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Kähler potentials of the Weil-Petersson metric

ABSTRACT. The Weil-Petersson metric on Teichmüller space is Kähler. We survey explicit constructions of Kähler potentials that have been obtained by various authors. The physical interpretation of the problem in the context of bosonic string theory is discussed.

In the context of number theory, Hans Petersson introduced an inner product on the spaces of modular forms of arbitrary weight. The case of weight 2 corresponds to holomorphic quadratic differentials, i.e., deformations of Teichmüller space. That Petersson's inner product should provide Teichmüller space with an interesting metric was suggested by André Weil in a letter to Lars Ahlfors. The metric was introduced by Weil in [11]. Ahlfors [1] established that the Weil-Petersson metric is Kähler.

Let X be a compact oriented Riemann surface of genus $\gamma > 1$ and $\mathcal{T}_\gamma$ the corresponding Teichmüller space. Its holomorphic tangent and cotangent spaces $T_{[X]}\mathcal{T}_\gamma$ and $T^*_{[X]}\mathcal{T}_\gamma$ at a point $[X] \in \mathcal{T}_\gamma$ can be identified, respectively, with the space of harmonic forms $\mathcal{H}^{0,1}(X, TX)$ and $\mathcal{H}^{1,0}(X, T^*X)$. The corresponding pairing $(\,.\,,\,.\,) : \mathcal{H}^{0,1}(X, TX) \otimes \mathcal{H}^{1,0}(X, T^*X) \longrightarrow \mathbb{C}$ is given by the integral

$$(p, q) = \int_X pq$$

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where $p \in \mathcal{H}^{0,1}(X, TX), q \in \mathcal{H}^{1,0}(X, T^*X)$, and the integration is with respect to the Poincaré metric, i.e., the unique metric of constant curvature -1 on X. The Weil-Petersson Hermitian metric is defined as $\langle\, .\, ,\, .\, \rangle_{W-P} = (\, .\, , *\, .\,)$ where the Hodge star $*$ corresponds to the Poincaré metric on X. The corresponding symplectic form is denoted by ω_{W-P}.

It is a problem of some interest to search for an explicit Kähler potential of the Weil-Petersson metric, i.e., a complex function K such that

$$\omega_{W-P} = \partial\overline{\partial}\, K$$

either locally or globally. Equivalently, in real terms, we may look for a real-valued energy function E such that the Hessian of E coincides with $\langle .,. \rangle_{W-P}$. Here we survey the answers obtained for this geometrical problem and briefly explain its physical interpretation.

A local potential on Teichmüller space. The first result was obtained by Tromba in 1982. The Teichmüller space $\mathcal{T}_\gamma$ of a compact Riemann surface of genus $\gamma > 1$ can also be described in purely Riemannian terms. Let $\mathcal{M}_{-1}$ be the Fréchet manifold of Riemannian metrics of constant negative curvature on X. The tangent space of $\mathcal{M}_{-1}$ at a metric g consists of those covariant 2-tensors h on M satisfying the equation

$$\triangle(\mathrm{tr}_g\, h) + \delta_g \delta_g h + \frac{1}{2}(\mathrm{tr}_g\, h) = 0$$

where $\mathrm{tr}_g\, h = g^{ij} h_{ij}$ is the trace of h with respect to g, the term $\delta_g\delta_g h = D_i D_j h_{ij}$ contains twice the divergence, and $\triangle = -D_i D_i$ is the Laplace operator on functions.

Let $\mathcal{D}_0$ be the Fréchet Lie group of diffeomorphisms of X which are homotopic to the identity. Then $\mathcal{D}_0$ acts on $\mathcal{M}_{-1}$ by pull-back, i.e., $f \mapsto f^*g$, where $g \in \mathcal{M}_{-1}$ and $f \in \mathcal{D}_0$. Teichmüller space is then simply defined as the moduli space

$$\mathcal{T}_\gamma = \mathcal{M}_{-1}/\mathcal{D}_0.$$

An L^2 metric on $\mathcal{M}_{-1}$ is given by the inner product

$$\langle h, k \rangle = \int_X \mathrm{trace}(H \circ K) d\mu_g$$

where $H = g^{-1}h, K = g^{-1}k$ are the (1,1) tensors on X obtained from h and k via the metric g, or by raising an index, i.e., $H^i_j = g^{ik} h_{kj}$ and similarly for K. Finally, $d\mu_g$ is the volume element induced on X by g and by the given orientation.

The above inner product is $\mathcal{D}_0$ invariant. Thus $\mathcal{D}_0$ acts smoothly on $\mathcal{M}_{-1}$ as a group of isometries with respect to this metric, and consequently we have an induced metric on $\mathcal{T}_\gamma$. It coincides with the Weil-Petersson metric up to some constant factor. This formalism yields an alternative proof for the Kählerity of the Weil-Petersson metric [5, 10].

Next we fix some $g_0 \in \mathcal{M}_{-1}$ and suppose that $s : (X, g) \to (X, g_0)$ is a smooth C^1 map homotopic to the identity which is viewed as a map from X with some arbitrary metric $g \in \mathcal{M}_{-1}$ to X with its g_0 metric.

Define the Dirichlet energy of s by the formula

$$E_g(s) = \int_X |ds|^2 d\mu_g$$

where $|ds|^2 = \operatorname{trace} ds \otimes ds$ depends on both g and g_0. For fixed g, the critical points of the functional E are said to be harmonic maps. The following result is due to Schoen and Yau [7].

Theorem 1. *For given metrics g and g_0, in $\mathcal{M}_{-1}$ there exists a unique harmonic map $s(g) : (X, g) \to (X, g_0)$ which is homotopic to the identity. Moreover, $s(g)$ depends smoothly on g.*

Consider now the function $g \mapsto E_g(s(g))$. It is $\mathcal{D}_0$ invariant on $\mathcal{M}_{-1}$ and therefore descends to a function on Teichmüller space. To see this, one must show that

$$E_{f^*g}(s(f^*g)) = E_g(s(g)).$$

Let $c(g)$ be the complex structure associated to g, and induced by a conformal coordinate system for g. For $f \in \mathcal{D}_0$, $f : (X, f^*c(g)) \longrightarrow (X, c(g))$ is holomorphic and consequently since the composition of a harmonic map and a holomorphic map is still harmonic we may conclude, by uniqueness, that

$$s(f^*g) = s(g) \circ f.$$

Since Dirichlet's functional is invariant under complex holomorphic changes of coordinates it follows immediately that

$$E_{f^*g}(s(g) \circ f) = E_g(s(g)).$$

Consequently, for $[g] \in \mathcal{M}_{-1}/\mathcal{D}_0$ define the smooth function $\bar{E} : \mathcal{M}_{-1}/\mathcal{D}_0 \to \mathbb{R}$ by $\bar{E}[g] = E_g(s(g))$. We may now state Tromba's result [9]:

Theorem 2. *$[g_0]$ is the only critical point of $\tilde{E}$. The Hessian of $\tilde{E}$ at $[g_0]$ is given by*

$$\text{Hess}\, \tilde{E}[g_0](h,k) = \langle h,k\rangle_{W-P}$$

where $h, k \in T_{[g_0]}\mathcal{T}(X)$.

In complex terms, the functional $\tilde{E}$ gives a local Kähler potential for the Weil-Petersson metric on Teichmüller space.

The above construction was independently discovered by Jost [3]. Essentially the same construction also occurs in Wolf [12] where the energy is studied as a function of the image rather than of the domain metric. It turns out that the Hessian of the energy with respect to the image metric also yields the Weil-Petersson metric.

A global potential on Schottky space. We first review some classical material concerning Schottky uniformization [8]. Let X be a marked compact Riemann surface, i.e., a Riemann surface X with a system of generators $\{\alpha_1, ..., \alpha_\gamma, \beta_1, ..., \beta_\gamma\}$ of the fundamental group $\pi_1(X, x_0), x_0 \in X$, satisfying the relation

$$\prod_{i=1}^{\gamma} \alpha_i^{-1}\beta_i^{-1}\alpha_i\beta_i = 1.$$

A marked Riemann surface X can be represented as a quotient Ω/Σ where $\Omega \subset \overline{\mathbb{C}}$ is the domain of discontinuity of some Schottky group Σ. Moreover, the covering map $\pi_\Sigma : \Omega \longrightarrow X$ can be chosen in such a way that the covering group coincides with the minimal normal subgroup $\mathcal{N} \subset \pi_1(X)$ generated by the elements $\alpha_1, ..., \alpha_\gamma$. The group Σ is then isomorphic to the quotient group $\pi_1(X)/\mathcal{N}$ and is uniquely determined up to conjugation in $\text{PSL}(2, \mathbb{C})$.

By a Schottky group we mean here a finitely generated strictly loxodromic free Kleinian group. The marked Schottky group Σ of rank $\gamma > 1$ is a Schottky group with a fixed system $L_1, ..., L_\gamma$ of free generators. The domain of discontinuity Ω of a Schottky group is the connected complement of a Cantor set in $\overline{\mathbb{C}}$; the fundamental domain $D = \Omega/\Sigma$ of a marked Schottky group can be chosen to be a domain in $\overline{\mathbb{C}}$ bounded by 2γ non-intersecting Jordan curves $C_1, ..., C_\gamma, C_1', ..., C_\gamma'$ such that $C_i' = -L_i(C_i), i = 1, ..., \gamma$. Each element $L_i \in \Sigma$ can be represented in normal form

$$\frac{L_i w - a_i}{L_i w - b_i} = \lambda_i \frac{w - a_i}{w - b_i}, \quad w \in \overline{\mathbb{C}},$$

where a_i and b_i are attracting and repelling points, respectively, and $0 < |\lambda_i| < 1$, $i = 1, \ldots, \gamma$.

A Schottky group Σ is called normalized if $a_1 = 0, b_1 = \infty$, and $a_2 = 1$. The mapping $(\Sigma; L_1, ..., L_\gamma) \mapsto (a_3, ..., a_\gamma, b_2, ..., b_\gamma, \lambda_1, ..., \lambda_\gamma)$ establishes a one-one correspondence between the normalized marked Schottky groups and a certain set $\mathcal{S}_\gamma \subset \overline{\mathbb{C}}^{3\gamma-3}$. In fact, this set is a domain in $\overline{\mathbb{C}}^{3\gamma-3}$ and is called the Schottky space. Thus we have a map $\psi : \mathcal{T}_\gamma \to \mathcal{S}_\gamma$ which is a complex analytic covering.

On the domain of discontinuity Ω of a marked normalized Schottky group Σ there exists a Γ-invariant Poincaré metric, i.e., a complete conformal metric of constant negative curvature -1. This metric has the form $e^{\varphi(w)}|dw|^2$, where the real-valued function φ satisfies the Liouville differential equation

$$\varphi_{w\overline{w}} = \frac{1}{2} e^{\varphi}.$$

Define the function $\zeta : \mathcal{S}_\gamma \to \mathbb{R}$ by the formula

$$\begin{aligned}\zeta(\Sigma) = &\int_D |\varphi_w|^2 \left| \frac{dw \wedge d\overline{w}}{2} \right| \\ &+ \frac{1}{2i} \sum_{j=2}^{\gamma} \int_{C_j} \left(\varphi \left(\frac{\overline{L_j''}}{\overline{L_j'}} d\overline{w} - \frac{L_j''}{L_j'} d\overline{w} \right) - \log |L_j'|^2 \frac{\overline{L_j''}}{\overline{L_j'}} d\overline{w} \right) \\ &+ 4\pi \sum_{j=2}^{\gamma} \log |l_j|^2,\end{aligned}$$

where $D = \Omega/\Sigma$ is the fundamental domain of a marked normalized Schottky group Σ, $\partial D = \cup_{j=1}^{\gamma} (C_j \cup (-L_j(C_j)))$, and l_j is the left bottom matrix element of the matrix $L_j \in \mathrm{PSL}(2, \mathbb{C}), j = 1, ..., \gamma$. The following result is due to Zograf and Takhtajan [15].

Theorem 3. *The function $-\zeta$ is a Khler potential of the Weil-Petersson metric on the Schottky space $\mathcal{S}_\gamma$ while the function $-\zeta \circ \psi$ plays the same role for the Teichmüller space $\mathcal{T}_\gamma$.*

A global potential on Torelli space. Along with the above-discussed marked Riemann surfaces one can consider the Torelli marking of Riemann surfaces X of genus $\gamma > 0$. This means that we fix for X a canonical basis $\{\alpha_1, \beta_1, ..., \alpha_\gamma, \beta_\gamma\}$ of its homology group $H_1(X, \mathbb{Z})$. The set of isomorphic Riemann surfaces marked in the Torelli sense is called Torelli space and will be denoted by $\mathcal{U}_\gamma$. The natural map $\mathcal{T}_\gamma \to \mathcal{U}_\gamma$ is a complex analytic covering, so the Weil-Petersson metric projects onto $\mathcal{U}_\gamma$. One can try to look for a Kähler potential for this metric on the Torelli space $\mathcal{U}_\gamma$.

To state the result, we introduce some additional concepts and notations. The canonical basis of $H_1(X,\mathbb{Z})$ has the intersection numbers

$$\begin{cases} \alpha_i \cdot \beta_j = \delta_{ij} \\ \alpha_i \cdot \alpha_j = 0 = \beta_i \cdot \beta_j \end{cases}$$

for $1 \leq i, j \leq \gamma$. By Riemann-Roch's theorem, the complex vector space of holomorphic 1-forms on X has dimension γ. We may choose a basis $\{\omega_1, ..., \omega_\gamma\}$ of such forms normalized so that $\int_{\alpha_j} \omega_i = \delta_{ij}$. Then the entries

$$\tau_{ij} = \int_{\beta_j} \omega_i$$

form the period matrix $\tau = \tau(X)$. It can be shown that τ is symmetric and has positive-definite imaginary part $\operatorname{Im}\tau$.

The Selberg zeta function $Z(s)$ of a Riemann surface X is defined for $\operatorname{Re} s > 1$ by the absolutely convergent product

$$Z(s) = \prod_{\{l\}} \prod_{k=0}^{\infty} (1 - e^{-(s+k)|l|}),$$

where l runs over the set of all simple closed geodesics on X supplied with the Poincaré metric, and $|l|$ is the length of a geodesic l. The function $Z(s)$ has a meromorphic continuation to the entire complex s-plane with a simple zero at $s = 1$. Another result due to Zograf and Takhtajan [13, 16] now states:

Theorem 4. *Let τ be the period matrix of a Torelli marked Riemann surface X and $Z(s)$ the Selberg zeta-function for X. Then the function*

$$K = \log \frac{Z'(1)}{\det \operatorname{Im} \tau},$$

up to a constant factor, is a Kähler potential for the Weil-Petersson metric on $\mathcal{U}_\gamma$.

The cases of genus 0 and 1. Let X be a Riemann surface of genus $\gamma = 0$ with n punctures. The case $n \leq 3$ is trivial because Teichmüller space then is a point. In the case $n > 3$ a formula analogous to that of Theorem 3 has been derived in [14] but it is somewhat too technical to be stated concisely here.

In the case of a compact surface of genus $\gamma = 1$, the Teichmüller space $\mathcal{T}_1$ is the upper half-plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} \,|\, y > 0\}$. The Weil-Petersson metric coincides with the standard hyperbolic metric. A well-known computation yields the Kähler potential

$$K = \log(y|\eta(z)|^4)$$

where

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})$$

is the Dedekind eta function.

The case of genus $\gamma \geq 1$ with punctures seems not to have been considered in the literature.

Comments on the physical interpretation. In [6] we discussed the relevance of universal Teichmüller space $T(1)$ for the purposes of bosonic string theory. On the other hand, we ignore if there exists a viable notion of universal Torelli space. According to the tenets of geometric quantization, it would be of interest to have a notion of universal Kähler potential K_∞ which should be the relevant Action Principle. It is for this reason that we have reviewed above the various finite-dimensional constructions. Unfortunately, we do not see how to make use of any of them for a construction of K_∞.

There exists, however, one set-up where K_∞ has been defined. Recall first the description of $T(1)$ as the space of quasisymmetric (QS) homeomorphisms of the unit circle S^1 modulo Möbius moves,

$$T(1) = \mathrm{QS}(S^1)/\,\mathrm{M\ddot{o}b}(S^1).$$

A well-understood, holomorphically embedded slice of this space is

$$M = \mathrm{Diff}(S^1)/\,\mathrm{M\ddot{o}b}(S^1).$$

The slice M carries a canonical Kähler form ω which, in terms of the Fourier modes

$$L_n = e^{in\theta}\frac{d}{d\theta}, \quad z = e^{i\theta},$$

at the origin and up to a constant factor, is given by

$$\omega(L_m, L_n) = (m^3 - m)\delta_{m,-n}, \quad m, n \in \mathbb{Z} \setminus \{0, \pm 1\}.$$

This expression converges when applied to tangent vectors to $T(1)$ which are $C^{3/2+\epsilon}$ smooth. Deformations of quasisymmetric maps are merely in a

Zygmund class so that $C^{3/2+\epsilon}$ smoothness is not guaranteed. Nonetheless, in a sense explained in [4], $\omega = \omega_{W-P,\infty}$ can be interpreted as the universal Weil-Petersson Kähler form.

Another possible description of $T(1)$ is as the space of univalent (=holomorphic and injective) functions f in the unit disk such that $f(0) = 0$ and $f'(0) = 1$ and allowing quasiconformal extension to the whole plane. Every element of $T(1)$ then has an expansion of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

The coefficients a_k, which according to Bieberbach-de Branges' theorem satisfy $|a_k| \leq k$, can be thought of as coordinates on $T(1)$.

In [2], it is shown that, at the origin and up to a constant factor, the expression

$$K_\infty = \sum_{k=1}^{\infty} (k^3 - k)|a_{k+1}|^2,$$

whenever it converges, is a Kähler potential of $\omega_{W-P,\infty}$.

Let us finally mention that the motivation for Theorem 3 and its counterpart with punctures also stems from bosonic string theory. Indeed, the functional ζ in Theorem 3 is closely related to the conformal anomaly in Polyakov's string theory.

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