha(z_0, z)[\mathcal{U}] = \rho_Y^\alpha(f(z_0), f(z))[\mathcal{V}] \quad \text{for } z_0, z \in X.$$

Propositions 1-2 motivate the following definition. Let X be a (k, m)-hyperbolic n-dimensional complex manifold for $k = 1, 2$ and a fixed $m \geq n$, and let $\alpha = 1 - n/m$. If, for a locally finite open covering $\mathcal{U}$ of X, $\rho_X^\alpha(\ ,\ )[\mathcal{U}]$ is a distance, i.e. $\rho_X^\alpha(z_0, z)[\mathcal{U}] > 0$ for $z_0 \neq z$, then X is called an $(\alpha, \mathcal{U})$-*almost hyperbolic manifold.* An $(\alpha, \mathcal{U})$-almost hyperbolic manifold X is said to be *complete* if it is complete with respect to $\rho_X^\alpha(\ ,\ )[\mathcal{U}]$. Almost hyperbolic manifolds are — in general — not hyperbolic-like in the sense of [4] and vice versa. Hyperbolic manifolds in the sense of [6] are simultaneously $(\alpha, \mathcal{U})$-almost hyperbolic and $(\alpha, \mathcal{U})$-hyperbolic-like.

3. The expression $\rho_X^\alpha(z_0, z)$ as an almost hyperbolic pseudodistance. We start with proving

Proposition 3. *Let $X, \mathcal{U}$ and $(Y, \mathcal{V})$ be two (k, m)-hyperbolic n-dimensional complex manifolds with k, m, α as in Proposition 1, locally finite open coverings $\mathcal{U}$ and $\mathcal{V}$, and admissible families $F[U]$ and $F[\mathcal{V}]$ of pluriharmonic functions. Let $f : X \to Y$ be a proper holomorphic mapping such that $f^{-1}[\mathcal{V}] \subset \mathcal{U}$. Then*

$$(3) \qquad \rho_X^\alpha(z_0, z)[\mathcal{U}] \geq \rho_Y^\alpha(f(z_0), f(z))[\mathcal{U}] \quad \textit{for } z_0, z \in X.$$

Proof. Given $u \in F[\mathcal{U}]$, we have $u \circ f \in F[\mathcal{U}]$. For each elementary chain Γ_j either $f[\Gamma_j]$ is one point or, since the image of any elementary chain passing through points z_0, z of X, is a bordered holomorphic chain passing through the points $f(z_0), f(z)$ of $Y = f[X]$ [1] (Lemma 1), $f[\Gamma_j]$ is a one-dimensional complex variety and the restriction $f|\Gamma_j : \Gamma_j \to f[\Gamma_j]$ is a finite ramified covering. By the definition of ρ_Y^α and the above observation, taking into account the suprema over $V \in F[\mathcal{V}]$ and $U \in F[\mathcal{U}]$, we arrive at (3), as desired.

Besides, arguing as in the proof of Proposition 4 in [4], we get

Proposition 4. *Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be two (k, m)-hyperbolic m-dimensional manifolds as in* Proposition 3 *such that is a finite-sheeted covering manifolds of X with covering projection $\pi : X \to Y$, every $u \in \mathcal{U}$ is well covered by π, and $\mathcal{V} = \pi^{-1}[\mathcal{U}]$. Let $z_0, z \in X$ and $S_0, S \in Y$ so that $\pi(s_0) = z_0$ and $\pi(s) = z$. Then*

$$\rho_X^\alpha(z_0, z)[\mathcal{U}] = \min\{\rho_Y^\alpha(s_0, s)[\mathcal{V}] : s \in Y,\ \pi(s_0) = z_0 \textit{ and } \pi(s) = z\}.$$

Since a (k, m)-hyperbolic n-dimensional complex manifold $(X, \mathcal{U})$ with a locally finite open covering $\mathcal{U}$ induces a locally finite open covering $\mathcal{U}'$ on any submanifold X' of X, we also have:

Proposition 5. *An n-dimensional complex submanifold $(X, \mathcal{U})$ of a complete hyperbolic n-dimensional complex manifold X is $(\alpha, \mathcal{U}')$-almost hyperbolic provided that $\mathcal{U}' = \mathcal{U} \cap X'$. If, in addition, X' is closed, it is also complete.*

Corollary. *The submanifold $X' = \{z \in X : f(z) = 0\}$ of an $(\alpha, \mathcal{U})$-almost hyperbolic manifold X, where f is a holomorphic function on X, is $(\alpha, \mathcal{U}')$-almost hyperbolic.*

Proof. It is sufficient to observe that the embedding of X' into X is proper holomorphic since X' is a closed submanifold of X, and to apply Proposition 1 of [1] in its modified version corresponding to (k,m)-hyperbolic manifolds.

The next step is to prove the following analogue of Theorem 4.10 in [6]:

Proposition 6. *Let X be an $(\alpha,\mathcal{U})$-almost hyperbolic manifold and f a holomorphic function on X. Then the open submanifold $X' = \{z \in X : f(z) \neq 0\}$ of X is $(\alpha,\mathcal{U}')$-almost hyperbolic manifold.*

Proof. It is easy to observe that X' is an open complex submanifold of X and thus, it is holomorphically embedded by a holomorphic inclusion mapping $i_X : X' \to X$. Next, by Proposition 3, we arrive at the statement.

References

[1] Boryczka, G. and L.M. Tovar, *Hyperbolic-like manifolds. Geometrical properties and holomorphic mappings*, in: Generalizations of Complex Analysis and Their Applications in Physics, ed. by J. Lawrynowicz, Banach Center Publications 37, PWN — Polish Scientific Publishers, Warsaw, 1996, pp. 53-66.

[2] Chern, S.S., H.I. Levine and L. Nirenberg, *Intrinsic norms on a complex manifold*, in: Global Analysis, Papers in honor of K. Kodaira, ed. by D.C. Spencer and S. Iynaga, Univ. of Tokyo Press and Princeton Univ. Press, Tokyo 1969, pp. 119-139; reprinted in S.S. Chern: Selected Papers, Springer, New York-Heidelberg-Berlin 1978, pp. 371-391.

[3] Dolbeault, P., *Sur les chaines maximalement complexes au bord donné*, Proc. Sympos. Pure Math. **44** (1986), 171-205.

[4] Dolbeault, P. and J. Lawrynowicz, *Holomorphic chains and extendability of holomorphic mappings*, in: Deformations of Mathematical Structures. Complex Analysis with Physical Applications. Selected papers from the Seminar on Deformations, Łódź-Lublin 1985/87. Edited by J.Lawrynowicz, Kluwer Academic Publishers, Dordrecht–Boston–London 1989, pp. 191-204.

[5] Eisenman, E. (now Pelles), *Intrinsic Measures on Complex Manifolds and Holomorphic Mappings*, Mem. Amer. Math. Soc. **96** (1970), ii + 80 pp.

[6] Kobayashi, S., *Hyperbolic Manifolds and Holomorphic Mappings* (Pure and Appl. Math. 2), Marcel Dekker, New York 1970, x + 148 pp.

[7] Pelles, D.A. (formerly Eisenman), *Holomorphic maps which preserve intrinsic measure*, Amer. J. Math. **97** (1975), 1-15.

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