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Integral means of derivatives of locally univalent Bloch functions

ABSTRACT. In this paper we give examples of locally univalent Bloch functions f_k, $(k = 0, 1, 2, \dots)$, such that for $p \geq 1/2$ the integral means $I_p(r, f_k)$ behave like $(1-r)^{1/2-p}(-\log(1-r))^k$ for $r \to 1^-$.

For a function $\varphi(z)$ analytic in the unit disk $\Delta = \{z : |z| < 1\}$ and $p > 0$, define its p*-integral mean* by the formula

$$I_p(r, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} |\varphi(re^{i\theta})|^p d\theta, \quad r \in (0,1).$$

There are many papers dealing with the integral means in various classes of functions. In particular asymptotic behaviour of integral means for $r \to 1-$ was investigated. For example, in the class S of functions $g(z) = z + \dots$ analytic and univalent in Δ sharp estimate $I_p(r, g') = O(\frac{1}{(1-r)^{3p-1}})$ for $p \geq 2/5$ ([F-MG]) was obtained. Since the derivative of functions in the class S satisfies sharp inequality $|g'(z)| \leq (1+|z|)(1-|z|)^{-3}$, $z \in \Delta$, the order of growth of the integral means of functions decreases by 1 as compared with the order of growth of the derivative of functions in S. A function f analytic in Δ belongs to the *Bloch class* $\mathcal{B}$, if it has a finite Bloch norm

$$||f||_{\mathcal{B}} = |f(0)| + \sup_{z \in \Delta} [(1 - |z|^2)|f'(z)|].$$

Hence the exact estimates

$$|f'(z)| = O((1-|z|)^{-1}), \quad |f(z)| = O(-\log(1-|z|)), \quad z \in \Delta,$$

follow. Also for Bloch functions the reduction of growth after integration on circles can be observed, (see [C-MG], [M]). In fact, for $f \in \mathcal{B}$ and $p > 0$ we have $I_p(r, f) = O((\log \frac{1}{1-|z|})^{p/2})$, as $r \to 1$. But for derivatives of Bloch functions have no similar property. In particular from Theorem 4 of [G] it follows, that there exists a function $f \in \mathcal{B}$ for which

$$I_p(r, f') \geq c^p(1-r)^{-p}, \quad 0 \leq r < 1, \quad p > 0;$$

where $c = c(f)$ is a constant.

Now, let us denote by $\mathcal{B}'$ the subclass of locally univalent functions in $\mathcal{B}$. Investigation of $I_p(r, f')$, $f \in \mathcal{B}'$, is motivated by the behaviour of Taylor coefficients of functions from $\mathcal{B}'$ ([P1], p.690).

In this paper we construct for every $k = 0, 1, 2, \ldots$ and every $p > 1$ examples of functions $F_k \in \mathcal{B}'$, such that

$$I_p(r, F_k') \geq \frac{c(k,p)}{(1-r)^{p-1/2}} \log^k \frac{1}{1-r}, \quad 1 > r \geq \rho_k(p) > 0,$$

where $c(k,p)$ is a constant independent of r. We will use the following two lemmas. Suppose $\mathcal{B}_M = \{f \in \mathcal{B} : \|f(z) - f(0)\|_{\mathcal{B}} \leq M\}$.

Lemma 1. *If $f \in \mathcal{B}_M$ and $\omega(z)$ is analytic in Δ with $|\omega(z)| < 1$ for $z \in \Delta$, then $F = f \circ \omega$ belongs to $\mathcal{B}_M$.*

Proof. By the Schwarz Lemma ([Gol], p. 319-320) we have

$$|\omega'(z)| \leq \frac{1-|\omega(z)|^2}{1-|z|^2} \quad \text{for } z \in \Delta.$$

Thus $|F'(z)|(1-|z|^2) \leq |f'(\omega(z))|(1-|\omega(z)|^2)$, i.e. $\|F(z) - F(0)\|_{\mathcal{B}} \leq \|f(z) - f(0)\|_{\mathcal{B}}$ and consequently $F \in \mathcal{B}_M$. □

Lemma 2. *Let $\Gamma = \{\Gamma(\theta) = r(\theta)e^{i\theta} : \theta \in [-\pi, \pi]\}$ be a closed, piecewise smooth curve contained in Δ, symmetric with respect to the real axis. Moreover, assume that $r(\theta) > 0$ increases on $[0, \pi]$ from r_0 to $r^0 > r_0$. If f is analytic in Δ with $|f(z)|(1-|z|^2) \leq 1$ in Δ, then for $\lambda > 1$*

$$(1) \qquad \int_\Gamma |f(z)|^\lambda |dz| \geq \frac{1}{\sqrt{2}} \Big[\int_{|z|=r_0} |f(z)|^\lambda |dz| - \frac{16}{\lambda-1}((1-r^0)^{1-\lambda} - (1-r_0)^{1-\lambda}) \Big],$$

and for $\lambda = 1$

$$\int_\Gamma |f(z)||dz| \geq \frac{1}{\sqrt{2}} \int_{|z|=r_0} |f(z)||dz| - 4\sqrt{2} r_0 \log \frac{1-r_0}{1-r^0}.$$

If $f(z) \neq 0$ *in* Δ, *then for* $\lambda \in (0,1)$

$$\int_\Gamma |f(z)|^\lambda |dz| \geq \frac{1}{\sqrt{2}} \int_{|z|=r_0} |f(z)|^\lambda |dz|$$

(1′)

$$- \frac{4\sqrt{2}(1+\lambda)}{\lambda(1-\lambda)} \left[(1-r_0)^{1-\lambda} - (1-r^0)^{1-\lambda}) - (1-\lambda)(r^0 - r_0)\right].$$

Proof. We may suppose that $r(\theta)$ increases on $[0,\pi]$. If $\theta \in [-\pi, 0]$, consider $\int_{-\Gamma} |f(-z)|^\lambda |dz|$, where the curve $-\Gamma$ has the parametrization $-\Gamma(\theta)$. Let us divide the interval $[-\pi,\pi]$ into $2n$ equal intervals $0 < \theta_0 < \theta_1 < \ldots < \theta_n = \pi$, $0 = \theta_0 > \theta_{-1} > \ldots > \theta_{-n} = -\pi$. Put $r_j = r(\theta_j)$, $j = -n, \ldots, n$; r_j is increasing with respect to $|j|$. Now let us consider the piecewise smooth curve $\Gamma^{(n)}$, which is the union of circular arcs $\{z = r_j e^{i\theta} : \theta \in [\theta_{j-1}, \theta_j]\}$, $j = -n+1, -n+2, \ldots, n$ and segments of radii $\{z = re^{i\theta_{j-1}} : r \in [r_{j-1}, r_j]\}$, $j = -n+1, -n+2, \ldots, n$. Put $\Delta\theta_j = \theta_j - \theta_{j-1}$, $\Delta r_j = |r_j - r_{j-1}|$, $z_j = r_j e^{i\theta_j}$, $j = -n+1, -n+2, \ldots, n$,

$$\Gamma_j = \{z \in \Gamma : z = r(\theta)e^{i\theta},\ \theta \in [\theta_{j-1}, \theta_j]\},$$

$$\Gamma_j^{(n)} = \{re^{i\theta} \in \Gamma^{(n)} : \theta \in [\theta_{j-1}, \theta_j]\}.$$

The length of the above curves $\Gamma, \Gamma^{(n)}, \Gamma_j, \Gamma_j^{(n)}$ will be denoted by the same symbols, respectively. The uniform continuity of $|f(z)|^\lambda$ in the disk $K = \{z : |z| \leq r^0\}$ implies for every $\varepsilon > 0$ the existence of $\eta = \eta(\varepsilon) > 0$, such that

$$||f(z')|^\lambda - |f(z'')|^\lambda| < \varepsilon \tag{2}$$

for every $z', z'' \in K$, $|z' - z''| < \eta$. Since $\sqrt{2}|d\Gamma(\theta)| \geq |dr(\theta)| + r(\theta)d\theta$ with $\theta \in [-\pi,\pi]$, we have for every fixed $\delta > 0$ and sufficiently large n

$$(\delta + \sqrt{2})\Gamma_j \geq \Delta r_j + r_j \Delta\theta_j = \Gamma_j^{(n)}, \quad j = -n+1, \ldots, n. \tag{3}$$

Then diameters of the curves Γ_j and $\Gamma_j^{(n)}$ will be less than η. Therefore by (2) and (3) we obtain

$$(\delta + \sqrt{2}) \int_\Gamma |f(z)|^\lambda |dz| - \int_{\Gamma^{(n)}} |f(z)|^\lambda |dz|$$

$$= \sum_{j=1-n}^{n} \left[(\delta + \sqrt{2}) \int_{\Gamma_j} |f(z)|^\lambda |dz| - \int_{\Gamma_j^{(n)}} |f(z)|^\lambda |dz|\right]$$

$$= \sum_{j=1-n}^{n} \Big[(\delta + \sqrt{2}) \int_{\Gamma_j} (|f(z)|^\lambda - |f(z_j)|^\lambda)|dz| - \int_{\Gamma_j^{(n)}} (|f(z)|^\lambda$$

$$- |f(z_j)|^\lambda)|dz| + (\delta + \sqrt{2})|f(z_j)|^\lambda \Gamma_j - |f(z_j)|^\lambda \Gamma_j^{(n)} \Big]$$

$$\geq -\varepsilon[(\sqrt{2} + \delta)\Gamma + \Gamma^{(n)}].$$

The number ε can be chosen so that the last expression will be greater than $-\delta(\sqrt{2} - 1) \int_\Gamma |f(z)|^\lambda |dz|$. Thus

$$\sqrt{2}(\delta + 1) \int_\Gamma |f(z)|^\lambda |dz| \geq \int_{\Gamma^{(n)}} |f(z)|^\lambda |dz|. \tag{4}$$

For the parameter $t \in [0, 1]$ let us consider a family of curves

$$\Gamma(n,t) = \{tz : z \in \Gamma^{(n)}\}, \quad \Gamma(n,1) = \Gamma^{(n)}, \quad \Gamma(n,0) = 0.$$

Then

$$\int_{\Gamma(n,t)} |f(z)|^\lambda |dz| = t \sum_{j=1-n}^{n} \Big(\int_{\theta_{j-1}}^{\theta_j} |f(tr_j e^{i\theta})|^\lambda r_j d\theta$$

$$+ \frac{1}{t} \int_{tr_{j-1}}^{tr_j} |f(re^{i\theta_{j-1}})|^\lambda |dr| \Big)$$

$$\geq tr_0 \sum_{j=1-n}^{n} \Big(\int_{\theta_{j-1}}^{\theta_j} |f(tr_j e^{i\theta})|^\lambda r_j d\theta + \frac{1}{tr_0} \int_{tr_{j-1}}^{tr_j} |f(re^{i\theta_{j-1}})|^\lambda |dr| \Big)$$

(5)

$$= tr_0 \times \sum_{j=1-n}^{n} \Big(\int_{\theta_{j-1}}^{\theta_j} |f(tr_j e^{i\theta})|^\lambda d\theta$$

$$- \int_0^t \frac{\lambda}{\tau} \Big[\int_{\tau r_{j-1}}^{\tau r_j} |f(re^{i\theta_{j-1}})|^{\lambda-1} \frac{\partial |f|}{\partial \theta}(re^{i\theta_{j-1}}) \frac{|dr|}{r} \Big] d\tau \Big)$$

$$+ tr_0 \sum_{j=1-n}^{n} \int_0^t \frac{\lambda}{\tau} \Big[\int_{\tau r_{j-1}}^{\tau r_j} |f(re^{i\theta_{j-1}})|^{\lambda-1} \frac{\partial |f|}{\partial \theta}(re^{i\theta_{j-1}}) \frac{|dr|}{r} \Big] d\tau$$

$$+ \sum_{j=1-n}^{n} \int_{tr_{j-1}}^{tr_j} |f(re^{i\theta_{j-1}})|^\lambda |dr|.$$

The first of the last three sums should be denoted by $I(t)$ and the components of the second and third sums for $t = 1$ by B_j and A_j, respectively.

Then

(6)
$$I(t)=\int_0^t \frac{\lambda}{\tau}\sum_{J=1-n}^{n}\left[\int_{\theta_{j-1}}^{\theta_j}|f(\tau r_j e^{i\theta})|^{\lambda-1}\frac{\partial|f|}{\partial r}(\tau r_j e^{i\theta})\tau r_j d\theta \right.$$
$$\left. -\int_{\tau r_{j-1}}^{\tau r_j}|f(re^{i\theta_{j-1}})|^{\lambda-1}\frac{\partial|f|}{\partial\theta}(re^{i\theta_{j-1}})\frac{|dr|}{r}\right]d\tau.$$

If $f\equiv 0$ then the lemma holds. Suppose f is not identically zero. The function f may have a finite set of zeros on the disk K. One can assume that for fixed n there exists a finite family of curves $\Gamma(n,t)$, containing those zeros. Otherwise instead of f one can consider $f(ze^{i\gamma})$ with small $\gamma\in\mathbb{R}$. Next let us consider such $t\in[0,1]$ that the curves $\Gamma(n,t)$ do not contain zeros of f. For $z=re^{i\theta}\in\Gamma(n,t)$ let $\Phi(z)=\arg f(z)$. By the Cauchy-Riemann equations we have

$$r\frac{\partial|f|}{\partial r}=|f|\frac{\partial\Phi}{\partial\theta},\qquad r|f|\frac{\partial\Phi}{\partial r}=-\frac{\partial|f|}{\partial\theta}.$$

Thus by (6) we obtain

$$I'(t)=\frac{\lambda}{t}\sum_{J=1-n}^{n}\left[\int_{\theta_{j-1}}^{\theta_j}|f(tr_je^{i\theta})|^{\lambda}d\Phi(tr_je^{i\theta})\right.$$
$$\left.+\int_{\tau r_{j-1}}^{\tau r_j}|f(re^{i\theta_{j-1}})|^{\lambda}d\Phi(re^{i\theta_{j-1}})\right]=\frac{\lambda}{t}\int_a^b|f(\gamma(\xi))|^{\lambda}d\Phi(\gamma(\xi)),$$

where $\gamma(\xi)$, $\xi\in[a,b]$, is a piecewise parametrization of the curve $\Gamma(n,t)$ which gives the positive orientation on $\Gamma(n,t)$. Let

$$L=L(\xi)=x(\xi)+iy(\xi)=|f(\gamma(\xi))|^{\lambda/2}e^{i\Phi(\gamma(\xi))}.$$

Then

$$x(\xi)dy(\xi)-y(\xi)dx(\xi)=|f(\gamma(\xi))|^{\lambda}d\Phi(\gamma(\xi))$$

and the Green formula implies

$$I'(t)=\frac{\lambda}{t}\int_L xdy-ydx=\frac{2\lambda}{t}S(n,t),$$

where $S(n,t)$ is the area of the image (generally many sheeted) of the compact set with the boundary $\Gamma(n,t)$ under the function

(7)
$$\begin{cases}|f(z)|^{\lambda/2}e^{i\Phi(z)}, & f(z)\neq 0,\\ 0, & f(z)=0.\end{cases}$$

Now, let

$$r_0 t\mathcal{I}(t) = r_0 t\int_{-\pi}^{\pi} |f(r_0 te^{i\theta})|^\lambda d\theta = \int_{|z|=r_0 t} |f(z)|^\lambda |dz|.$$

Then we get

$$\begin{aligned}\mathcal{I}'(t) &= \lambda\int_{-\pi}^{\pi} |f(r_0 te^{i\theta})|^{\lambda-1}\frac{\partial|f|}{\partial r}(r_0 te^{i\theta})r_0 d\theta\\ &= \frac{\lambda}{t}\int_{-\pi}^{\pi} |f(r_0 te^{i\theta})|^\lambda d\Phi(r_0 te^{i\theta}) = \frac{2\lambda}{t}S(r_0 t),\end{aligned}$$

where $S(r_0 t)$ is the area of the image of the disk $\{z : |z| \le r_0 t\}$ under the function (7). Thus the inequality $I'(t) \ge \mathcal{I}'(t)$ holds for all $t \in [0,1]$, possibly except for a finite set of t. Therefore by continuity of $I(t)$ and $\mathcal{I}(t)$ in $[0,1]$ we obtain $I(1) - I(0) \ge \mathcal{I}(1) - \mathcal{I}(0)$. But $\mathcal{I}(0) = I(0) = 2\pi|f(0)|^\lambda$, because for sufficiently small r the quantity $|f(re^{i\theta})|^\lambda|\frac{\partial\Phi}{\partial r}(re^{i\theta})|$ is bounded by a constant C. Thus by the Cauchy-Riemann equations

$$\left|\int_0^t \frac{1}{\tau}\int_{\tau r_{j-1}}^{\tau r_j} |f(re^{i\theta_{j-1}})|^{\lambda-1}\frac{\partial|f|}{\partial\theta}(re^{i\theta_{j-1}})\frac{dr}{r}d\tau\right|$$

$$= \left|\int_0^t \frac{1}{\tau}\int_{\tau r_{j-1}}^{\tau r_j} |f(re^{i\theta_{j-1}})|^\lambda\frac{\partial\Phi}{\partial r}(re^{i\theta_{j-1}})drd\tau\right| \le C\int_0^t (r_j - r_{j-1})d\tau \to 0,$$

as $t \to 0$. Consequently $I(1) \ge \mathcal{I}(1)$. Then

(8)

$$\begin{aligned}\int_{\Gamma^{(n)}} |f(z)|^\lambda |dz| &\ge r_0 I(1) + \sum_{j=1-n}^{n}(A_j + B_j)\\ &\ge r_0\mathcal{I}(1) + \sum_{j=1-n}^{n} B_j = \int_{|z|=r_0} |f(z)|^\lambda|dz| + \sum_{j=1-n}^{n} B_j.\end{aligned}$$

Now, observe that

$$\left|\frac{\partial|f|}{\partial\theta}\right| = \left|\frac{\partial\exp(\mathrm{Re}\ \log f)}{\partial\theta}\right| \le |zf'(z)| \le \frac{4|z|}{(1-|z|^2)^2}$$

(cf. [W]). Thus, in order to obtain an estimate of $|B_j|$ we deal with

$$B = r_0\int_0^1 \frac{\lambda}{\tau}\int_{\tau\rho_1}^{\tau\rho_2} |f(re^{i\theta_{j-1}})|^{\lambda-1}|f'(re^{i\theta_{j-1}})|drd\tau, \quad 0 < \rho_1 < \rho_2 < 1.$$

From our assumptions we obtain

$$|B| \le r_0 \int_0^1 \frac{\lambda}{\tau} \int_{\tau\rho_1}^{\tau\rho_2} \frac{4}{(1-r)^{\lambda+1}} dr d\tau$$
$$= 4r_0 \int_0^1 \frac{1}{t} \left[\frac{1}{(1-t\rho_2)^\lambda} - \frac{1}{(1-t\rho_1)^\lambda} \right] dt.$$

Now, let $\varphi(t)$ be the function appearing in the last integral. We have

$$\varphi(t) = \lambda(\rho_2 - \rho_1) + \frac{\lambda(\lambda+1)}{2}(\rho_2^2 - \rho_1^2)t$$
$$+ \frac{\lambda(\lambda+1)(\lambda+2)}{3!}(\rho_2^3 - \rho_1^3)t^2 + \dots .$$

Since the radius of convergence is greater than 1, we obtain

$$\int_0^1 \varphi(t)dt = \lambda(\rho_2 - \rho_1) + \dots + \frac{\lambda(\lambda+1)\dots(\lambda+k-1)}{k!} \frac{\rho_2^k - \rho_1^k}{k} + \dots .$$

However,

$$\frac{\rho_2^k - \rho_1^k}{k} = \frac{\rho_2^k - \rho_1^k}{k+1} \frac{k+1}{k} \le \frac{2}{k+1} \frac{\rho_2^{k+1} - \rho_1^{k+1}}{\rho_2}$$

and hence for $\lambda > 1$ we get

$$\int_0^1 \varphi(t)dt \le \frac{2}{\rho_2(\lambda-1)}$$
$$\times \left[\frac{(\lambda-1)\lambda}{2!}(\rho_2^2 - \rho_1^2) + \dots + \frac{(\lambda-1)\lambda\dots(\lambda+k-1)}{(k+1)!}(\rho_2^{k+1} - \rho_1^{k+1}) + \dots \right]$$
$$= \frac{2}{\rho_2(\lambda-1)} \left[((1-\rho_2)^{1-\lambda} - 1 - (\lambda-1)\rho_2) - ((1-\rho_1)^{1-\lambda} - 1 - (\lambda-1)\rho_1) \right]$$
$$\le \frac{2}{\rho_2(\lambda-1)}((1-\rho_2)^{1-\lambda} - (1-\rho_1)^{1-\lambda}),$$

so that $|B| \le 4r_0 \int_0^1 \varphi(t)dt \le \frac{8r_0}{\rho_2(\lambda-1)}((1-\rho_2)^{1-\lambda} - (1-\rho_1)^{1-\lambda})$. Thus

$$\left| \sum_{j=1-n}^{n} B_j \right| \le \sum_{j=1-n}^{n} |B_j| \le \frac{16}{\lambda-1} \sum_{j=1}^{n} ((1-r_j)^{1-\lambda} - (1-r_{j-1})^{1-\lambda})$$
$$= \frac{16}{\lambda-1}((1-r^0)^{1-\lambda} - (1-r_0)^{1-\lambda})$$

and by (8) we have

$$\int_{\Gamma^{(n)}} |f(z)|^\lambda |dz| \geq \int_{|z|=r_0} |f(z)|^\lambda |dz| - \frac{16}{\lambda-1}((1-r^0)^{1-\lambda} - (1-r_0)^{1-\lambda}).$$

Then from (4) we obtain

$$\int_\Gamma |f(z)|^\lambda |dz| \geq \frac{1}{\sqrt{2}(\delta+1)} \times \left[\int_{|z|=r_0} |f(z)|^\lambda |dz| - \frac{16}{\lambda-1}((1-r^0)^{1-\lambda} - (1-r_0)^{1-\lambda})\right]$$

Since δ is any positive number, we get our Lemma for $\lambda > 1$. If $\lambda = 1$ then

$$\int_0^1 \varphi(t)dt = \log\frac{1-\rho_1}{1-\rho_2}, \quad B \leq 4r_0 \log\frac{1-\rho_1}{1-\rho_2}.$$

Thus

$$|\sum_{j=1-n}^{n} B_j| \leq 8r_0 \log\frac{1-r_0}{1-r^0}$$

and

$$\int_\Gamma |f(z)||dz| \geq \frac{1}{\sqrt{2}} \int_{|z|=r_0} |f(z)||dz| - 4\sqrt{2}r_0 \log\frac{1-r_0}{1-r^0}.$$

Now, let $\lambda \in (0,1)$ and $f(z) \neq 0$ in Δ. Then the function $f_\lambda(z) = f^\lambda(z)$ is analytic in Δ and $|f_\lambda(z)|(1-|z|^2)^\lambda \leq 1$. For such functions $f_\lambda(z)$ K. J. Wirths ([W]) showed that

$$|f'_\lambda(z)|(1-|z|^2)^{\lambda+1} \leq 2(\lambda+1).$$

Therefore

$$\begin{aligned} B &= r_0 \int_0^1 \frac{1}{\tau} \int_{\tau\rho_1}^{\tau\rho_2} |f'_\lambda(re^{i\theta_{j-1}}|drd\tau \\ &\leq 2r_0(\lambda+1) \int_0^1 \frac{1}{\tau} \int_{\tau\rho_1}^{\tau\rho_2} \frac{drd\tau}{(1-r)^{\lambda+1}} \\ &= \frac{2r_0(\lambda+1)}{\lambda} \int_0^1 \frac{1}{t}[(1-t\rho_2)^{-\lambda} - (1-t\rho_1)^{-\lambda}]dt. \end{aligned}$$

As in the case $\lambda > 1$ we estimate the last integral by

$$\frac{2}{\rho_2(1-\lambda)}((1-\rho_1)^{1-\lambda} - (1-\rho_2)^{1-\lambda} + (1-\lambda)(\rho_1 - \rho_2)),$$

i.e.

$$B \leq \frac{4r_0(1+\lambda)}{\rho_2\lambda(1-\lambda)}((1-\rho_1)^{1-\lambda} - (1-\rho_2)^{1-\lambda} + (1-\lambda)(\rho_1 - \rho_2)).$$

Thus

$$\left|\sum_{j=1-n}^{n} B_j\right| \leq \frac{8(1+\lambda)}{\lambda(1-\lambda)}((1-r_0)^{1-\lambda} - (1-r^0)^{1-\lambda} - (1-\lambda)(r^0 - r_0)).$$

Then by (4) and (8) we obtain

$$\int_\Gamma |f(z)|^\lambda |dz| \geq \frac{1}{\sqrt{2}(\delta+1)}\left[\int_{|z|=r_0} |f(z)|^\lambda |dz| - \frac{8(1+\lambda)}{\lambda(1-\lambda)}((1-r_0)^{1-\lambda} - (1-r^0)^{1-\lambda} - (1-\lambda)(r^0-r_0))\right].$$

Since δ is an arbitrary positive number, we get our Lemma for $\lambda \in (0,1)$. □

Remark. Lemma 2 holds also for monotonic $r(\theta)$ in $[\theta_0, \theta^0]$ and $[\theta^0, \theta_0 + 2\pi]$. It can be generalized for a piecewise monotonic and continuous function $r(\theta)$. In the case $\lambda > 1$ the coefficient $16/(\lambda - 1)$ from Lemma must be replaced by $8k/(\lambda-1)$. Similarly we can consider the case $\lambda \in (0,1]$.

Let us now consider $f(z) = \log(1-z) \in \mathcal{B}_2$ and $\omega(z) = \exp\left(-\pi\frac{1+z}{1-z}\right)$. Since $|\omega| < 1$ in Δ, one can define functions

(9) $$F_0 = f \circ \omega, \; F_k = F_{k-1} \circ \omega, \qquad k \in \mathrm{N},$$

analytic in Δ.

Theorem. *The functions F_k defined by (9) belong to $\mathcal{B}_2 \cap \mathcal{B}'$. Moreover, the inequality*

$$I_p(r, F_k') \geq \frac{c(k,p)}{(1-r^2)^{p-1/2}} \log^k \frac{1}{1-r^2} \quad \textit{for } 0 \leq \rho_k(p) < 1,$$

holds for every $k = 0, 1, 2, \ldots$ and every $p > 1/2$ with the constants $c(k,p)$ defined as follows.
If $p > 1$ then

$$c(0,p) = \frac{ce^{-\pi p}}{2\pi 10^{p-1}}\left(\frac{2}{5}\right)^{p-1/2},$$

where $0 < c = c(p) = \inf_{r\in[0,1)}[(1-r)^{1-p}\int_0^{2\pi}|1-re^{it}|^{-p}dt]$, *and*

$$c(k,p) = \frac{c(0,p)}{k!(2^{(k+3)/2}\sqrt{\pi}10^p)^k}, \quad \rho_0(p) = 1/\sqrt{2}.$$

If $p \in (1/2, 1]$ *then*

$$c(0,p) = \frac{c(p)e^{-\pi}}{3(2\pi)^{2-p}(2p-1)}$$

with $c(p) = \inf_{r\in[0,1)}\int_0^{2\pi}\frac{dt}{|1-re^{it}|^p} > 0$ *and*

$$c(k,p) = \frac{c(0,p)}{(10\sqrt{\pi})^k k!}.$$

For $p = 1/2$ *we have*

$$I_{1/2}(r, F_k') \geq \frac{c(0,1/2)}{(10\sqrt{\pi})^k(k+1)!}\log^{k+1}\frac{1}{1-r^2},$$

where $c(0,1/2)$ *is given by the same formula as in the case* $p \in (1/2, 1]$.

Proof. From the definition of F_k it follows that $F_k \in \mathcal{B}'$. By Lemma 1 we get $F_k \in \mathcal{B}_2$ for every k, since $\log(1-z) \in \mathcal{B}_2$.

For positive integers N consider the sequence $r_N = \frac{N}{\sqrt{N^2+1}} \xrightarrow[N\to\infty]{} 1$. Put $\delta_N = \arccos r_N$. Then

$$\operatorname{Re}\frac{1+r_Ne^{i\delta_N}}{1-r_Ne^{i\delta_N}} = \frac{1-r_N^2}{1-2r_N\cos\delta_N + r_N^2} = 1,$$

$$\operatorname{Im}\frac{1+r_Ne^{i\delta_N}}{1-r_Ne^{i\delta_N}} = \frac{2r_N\sin\delta_N}{1-2r_N\cos\delta_N+r_N^2} = \frac{2r_N\sqrt{1-r_N^2}}{1-r_N^2} = \frac{2r_N}{\sqrt{1-r_N^2}} = 2N.$$

Now let $\delta_m \in [0,\pi]$ be a solution of the equation

$$\operatorname{Im}\frac{1+r_Ne^{i\delta_m}}{1-r_Ne^{i\delta_m}} = \frac{2r_N\sin\delta_m}{1-2r_N\cos\delta_m+r_N^2} = 2m,$$

where $m \in [0, N]$ is an integer. Setting $\gamma = \cos\delta_m$ we obtain a quadratic equation $\gamma^2(4r_N^2m^2 + r_N^2) - 4m^2r_N(1+r_N^2)\gamma + m^2(1+r_N^2)^2 - r_N^2 = 0$. Hence

$$\gamma = \cos\delta_m = \frac{2m^2}{1+4m^2}\frac{1+2N^2}{N\sqrt{1+N^2}} - \frac{1}{1+4m^2}\sqrt{1-\frac{m^2}{N^2(N^2+1)}}.$$

Let us introduce the expression

$$x_m = \operatorname{Re}\frac{1+r_N e^{i\delta_m}}{1-r_N e^{i\delta_m}} = \frac{1-r_N^2}{1-2r_N\cos\delta_m + r_N^2}$$
$$= \frac{4m^2+1}{2N^2+1+\sqrt{(2N^2+1)^2-4m^2-1}}.$$

First consider the case $p > 1$ and use the induction with respect to $k = 0,1,2,\dots$.
a) For $k = 0$ we have

$$I_p(r_N, F_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |F_0'(r_N e^{it})|^p dt = \frac{1}{\pi}\int_0^{\pi} |F_0'(r_N e^{it})|^p dt$$

$$\geq \frac{1}{\pi}\sum_{m=N}^{1}\int_{\delta_m}^{\delta_{m-1}} |f'[\omega(r_N e^{it})]|^p |\omega(r_n e^{it})|^p \left|\frac{2\pi}{(1-r_N e^{it})^2}\right|^p dt.$$

Note that for $t \in [\delta_m, \delta_{m-1}]$ we have $|1-r_N e^{it}| \leq |1-r_N e^{i\delta_{m-1}}|$ and

$$R(t) = |\omega(r_N e^{it})| \geq R_m = e^{-\pi x_m}.$$

Moreover,

$$|1-\omega(r_N e^{it})| = |1-R(t)e^{i\theta(t)}| \leq |1-R_m e^{i\theta(t)}| + (R(t)-R_m)$$
$$\leq |1-R_m e^{i\theta}| + (1-R_m) \leq 2|1-R_m e^{i\theta}|.$$

The interval $[\delta_m, \delta_{m-1}]$ is mapped by $\omega(r_N e^{it})$ onto one branch of the spiral $\omega = R(t)e^{i\theta(t)} = \rho(\theta)e^{i\theta}$, $\theta \in [-2\pi m, -2\pi(m-1)]$ and $\rho(\theta)$ increases from R_m to R_{m-1}. The element of length $|d\omega| = |d(\rho(\theta)e^{i\theta})|$ of the spiral is not less than the element of length $|d(R_m e^{i\theta})|$ of the circle $\{|\omega| = R_m\}$. In this way we get

$$r_N I_p(r_N, F_0) \geq \frac{1}{\pi}\sum_{m=N}^{1}\frac{(2\pi)^{p-1}R_m^{p-1}}{|1-r_N e^{i\delta_{m-1}}|^{2(p-1)}2^p}\int_{|\omega|=R_m}\frac{|d\omega|}{|1-\omega|^p}.$$

Since for $p > 1$ (cf.e.g. [MOS], p. 157)

$$u(r) = (1-r)^{p-1}\int_0^{2\pi}\frac{dt}{|1-re^{it}|^p} \underset{r\to 1}{\longrightarrow} \sqrt{\pi}\,\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{p}{2}\right),$$

this means that the function $u(r)$ is positive and continuous on $[0,1]$ with $u(1) = \sqrt{\pi}\,\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{p}{2}\right)$.

Therefore $u(r) \geq c > 0$ for $r \in [0,1]$ and consequently

$$r_N I_p(r_N, F_0) \geq \frac{c\pi^{p-2}}{2} \sum_{m=N}^{1} \frac{R_m^p}{|1 - r_N e^{i\delta_{m-1}}|^{2(p-1)}(1-R_m)^{p-1}}.$$

For any integer $m \in [0, N]$

$$\frac{x_{m-1}}{x_m} = \frac{4(m-1)^2+1}{4m^2+1} \frac{2N^2+1+\sqrt{(2N^2+1)^2-(4m^2+1)}}{2N^2+1+\sqrt{(2N^2+1)^2-4(m-1)^2-1}}$$

$$= \left(1 - \frac{8m-4}{4m^2+1}\right) \frac{1+\sqrt{1-\frac{4m^2+1}{(2N^2+1)^2}}}{1+\sqrt{1-\frac{4(m-1)^2+1}{(2N^2+1)^2}}} \geq \left(1 - \frac{8m-4}{4m^2+1}\right)\frac{1}{2} > \frac{1}{10}.$$

Thus

$$\frac{1}{|1 - r_N e^{i\delta_{m-1}}|^2} = \frac{x_{m-1}}{1-r_N^2} \geq \frac{x_m}{1-r_N^2}\frac{1}{10} \tag{10}$$

and

$$r_N I_p(r_N, F_0) \geq \frac{c\pi^{p-2}}{2 10^{p-1}} \frac{1}{(1-r_N^2)^{p-1}} \sum_{m=1}^{N} \frac{R_m^p x_m^{p-1}}{(1-R_m)^{p-1}}.$$

Because $x_m \in [0,1]$, we have $R_m \geq e^{-\pi}$, $1 - e^{-\pi x_m} \leq \pi x_m$. Thus

$$r_N I_p(r_N, F_0) \geq \frac{c\pi^{p-2} e^{-\pi p}}{2 10^{p-1} \pi^{p-1}} \frac{N}{(1-r_N^2)^{p-1}} = \frac{c e^{-\pi p}}{2\pi 10^{p-1}} \frac{r_N}{(1-r_N^2)^{p-1/2}}.$$

The integral means $I_p(r, \varphi)$ are increasing with respect to $r \in [0,1)$ for every function φ analytic in Δ (cf.e.g. [H], Theorem 3.1). Therefore $I_p(r, F_0) \geq I_p(r_N, F_0)$ for $r \in [r_N, r_{N+1}]$. Thus for $r \in [r_N, r_{N+1}]$

$$\begin{aligned} I_p(r, F_0) &\geq \frac{c e^{-\pi p}}{2\pi 10^{p-1}} \frac{1}{(1-r^2)^{p-1/2}} \left(\frac{1-r_{N+1}^2}{1-r_N^2}\right)^{p-1/2} \\ &\geq \frac{c e^{-\pi p}}{2\pi 10^{p-1}} \left(\frac{2}{5}\right)^{p-1/2} \frac{1}{(1-r^2)^{p-1/2}} \end{aligned} \tag{11}$$

for $N \geq 1$. Since N is arbitrary, the inequality (11) holds for $r \in [1/\sqrt{2}, 1)$.

b) Now, suppose that the theorem holds for any fixed positive integer $k \geq 0$, i.e.

$$I_p(r, F_k') \geq \frac{c_k}{(1-r^2)^{p-1/2}} \log^k \frac{1}{1-r^2} \quad \text{for } 1 > r \geq \rho_k \in (0,1). \tag{12}$$

We show that it holds for $k+1$. For $m = 1, \dots, N$ write

$$L_m = \{\omega(r_N e^{it}) : t \in [\delta_m, \delta_{m-1}]\},$$
$$L_{-m} = \{\omega(r_N e^{it}) : t \in [-\delta_m, -\delta_{m-1}]\},$$

where L_m is a spiral-like curve which winds once around the point $z = 0$. For $t \in [\delta_m, \delta_{m-1}]$ the quantity $|\omega(r_N e^{it})|$ increases with respect to t. L_{-m} is a curve symmetric to L_m with respect to the real axis. Therefore for every $m = 1, \dots, N$ the curve $L_m \cup L_{-m}$ may be represented as a union of two piecewise smooth closed curves $\Gamma_m \cup \Gamma'_m$, where Γ_m consists of the upper part of L_m and the lower part of L_{-m}, and $\Gamma'_m = L_m \cup L_{-m} \setminus \Gamma_m$. Both curves Γ_m and Γ'_m fulfil the assumptions of Lemma 2 with $r_0 \geq R_m$ and $r^0 \leq R_{m-1}$. Thus by (10) and (1) we obtain

$$r_n I_p(r_N, F'_{k+1}) \geq \frac{r_N}{2\pi} \sum_{m=N}^{1-N} \int_{\delta_m}^{\delta_{m-1}} |F'_{k+1}(r_N e^{it})|^p dt$$

$$= \frac{1}{2\pi} \sum_{m=N}^{1-N} \int_{\delta_m}^{\delta_{m-1}} |F'_k[\omega(r_N e^{it})]|^p |\omega'(r_N e^{it})|^{p-1} |d\omega(r_N e^{it})|$$

$$\geq \frac{1}{2\pi} \sum_{m=1}^{N} \frac{(2\pi R_m)^{p-1}}{|1 - r_N e^{i\delta_{m-1}}|^{2(p-1)}} \int_{L_m \cup L_{-m}} |F'_k(\omega)|^p |d\omega|$$

$$\geq \frac{(2\pi)^{p-2}}{10^{p-1}(1-r_N^2)^{p-1}} \sum_{m=1}^{N} (x_m R_m)^{p-1} \int_{\Gamma_m \cup \Gamma'_m} |F'_k(\omega)|^p |d\omega|$$

$$\geq \frac{(2\pi)^{p-2}\sqrt{2}}{10^{p-1}(1-r_N^2)^{p-1}} \times \sum_{m=1}^{N} (x_m R_m)^{p-1} \Big[2\pi R_m I_p(R_m, F'_k) - \frac{2^{p+4}}{p-1} \left((1-R_{m-1})^{1-p} - (1-R_m)^{1-p} \right) \Big],$$

since by Lemma 1 the functions F_k belong to $\mathcal{B}_2$, i.e. $|F'_k(z)|(1-|z|^2) \leq 2$ for $z \in \Delta$.

Because $\frac{x_{m-1}}{x_m} > \frac{1}{10}$ for integers $m \in [0, N]$, we have

$$x_m < 10 x_{m-1} \Longrightarrow \pi(x_m - x_{m-1}) < 9\pi x_{m-1} < 9(e^{\pi x_{m-1}} - 1)$$
$$\Longrightarrow R_{m-1}\pi(x_m - x_{m-1}) < 9(1 - R_{m-1})$$
$$\Longleftrightarrow \frac{1 - R_{m-1}(1 - \pi(x_m - x_{m-1}))}{1 - R_{m-1}} < 10$$
$$\Longrightarrow \frac{1 - R_{m-1} e^{-\pi(x_m - x_{m-1})}}{1 - R_m - 1} < 10 \Longleftrightarrow \frac{1 - R_m}{1 - R_{m-1}} < 10.$$

Therefore (see (12)) for $R_m \in (\rho_k, 1)$

$$(13) \qquad 2\pi R_m I_p(R_m, F'_k) - \frac{2^{p+4}}{p-1}((1-R_{m-1})^{1-p} - (1-R_m)^{1-p})$$
$$> \frac{2\pi c_k R_m}{(1-R_m^2)^{p-1/2}} \log^k \frac{1}{1-R_m^2} - \frac{2^{p+4}}{p-1}((1-R_{m-1})^{1-p} - (1-R_m)^{1-p})$$
$$= (1-R_m^2)^{1-p}\left[\frac{2\pi c_k R_m}{\sqrt{1-R_m^2}} \log^k \frac{1}{1-R_m^2} - \frac{2^{p+4}}{p-1}\left(\left(\frac{1-R_m}{1-R_{m-1}}\right)^{p-1} - 1\right)\right]$$
$$> (1-R_m^2)^{1-p}\left[\frac{2\pi c_k R_m}{\sqrt{1-R_m^2}} \log^k \frac{1}{1-R_m^2} - \frac{2^{p+4}}{p-1} 10^{p-1}\right]$$
$$> \frac{\pi c_k R_m \log^k \frac{1}{1-R_m^2}}{(1-R_m^2)^{p-1/2}}$$

for R_m sufficiently close to 1, i.e. for $R_m > 1 - \varepsilon_k \geq \rho_k$, $\varepsilon_k \in (0,1)$.

$$R_m > 1 - \varepsilon_k \iff x_m < \frac{1}{\pi} \log \frac{1}{1-\varepsilon_k} = 2\eta_k^2 \ (0 < \eta_k < 1)$$
$$\iff \frac{4m^2+1}{2N^2+1+\sqrt{(2N^2+1)^2 - 4m^2 - 1}} \leq 2\eta_k^2$$
$$\iff 4m^2 + 1 \leq (2N^2+1)4\eta_k^2 - 4\eta_k^4.$$

The last condition holds for $m \leq N\eta_k$, with $N > 1/(2\eta_k)$. Now, suppose that N is sufficiently large ($N \geq 2/\eta_k^2$). Then the inequality (13) holds for $1 \leq m \leq N\eta_k$ and for $N \geq 2/\eta_k^2$

$$r_N I_p(r_N, F'_{k+1}) \geq \frac{\pi^{p-1} c_k}{\sqrt{2}5^{p-1}(1-r_N^2)^{p-1}} \sum_{m=1}^{N\eta_k} \frac{x_m^{p-1} R_m^p}{(1-R_m^2)^{p-1/2}} \log^k \frac{1}{1-R_m^2}.$$

As stated above, $1 - R_m^2 \leq 2\pi x_m$ for every m. Moreover, $R_m > 1 - \varepsilon_k$ for $m \in [1, N\eta_k]$. Consequently

$$r_N I_p(r_N, F'_{k+1}) \geq \frac{c_k(1-\varepsilon_k)^p}{2\sqrt{\pi}10^{p-1}(1-r_N^2)^{p-1}} \sum_{m=1}^{N\eta_k} \frac{1}{\sqrt{x_m}} \log^k \frac{1}{2\pi x_m}.$$

Since x_m increases with respect to m, each term in the last sum decreases with respect to m (we can assume that η_k is sufficiently small and then $4\pi x_m < 1$). Therefore

$$r_N I_p(r_N, F'_{k+1}) \geq \frac{c_k(1-\varepsilon_k)^p}{2\sqrt{\pi}10^{p-1}(1-r_N^2)^{p-1}} \int_1^{N\eta_k} \frac{1}{\sqrt{x_m}} \log^k \frac{1}{2\pi x_m} dm.$$

The change of variables in the integral

$$(14)\qquad \begin{aligned} x_m &= \frac{(2N^2+1)u}{1+\sqrt{1-u}},\\ u &= \frac{4m^2+1}{(2N^2+1)^2} \in \Big[\frac{5}{(2N^2+1)^2}, \frac{4(N\eta_k)^2+1}{(2N^2+1)^2}\Big] = [A, B] \end{aligned}$$

yields $2m = \sqrt{(2N^2+1)^2u - 1} \le (2N^2+1)\sqrt{u}$, and $dm = \frac{(2N^2+1)^2}{8m}du \ge \frac{2N^2+1}{4\sqrt{u}}du$.

Consequently

$$\begin{aligned} \int_1^{N\eta_k} & \frac{1}{\sqrt{x_m}} \log^k \frac{1}{2\pi x_m} dm \\ &\ge \int_A^B \frac{\sqrt{1+\sqrt{1-u}}}{\sqrt{(2N^2+1)u}} \frac{2N^2+1}{4\sqrt{u}} \log^k \frac{1+\sqrt{1-u}}{2\pi(2N^2+1)u} du \\ &\ge \frac{\sqrt{2N^2+1}}{4} \int_A^B \log^k \frac{1}{2\pi(2N^2+1)u} \frac{du}{u} \\ &= \frac{\sqrt{2N^2+1}}{4(k+1)} \log^{k+1} \frac{1}{2\pi(2N^2+1)u} \Big|_{u=B}^{u=A} \\ &= \frac{\sqrt{2N^2+1}}{4(k+1)} \Big[\log^{k+1} \frac{2N^2+1}{10\pi} - \log^{k+1} \frac{2N^2+1}{2\pi(4N^2\eta_k^2+1)}\Big] \\ &\ge \frac{\sqrt{2N^2+1}}{4(k+1)} \log^{k+1} \frac{4N^2\eta_k^2+1}{5}, \end{aligned}$$

since $a^k - b^k \ge (a-b)^k$ for $0 < b < a$ and any positive integers k. Because N is sufficiently large ($N\eta_k^2 \ge 2$), we obtain

$$\begin{aligned} \int_1^{N\eta_k} \frac{1}{\sqrt{x_m}} \log^k \frac{1}{2\pi x_m} dm &\ge \frac{\sqrt{N^2+1}}{4(k+1)} \log^{k+1} \sqrt{N^2+1} \\ &= \frac{\log^{k+1} \frac{1}{1-r_N^2}}{4(k+1)2^{k+1}\sqrt{1-r_N^2}}. \end{aligned}$$

In this way for sufficiently large N we have

$$r_N I_p(r_N, F'_{k+1}) \ge \frac{c_k(1-\varepsilon_k)^p}{8\sqrt{\pi}10^{p-1}(k+1)2^{k+1}} \frac{1}{(1-r_N^2)^{p-1/2}} \log^{k+1} \frac{1}{1-r_N^2}.$$

Now, if $r \in [r_N, r_{N+1}]$, $N\eta_k^2 \ge 2$, then

$$(15)\qquad \begin{aligned} rI_p(r, F'_{k+1}) &\ge r_N I_p(r_N, F'_{k+1}) \\ &\ge \frac{c_k(1-\varepsilon_k)^p c'}{8\sqrt{\pi}10^{p-1}(k+1)2^{k+1}} \frac{\log^{k+1} \frac{1}{1-r^2}}{(1-r^2)^{p-1/2}}. \end{aligned}$$

where

$$c' = c'(\eta_k) = \min_{N \ge 2/\eta_k^2} \left(\frac{1 - r_{N+1}^2}{1 - r_N^2} \right)^{p-1/2} \left(\frac{\log(1 - r_N^2)}{\log(1 - r_{N+1}^2)} \right)^{k+1} \underset{\eta_k \to 0}{\longrightarrow} 1.$$

In the above considerations we can take ε_k and η_k sufficiently close to 0. Therefore we can assume that $c'(\eta_k)(1 - \varepsilon_k)^p > 8/10$. Then

$$I_p(r, F'_{k+1}) \ge \frac{c_k}{2\sqrt{\pi} 10^p (k+1) 2^{k+1}} \frac{1}{(1 - r^2)^{p-1/2}} \log^{k+1} \frac{1}{1 - r^2}$$

for r sufficiently close to 1, i.e. for $r \ge \rho_{k+1} \ge 1/2$.

Now consider the case $1/2 \le p < 1$. As above, we also use the induction with respect to $k = 0, 1, \dots$. For $N \ge 1$

$$I_p(r_N, F'_0) \ge \frac{1}{\pi} \sum_{m=1}^{N} \int_{\delta_m}^{\delta_{m-1}} |F'_0(r_N e^{it})|^p \, dt.$$

The following inequalities

$$|\omega(r_N e^{it})| \le R_{m-1}, \quad |1 - r_N e^{it}|^{-2} \le |1 - r_N e^{i\delta_m}|^{-2} = \frac{x_m}{1 - r_N^2}$$

hold for $t \in [\delta_m, \delta_{m-1}]$. In a similar way as for $p > 1$ we obtain

$$r_N I_p(r_N, F'_0) \ge \frac{(1 - r_N^2)^{1-p}}{2\pi^{2-p}} \sum_{m=1}^{N} \frac{1}{(x_m R_{m-1})^{1-p}} \int_{|\omega| = R_m} \frac{|d\omega|}{|1 - \omega|^p}.$$

For $0 \le p \le 1$

$$u(r) = \int_0^{2\pi} \frac{dt}{|1 - re^{it}|^p} \ge \int_{\pi/2}^{3\pi/2} \frac{dt}{|1 - re^{it}|^p} > \frac{\pi}{(1 + r)^P} \underset{r \to 1}{\longrightarrow} \frac{\pi}{2^p}.$$

Therefore $c = c(p) = \inf_{r \in [0,1)} u(r) > 0$. Consequently

$$\begin{aligned} r_N I_p(r_N, F'_0) &\ge \frac{ce^{-\pi}}{2\pi^{2-p}} (1 - r_N^2)^{1-p} \sum_{m=1}^{N} x_m^{p-1} \\ &\ge \frac{ce^{-\pi}}{2\pi^{2-p}} (1 - r_N^2)^{1-p} \int_1^N \frac{dm}{x_m^{1-p}}. \end{aligned}$$

Using change of variables (14) in the integral with $u \in [\frac{5}{(2N^2+1)^2}, \frac{4N^2+1}{(2N^2+1)^2}] = [A, B]$ for $1/2 < p \le 1$ we get

$$\int_1^N \frac{dm}{x_m^{1-p}} \ge \frac{(2N^2 + 1)^p}{4} \int_A^B u^{p-3/2} du \ge \frac{(2N^2 + 1)^P}{2(2p - 1)} B^{p-1/2}$$

$$= \frac{(2N^2+1)^p}{2(2p-1)} \frac{2^{p-1/2}+o(1)}{(2N^2+1)^{p-1/2}} = \frac{\sqrt{2N^2+1}(1+o(1))}{2^{3/2-p}(2p-1)}$$

$$= \frac{1+o(1)}{2^{1-p}(2p-1)}(1-r_N^2)^{-1/2}, \quad \text{where } o(1) \underset{N\to\infty}{\longrightarrow} 0.$$

In the case $p = 1/2$, we obtain for sufficiently great N

$$\int_1^N \frac{dm}{x_m^{1/2}} \geq \frac{\sqrt{2N^2+1}}{4} \log \frac{4N^2+1}{5} > \frac{\sqrt{N^2+1}}{2\sqrt{2}} \log \sqrt{N^2+1}$$

$$= \frac{\log \frac{1}{1-r_N^2}}{2\sqrt{2}(1-r_N^2)^{1/2}}.$$

Moreover, for $N > N_0$ we have

$$r_N I_p(r_N, F_0') \geq \frac{ce^{-\pi}}{2(2\pi)^{2-p}(2p-1)} \frac{1}{(1-r_N^2)^{p-1/2}}, \qquad 1 \geq p > 1/2,$$

$$r_N I_p(r_N, F_0') \geq \frac{ce^{-\pi}}{2(2\pi)^{3/2}} \log \frac{1}{1-r_N^2}, \qquad p = 1/2.$$

Now let N be sufficiently great and $r \in [r_N, r_{N+1}]$. Then for $p \in (1/2, 1]$ we have a result similar to (15)

$$(16) \qquad I_p(r, F_0') \geq I_P(r_N, F_0') \geq \frac{ce^{-\pi}}{3(2\pi)^{2-p}(2p-1)} \frac{1}{(1-r^2)^{p-1/2}};$$

$$(17) \qquad I_{1/2}(r, F_0') \geq \frac{ce^{-\pi}}{3(2\pi)^{3/2}} \log \frac{1}{1-r^2}.$$

Therefore the inequalities (16) and (17) hold for $1 > r > \rho_0(p)$.

Now suppose that for some integer $k \geq 0$ the theorem is true, i.e.

$$(18) \qquad I_p(r, F_k') \geq \frac{c_k(\rho)}{(1-r^2)^{p-1/2}} \left(\log \frac{1}{1-r^2}\right)^k, \qquad 1 \geq p > \frac{1}{2};$$

$$(19) \qquad I_{1/2}(r, F_k') \geq c_k(1/2) \left(\log \frac{1}{1-r^2}\right)^{k+1}$$

hold for $1 > r > \rho_k(p)$. We show the theorem to be true for $k+1$.

As above

$$I_p(r_N, F'_{k+1}) \geq \frac{1}{2\pi}\sum_{k=N}^{1-N}\int_{\delta_m}^{\delta_{m-1}} |F'_k[\omega(r_n e^{it})]|^p \frac{|d\omega(r_N e^{it})|}{|\omega'(r_N e^{it})|^{1-p}}$$

$$\geq \frac{(1-r_N^2)^{1-p}}{(2\pi)^{2-p}}\sum_{k=1}^{N}(R_{m-1}x_m)^{p-1}\int_{\Gamma_m\cup\Gamma'_m} |F'_k(\omega)|^p|d\omega|.$$

Since $F_k \in \mathcal{B}'$, one can use Lemma 2 for the integrals over Γ_m and Γ'_m. By (1') with $r_0 \geq R_m$, $r^0 \leq R_{m-1}$ and $1/2 \leq p < 1$ we get

$$r_N I_p(r_N, F'_{k+1}) \geq \frac{(1-r_N^2)^{1-p}}{(2\pi)^{2-p}}\sqrt{2}\sum_{m=1}^{N}(R_{m-1}x_m)^{p-1}\left[\int_{|\omega|=R_m} |F'_k(\omega)|^p|d\omega|\right.$$

$$\left. - \frac{8(1+p)}{p(1-p)}((1-R_m)^{1-p}-(1-R_{m-1})^{1-p})\right].$$

With $p=1$ we have the following inequality

$$r_N I_1(r_N, F'_{k+1}) \geq \frac{\sqrt{2}}{2\pi}\sum_{m=1}^{N}\left[\int_{|\omega|=R_m} |F'_k(\omega)||d\omega| - 8R_m\log\frac{1-R_m}{1-R_{m-1}}\right].$$

From (18) and (19) it follows that for $1/2 < p < 1$

$$\frac{1}{2}\int_{|\omega|=R_m} |F'_k(\omega)|^p|d\omega| - \frac{8(1+p)}{p(1-p)} \geq 0,$$

$$\frac{1}{2}\int_{|\omega|=R_m} |F'_k(\omega)||d\omega| - 8\log 10 \geq 0,$$

where $R_m > \rho_k(p)$ and R_m is sufficiently close to 1, i.e. $R_m > 1-\varepsilon_k$, $\varepsilon_k = \varepsilon_k(p) \in (0,1)$. This is equivalent to $1 \leq m \leq N\eta_k$, $\eta_k = \eta_k(p) \in (0,1)$ where N is sufficiently great and $(N\eta_k^2 \geq 2)$. We have shown that

$$\frac{1-R_m}{1-R_{m-1}} < 10$$

as $m \in [0,N]$. Thus for $N \geq 2/\eta_k^2$ and $m \in [1, N\eta_k]$ we have the following inequality

$$(20)\qquad r_N \geq \frac{(1-r_N^2)^{1-p}}{(2\pi)^{2-p}}\sqrt{2}\sum_{m=1}^{N\eta_k}(R_{m-1}x_m)^{p-1}\pi I_p(R_m, F'_k).$$

This implies for $1/2 < p \le 1$ and $1 - R_m^2 \le 2\pi x_m$

$$r_N I_p(r_N, F'_{k+1}) \ge \frac{c_k(p)(1-r_N^2)^{1-p}}{(2\pi)^{1-p}\sqrt{2}} \sum_{m=1}^{N\eta_k} \frac{(R_{m-1}x_m)^{p-1}}{(1-R_m^2)^{p-1/2}} \left(\log \frac{1}{1-R_m^2}\right)^k$$

$$\text{(21)} \qquad \ge \frac{c_k(p)(1-r_N^2)^{1-p}}{(2\pi)^{1-p}\sqrt{2}(2\pi)^{p-1/2}} \sum_{m=1}^{N\eta_k} \frac{x_m^{p-1}}{x_m^{p-1/2}} \left(\log \frac{1}{2\pi x_m}\right)^k$$

$$\ge \frac{c_k(p)(1-r_N^2)^{1-p}}{2\sqrt{\pi}} \sum_{m=1}^{N\eta_k} \frac{1}{\sqrt{x_m}} \left(\log \frac{1}{2\pi x_m}\right)^k .$$

The last sum in (21) has the same form as in b) in the first part of the proof. Therefore for $N \ge 2/\eta_k^2$

$$r_N I_p(r_N, F'_{k+1}) \ge \frac{c_k(p)}{8\sqrt{\pi}(k+1)(1-r_N^2)^{p-1/2}} \log^{k+1} \frac{1}{1-r_N^2}.$$

Now, if $r \in [r_N, r_{N+1}]$, $N\eta_k^2 \ge 2$, then, similarly as above (see (15)) we obtain

$$\text{(22)} \qquad I_p(r, F'_{k+1}) \ge \frac{c_k(p)}{10\sqrt{\pi}(k+1)(1-r^2)^{p-1/2}} \log^{k+1} \frac{1}{1-r^2},$$

for N sufficiently great. This means that (22) holds with r sufficiently close to 1, i.e. $0 < \rho_{k+1}(p) < r < 1$. In this way the proof is complete for $1/2 < p \le 1$.

For $p = 1/2$ we obtain from (20)

$$r_N I_{1/2}(r_N, F'_{k+1}) \ge \frac{c_k(1/2)}{2\sqrt{\pi}} \sqrt{1-r_N^2} \sum_{m=1}^{N\eta_k} \frac{1}{\sqrt{x_m R_{m-1}}} \log^{k+1} \frac{1}{1-R_m^2}$$

$$\ge \frac{c_k(1/2)}{2\sqrt{\pi}} \sqrt{1-r_N^2} \sum_{m=1}^{N\eta_k} \frac{1}{\sqrt{x_m}} \log^{k+1} \frac{1}{2\pi x_m}.$$

We have obtained the sum of the same form as in (21). Thus for $N \ge 2/\eta_k^2$

$$r_N I_{1/2}(r_N, F'_{k+1}) \ge \frac{c_k(1/2)}{8\sqrt{\pi}(k+2)} \log^{k+2} \frac{1}{1-r_N^2}.$$

This implies (in a similar way as before) the following inequality

$$I_{1/2}(r, F'_{k+1}) \ge \frac{c_k(1/2)}{10\sqrt{\pi}(k+2)} \log^{k+2} \frac{1}{1-r^2}$$

for r sufficiently close to 1 which shows the theorem in the case $p = 1/2$. The proof of the theorem is complete. □

The idea of constructing the function F_k appears in [S], where the author considered the linearly invariant families $\mathcal{U}_\alpha$ of locally univalent functions $h(z) = z + \dots$ of the order α (cf. [P2]).

For $h \in \mathcal{U}_\alpha$ sharp inequality

$$|h'(z)| \le \frac{(1+|z|)^{\alpha-1}}{(1-|z|)^{\alpha+1}}, \quad z \in \Delta$$

was shown in [P2]. Hence

$$h \in \mathcal{U}_\alpha \implies h' = (f')^{\alpha+1}, \quad f \in \mathcal{B}', \tag{23}$$

and for functions $f \in \mathcal{B}'$, defined by (23) $I_{\alpha+1}(r, f') = I_1(r, h')$. For $h \in \mathcal{U}_\alpha$ the inequality

$$I_1(r, h') \le c(1-r)^{-1/2-\sqrt{\alpha^2-3/4}-\varepsilon},$$

where $c = \text{const}$ and $\varepsilon > 0$ sufficiently small, was given in [P3] (p. 182, Problem 5). Since $\alpha + 1/2 > \sqrt{\alpha^2 - 3/4} + 1/2 = \alpha + 1/2 + O(1/\alpha)$, we have $\alpha \to \infty$ and after integration of $|f'|^{\alpha+1}$ the order of the growth of $I_{\alpha+1}(r, f')$ is reduced, as compared with the growth

$$\max_{h \in \mathcal{U}_\alpha, |z|=r} |h'(z)| = \max_{f, |z|=r} |f'(z)|^{\alpha+1}$$

by more than 1/2.

Thus we obtain the following

Problem. Does there exist a function $f \in \mathcal{B}'$ for which $I_p(r, f')$ has an order of growth greater than that given in Theorem? For $p > 0$

$$\inf\{\beta > 0 : I_p(r, f') = O((1-r)^{-\beta}) \quad \forall f \in \mathcal{B}'\} = \beta(p).$$

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