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A conformally invariant dilatation of quasisymmetry

ABSTRACT. We discuss a conformally invariant modification of the Beurling-Ahlfors condition of quasisymmetry.

0. Introduction. Given a domain $\Omega \subset \hat{\mathbb{C}}$ and $K \geq 1$ let $\mathrm{QC}(\Omega; K)$ stand for the class of all K–quasiconformal (qc. for short) self–mappings of Ω and let

$$\mathrm{QC}(\Omega) := \bigcup_{K \geq 1} \mathrm{QC}(\Omega; K) .$$

Assume that Ω is a Jordan domain bounded by a Jordan curve Γ. A classical result says that each $F \in \mathrm{QC}(\Omega)$ has a homeomorphic extension F^* of the closure $\overline{\Omega} = \Omega \cup \Gamma$ onto itself; cf. [LV]. Then the restriction

$$\mathrm{Tr}[F] := F^*_{|\Gamma} \in \mathrm{Hom}^+(\Gamma) ,$$

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where $\mathrm{Hom}^+(\Gamma)$ denotes the class of all sense-preserving homeomorphic self-mappings of Γ. For $K \geq 1$ consider the class

$$\mathrm{Q}(\Gamma; K) := \{\mathrm{Tr}[F] : F \in \mathrm{QC}(\Omega; K)\}$$

and

$$\mathrm{Q}(\Gamma) := \{\mathrm{Tr}[F] : F \in \mathrm{QC}(\Omega)\} .$$

A natural problem appears to describe the class $\mathrm{Q}(\Gamma)$. The first such characterization in the case Ω is the upper half plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ and $F \in \mathrm{QC}(\mathbb{C}_+)$ satisfying $F^*(\infty) = \infty$ was given by Beurling and Ahlfors in [BA] by means of the so-called quasisymmetric functions. They showed that for every $F \in \mathrm{QC}(\mathbb{C}_+)$ such that $F^*(\infty) = \infty$,

$$\mathrm{Tr}[F] \in \mathrm{QS}(\overline{\mathbb{R}}),$$

where $\mathrm{QS}(\overline{\mathbb{R}}) := \bigcup_{M \geq 1} \mathrm{QS}(\overline{\mathbb{R}}; M)$ and

$$\mathrm{QS}(\overline{\mathbb{R}}; M) := \Big\{ f \in \mathrm{Hom}^+(\overline{\mathbb{R}}) : f(\infty) = \infty \text{ and } \frac{1}{M} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq M\,,\ x \in \mathbb{R}, t > 0 \Big\} .$$

Conversely, if $f \in \mathrm{QS}(\overline{\mathbb{R}})$, then f admits a qc. extension to $\mathbb{C}_+$, i.e., there exists $F \in \mathrm{QC}(\mathbb{C}_+)$ such that $\mathrm{Tr}[F] = f$. The Beurling–Ahlfors concept of quasisymmetric functions may be easily carried to the case of an oriented Jordan arc or an oriented Jordan curve $\Gamma \subset \mathbb{C}$ which is locally rectifiable. To be more precise we say that a homeomorphism $f \in \mathrm{Hom}^+(\Gamma)$ is M-quasisymmetric (qs. for short) provided the inequality

$$\frac{1}{M} \leq \frac{|f(I_1)|_1}{|f(I_2)|_1} \leq M$$

holds for all closed arcs $I_1, I_2 \subset \Gamma$ such that their intersection $I_1 \cap I_2$ is not empty and consists of at most two points (the arcs I_1 and I_2 are then said to be adjacent) and $0 < |I_1|_1 = |I_2|_1 < \infty$. Here and in the sequel $|I|_1$ stands for the arc length measure of an arc I. We write $\mathrm{QS}(\Gamma; M)$ for the class of all M-qs. homeomorphic self-mappings of Γ, $M \geq 1$, and we set

$$\mathrm{QS}(\Gamma) := \bigcup_{M \geq 1} \mathrm{QS}(\Gamma; M) .$$

According to these definitions $\mathrm{QS}(\overline{\mathbb{R}}) = \{f \in \mathrm{Hom}^+(\overline{\mathbb{R}}) : f_{|\mathbb{R}} \in \mathrm{QS}(\mathbb{R})\}$. It was shown by J. G. Krzyż in [K] that in the case where Γ is the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and Ω is the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$,

$$\mathrm{Q}(\mathbb{T}) = \mathrm{QS}(\mathbb{T}) .$$

The characterization of the class $Q(\Gamma)$ by means of the class $QS(\Gamma)$, $\Gamma := \overline{\mathbb{R}}$ or $\Gamma := \mathbb{T}$, requires only two real parameters which represent a common point $\zeta \in I_1 \cap I_2 \in \Gamma$ and the length $|I_1|_1$. On the other hand such description is not conformally invariant, i.e.,

$$\{h_1 \circ f \circ h_2 : f \in QS(\Gamma; M) \text{ and } h_1, h_2 \in Q(\Gamma; 1)\} \not\subset QS(\Gamma; M)$$

in general. A conformally invariant description of the class $Q(\Gamma)$ by means of quasihomographies is due to J. Zając even in the general case of a domain Ω bounded by a Jordan curve Γ; cf. [Z]. To define a K-quasihomography (qh. for brevity) he used the so-called harmonic cross-ratio $[z_1, z_2, z_3, z_4]_\Omega$ of a positively ordered, with respect to Ω, quadruple of distinct points $z_1, z_2, z_3, z_4 \in \Gamma$. If $\Gamma = \overline{\mathbb{R}}$ or $\Gamma = \mathbb{T}$, then the harmonic cross-ratio $[z_1, z_2, z_3, z_4]_\Omega$ is reduced to the square root of the following usual cross-ratio

$$[z_1, z_2, z_3, z_4] := \frac{z_2 - z_3}{z_1 - z_3} \cdot \frac{z_1 - z_4}{z_2 - z_4} .$$

According to [Z, p. 44 Definition], for $K \geq 1$ a homeomorphism $f \in \mathrm{Hom}^+(\Gamma)$ is said to be a K-qh. of Γ onto itself if the inequality

$$(0.1) \qquad \begin{aligned} \Phi_{1/K}(\sqrt{[z_1, z_2, z_3, z_4]})^2 &\leq [f(z_1), f(z_2), f(z_3), f(z_4)] \\ &\leq \Phi_K(\sqrt{[z_1, z_2, z_3, z_4]})^2 \end{aligned}$$

holds for all quadruples of distinct points $z_1, z_2, z_3, z_4 \in \Gamma$ ($\Gamma = \mathbb{R}, \mathbb{T}$) that are positively ordered with respect to Ω. Here Φ_K is the familiar Hersch–Pfluger distortion function; cf. [HP], [LV, pp. 53, 63]. For $K \geq 1$ write $QH(\Gamma; K)$ for the class of all K-qh.-s of Γ onto itself and let

$$QH(\Gamma) := \bigcup_{K \geq 1} QH(\Gamma; K) .$$

From [Z, Thm.-s 2.1 and 2.8] it follows that $Q(\Gamma) = QH(\Gamma)$. Since the harmonic cross-ratio is conformally invariant, we easily see that the class $QH(\Gamma; K)$ is conformally invariant for each $K \geq 1$, i.e.,

$$\{h_1 \circ f \circ h_2 : f \in QH(\Gamma; K) \text{ and } h_1, h_2 \in Q(\Gamma; 1)\} \subset QH(\Gamma; K) , \quad K \geq 1 .$$

However, (0.1) shows that Zając's description of the class $Q(\Gamma)$ requires four real parameters which represent $z_1, z_2, z_3, z_4 \in \Gamma$.

This paper aims at giving a three real parameters description of the class $Q(\Gamma)$ which is still conformally invariant. To this end we modify the classical Beurling–Ahlfors condition of quasisymmetry. Key tools in our case are notions of the *second module of a quadrilateral* and the *hyperbolic* *square* that are defined and studied in Sections 1 and 2. Then we introduce *generalized quasisymmetric* homeomorphisms of Γ onto itself in Section 2 and give a new description of the class $Q(\Gamma)$. In Section 3 we focus our attention on the simplest case where Γ is the closed real axis $\overline{\mathbb{R}}$ or the unit circle $\mathbb{T}$. Section 4 is devoted to applications of our description.

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1. The second module of a quadrilateral. Write $\omega(z,\Omega)[I]$ for the harmonic measure at the point $z \in \Omega$ of the arc $I \subset \Gamma$ with respect to a domain $\Omega \subset \hat{\mathbb{C}}$ bounded by a Jordan curve $\Gamma = \partial\Omega$. Given distinct points $z_1, z_2 \in \Gamma$ we denote by $\Gamma(z_1, z_2)$ the open arc from z_1 to z_2 according to the positive orientation of Γ with respect to Ω. We recall that a quadrilateral $Q := \Omega(z_1, z_2, z_3, z_4)$ is a Jordan domain $\Omega \subset \hat{\mathbb{C}}$ with distinct points z_1, z_2, z_3, z_4 lying on the boundary curve $\Gamma = \partial\Omega$ and ordered according to the positive orientation of Γ with respect to Ω; cf. [LV, pp. 8-9].

Lemma 1.1. *There exists a unique point $z \in \Omega$ with the following property*

$$\begin{aligned}(1.1)\qquad &\omega(z,\Omega)[\Gamma(z_1,z_2)] = \omega(z,\Omega)[\Gamma(z_3,z_4)] \quad \textit{and}\\ &\omega(z,\Omega)[\Gamma(z_2,z_3)] = \omega(z,\Omega)[\Gamma(z_4,z_1)]\,.\end{aligned}$$

Proof. By the Riemann and Taylor–Osgood–Carathéodory theorems there exists a homeomorphism φ of the closure $\overline{\Omega} = \Omega \cup \Gamma$ onto $\overline{\mathbb{D}}$ which is conformal on Ω and sends the points z_1, z_2, z_3 into $1, i, -1$, respectively. Let $\zeta := \varphi(z_4)$. For $a \in \mathbb{D}$ define

$$h_a(1/\overline{a}) := \infty, \quad h_a(\infty) := -1/\overline{a} \ \text{ and } \ h_a(u) := \frac{z-a}{1-\overline{a}z}\,, \quad u \in \mathbb{C}\setminus\{1/\overline{a}\}\,.$$

Obviously, $h_{a|\mathbb{D}} \in QC(\mathbb{D};1)$ for $a \in \mathbb{D}$. A simple calculation shows that there exists $t \in (-1,1)$ satisfying $h_t(\zeta) = -h_t(i)$. Since $h_t(1) = 1$ and $h_t(-1) = -1$ we have

$$\begin{aligned}(1.2)\qquad &\omega(0,\mathbb{D})[(h_t\circ\varphi(\Gamma(z_1,z_2))] = \omega(0,\mathbb{D})[h_t\circ\varphi(\Gamma(z_3,z_4))]\,,\\ &\omega(0,\mathbb{D})[h_t\circ\varphi(\Gamma(z_2,z_3))] = \omega(0,\mathbb{D})[h_t\circ\varphi(\Gamma(z_4,z_1))]\,.\end{aligned}$$

By the conformal invariance of the harmonic measure the equalities (1.1) hold with $z := (h_t\circ\varphi)^{-1}(0) \in \Omega$. Since (1.2) does not hold if 0 is replaced by any $a \in \mathbb{D}\setminus\{0\}$, it follows that z is a unique point satisfying (1.1). $\square$

Definition 1.2. The unique point $z \in \Omega$ satisfying (1.1) is said to be the *hyperbolic center* of a quadrilateral $Q := \Omega(z_1, z_2, z_3, z_4)$. We denote it by $\mathrm{c}(Q)$.

Lemma 1.1 justifies the following

Definition 1.3. The ratio

$$\mathrm{m}(Q) := \frac{\tan \pi\omega(\mathrm{c}(Q), \Omega)[\Gamma(z_1, z_2)]}{\tan \pi\omega(\mathrm{c}(Q), \Omega)[\Gamma(z_2, z_3)]}$$

is said to be the *second module* of a quadrilateral $Q := \Omega(z_1, z_2, z_3, z_4)$.

Example 1.4. Given $x_1, x_2, x_3 \in \mathbb{R}$, $x_1 < x_2 < x_3$, consider the quadrilateral $Q := \mathbb{C}_+(x_1, x_2, x_3, \infty)$. Then the hyperbolic center $\mathrm{c}(Q) = x_2 + iy$, where $y > 0$ is determined by the equation

$$\frac{1}{\pi} \arctan \frac{x_2 - x_1}{y} \frac{1}{2} - \frac{1}{\pi} \arctan \frac{x_3 - x_2}{y} .$$

This equation has a simple geometric interpretation: the vectors $[\mathrm{c}(Q), x_1]$ and $[\mathrm{c}(Q), x_3]$ are orthogonal. Hence $y^2 = (x_3 - x_2)(x_2 - x_1)$ and therefore

$$\mathrm{c}(Q) = x_2 + i\sqrt{(x_3 - x_2)(x_2 - x_1)} . \tag{1.3}$$

Consequently, the second module of Q is equal to

$$\mathrm{m}(Q) = \frac{x_2 - x_1}{x_3 - x_2} . \tag{1.4}$$

The second module $\mathrm{m}(Q)$ is related to the module $\mathrm{M}(Q)$ of Q as follows.

Theorem 1.5. *The second module* $\mathrm{m}(Q)$ *is conformally invariant and the equality*

$$\mathrm{M}(Q) = \frac{2}{\pi}\mu\left(\frac{1}{\sqrt{1 + \mathrm{m}(Q)}}\right) \tag{1.5}$$

holds for every quadrilateral $Q := \Omega(z_1, z_2, z_3, z_4)$, *where*

$$\mu(r) := 2\pi\, \mathrm{M}(\mathbb{D} \setminus [0, r]) , \quad 0 < r < 1 ,$$

and $\mathrm{M}(\mathbb{D} \setminus [0, r])$ *is the module of the Grötzsch extremal domain defined by means of the extremal length.*

Proof. Since the harmonic measure is conformally invariant, so are by Lemma 1.1 the hyperbolic center $\mathrm{c}(Q)$ and the second module $\mathrm{m}(Q)$.

As shown in the proof of Lemma 1.1, there exist a point $\eta \in \mathbb{T}(1,-1)$ and a homeomorphism φ of $\overline{\Omega}$ onto $\overline{\mathbb{D}}$ which is conformal on Ω and sends the points z_1, z_2, z_3, z_4 into the points $1, \eta, -1, -\eta$, respectively. Define

$$h(-\eta) := \infty \quad \text{and} \quad h(u) := i\frac{\eta - u}{\eta + u}\,, \quad u \in \mathbb{C} \setminus \{-\eta\}\,.$$

Then $h \circ \varphi$ maps conformally Ω onto $\mathbb{C}_+$ and sends the points z_1, z_2, z_3, z_4 into the points $x_1 := h(1)$, $x_2 := 0 = h(\eta)$, $x_3 := h(-1)$ and $x_4 := \infty = h(-\eta)$, respectively. Since the second module $\mathrm{m}(Q)$ is conformally invariant and, by [G, p. 13],

$$\omega(i, \mathbb{C}_+)[(x_1, 0)] = \frac{1}{\pi}\arctan(-x_1) \quad , \quad \omega(i, \mathbb{C}_+)[(0, x_3)] = \frac{1}{\pi}\arctan x_3\,,$$

we see that

$$\mathrm{m}(\mathbb{C}_+(x_1, x_2, x_3, x_4)) = -\frac{x_1}{x_3}\,. \tag{1.6}$$

On the other hand

$$\mathrm{M}(\mathbb{C}_+(x_1, x_2, x_3, x_4)) = \frac{2}{\pi}\mu\left(\sqrt{\frac{x_3}{x_3 - x_1}}\right) = \frac{2}{\pi}\mu\left(\frac{1}{\sqrt{1 - \frac{x_1}{x_3}}}\right). \tag{1.7}$$

Combining (1.6) with (1.7) and applying the conformal invariance of the module of a quadrilateral we obtain (1.5). □

Theorem 1.5 enables us to express the quasiconformality of a mapping by means of the second module of a quadrilateral and the Hersch–Pfluger distortion function Φ_K, $K > 0$, defined by the equalities

$$\Phi_K(r) := \mu^{-1}(\mu(r)/K)\,, \quad 0 < r < 1\,, \quad \Phi_K(0) := 0\,,\ \Phi_K(1) := 1\,, \tag{1.8}$$

where μ^{-1} denotes the inverse of the homeomorphism μ; cf. [HP], [LV]. Applying the identities ([AVV, Thm. 3.3])

$$\Phi_K(r)^2 + \Phi_{1/K}(\sqrt{1 - r^2}\,)^2 = 1\,, \quad 0 \le r \le 1\,, \tag{1.9}$$

and

$$\mathrm{m}(\Omega(z_1, z_2, z_3, z_4))\,\mathrm{m}(\Omega(z_2, z_3, z_4, z_1)) = 1 \tag{1.10}$$

for all quadrilaterals $\Omega(z_1, z_2, z_3, z_4)$, we immediately obtain

Corollary 1.6. *For every $K \geq 1$ a sense-preserving homeomorphism $\varphi : U \to U' = \varphi(U) \subset \hat{\mathbb{C}}$ is K-qc. on a domain $U \subset \hat{\mathbb{C}}$ iff the inequality*

$$(1.11) \qquad \frac{1}{\sqrt{1+\mathrm{m}(\varphi * Q)}} \leq \Phi_K\left(\frac{1}{\sqrt{1+\mathrm{m}(Q)}}\right)$$

holds for every quadrilateral $Q = \Omega(z_1, z_2, z_3, z_4)$ satisfying $\overline{\Omega} \subset U$, where $\varphi * Q := \varphi(\Omega)(\varphi(z_1), \varphi(z_2), \varphi(z_3), \varphi(z_4))$.

Remark 1.7. As a matter of fact, the inequality (1.11) is equivalent to the double one

$$\Phi_{1/K}\left(\frac{1}{\sqrt{1+\mathrm{m}(Q)}}\right) \leq \frac{1}{\sqrt{1+\mathrm{m}(\varphi * Q)}} \leq \Phi_K\left(\frac{1}{\sqrt{1+\mathrm{m}(Q)}}\right)$$

for all quadrilaterals $Q = \Omega(z_1, z_2, z_3, z_4)$ satisfying $\overline{\Omega} \subset U$, which is due to (1.9) and (1.10).

2. Generalized quasisymmetry. We are now in a position to give a conformally invariant description of the class $\mathrm{Q}(\Gamma)$ for a boundary curve Γ of a Jordan domain $\Omega \subset \hat{\mathbb{C}}$ in terms of the second module of a quadrilateral.

Definition 2.1. A quadrilateral $Q := \Omega(z_1, z_2, z_3, z_4)$ is said to be a *hyperbolic square* if $\mathrm{m}(Q) = 1$; in other words, if

$$\begin{aligned}\omega(\mathrm{c}(Q), \Omega)[\Gamma(z_1, z_2)] = \omega(\mathrm{c}(Q), \Omega)[\Gamma(z_2, z_3)] &= \omega(\mathrm{c}(Q), \Omega)[\Gamma(z_3, z_4)] \\ = \omega(\mathrm{c}(Q), \Omega)[\Gamma(z_4, z_1)] &= \frac{1}{4}\,.\end{aligned}$$

The class of all hyperbolic squares $\Omega(z_1, z_2, z_3, z_4)$ is denoted by $\mathrm{HS}(\Omega)$. For a given $z \in \Gamma$ we write $\mathrm{HS}_z(\Omega)$ for the class of all $\Omega(z_1, z_2, z_3, z_4) \in \mathrm{HS}(\Omega)$ such that $z_4 = z$. If $f \in \mathrm{Hom}^+(\Gamma)$ and $Q := \Omega(z_1, z_2, z_3, z_4)$ is a quadrilateral, then we use the notation $f * Q$ for the quadrilateral $\Omega(f(z_1), f(z_2), f(z_3), f(z_4))$.

Theorem 2.2. *For every homeomorphism $f \in \mathrm{Hom}^+(\Gamma)$, $f \in \mathrm{Q}(\Gamma)$ iff the inequality*

$$(2.1) \qquad \frac{1}{M} \leq \mathrm{m}(f * Q) \leq M\ , \quad Q \in \mathrm{HS}(\Omega)\ ,$$

holds for some $M \geq 1$. More precisely, if $f \in \mathrm{Q}(\Gamma; K)$ for some $K \geq 1$, then f satisfies (2.1) *with $M := \lambda(K)$; see* (2.3). *Conversely, if f satisfies* (2.1) *with some $M \geq 1$, then*

$$f \in \mathrm{Q}\left(\Gamma; \min\{M^{3/2}, 2M-1\}\right) . \tag{2.2}$$

Proof. Assume that $f \in \mathrm{Q}(\Gamma)$. Then there exist $K \geq 1$ and a homeomorphic self-mapping F of $\overline{\Omega}$ such that $F_{|\Omega} \in \mathrm{QC}(\Omega; K)$ and $F_{|\Gamma} = f$. Since Ω is a Jordan domain, we conclude from Corollary 1.6, Remark 1.7 and [LV, Lemma 5.1 in Chap. I] that for every $Q \in \mathrm{HS}(\Omega)$,

$$\Phi_{1/K}\left(\frac{1}{\sqrt{1+\mathrm{m}(Q)}}\right) \leq \frac{1}{\sqrt{1+\mathrm{m}(f*Q)}} \leq \Phi_K\left(\frac{1}{\sqrt{1+\mathrm{m}(Q)}}\right) .$$

Since $\mathrm{m}(Q) = 1$, we obtain

$$\Phi_{1/K}\left(\frac{1}{\sqrt{2}}\right) \leq \frac{1}{\sqrt{1+\mathrm{m}(f*Q)}} \leq \Phi_K\left(\frac{1}{\sqrt{2}}\right) .$$

Hence by (1.9) we see that $1/\lambda(K) \leq \mathrm{m}(f*Q) \leq \lambda(K)$, where

$$\lambda(K) := \Phi_K\left(\frac{1}{\sqrt{2}}\right)^2 \Phi_{1/K}\left(\frac{1}{\sqrt{2}}\right)^{-2} , \quad K > 0 , \tag{2.3}$$

is the distortion function introduced by Lehto, Virtanen and Väisäla in [LVV]; see also [LV], [Le]. Setting $M := \lambda(K)$ we obtain (2.1).

Assume now that (2.1) holds for some $M \geq 1$. By the Riemann and Taylor–Osgood–Carathéodory theorems there exist homeomorphisms $H_1 : \overline{\mathbb{C}_+} \to \overline{\Omega} = H_1(\overline{\mathbb{C}_+})$ and $H_2 : \overline{\Omega} \to \overline{\mathbb{C}_+} = H_2(\overline{\Omega})$ conformal on $\mathbb{C}_+$ and Ω, respectively, satisfying

$$H_2 \circ f \circ H_1(\infty) = \infty . \tag{2.4}$$

Set $g(t) := H_2 \circ f \circ H_1(t)$, $t \in \overline{\mathbb{R}}$. By (2.4) the mapping $g_{|\mathbb{R}}$ is an increasing homeomorphism of $\mathbb{R}$ onto itself. Fix $x \in \mathbb{R}$ and $y > 0$. Example 1.4 shows that the quadrilateral $Q := \mathbb{C}_+(x-y, x, x+y, \infty)$ is a hyperbolic square and $\mathrm{c}(Q) = x + iy$. Since the second module is conformally invariant, $H_1(Q) \in \mathrm{HS}(\Omega)$, and by (2.1) we have

$$\frac{1}{M} \leq \mathrm{m}(f * H_1(Q)) = \mathrm{m}(H_2 * (f * H_1(Q))) = \mathrm{m}(g * Q) \leq M . \tag{2.5}$$

By (2.4), $g(\infty)=\infty$. Combining (2.5) with (1.4) we have

$$\frac{1}{M} \le \frac{g(x+y)-g(x)}{g(x)-g(x-y)} \le M\ .$$

Since the above inequality holds for all $x \in \mathbb{R}$ and $y > 0$, we see that $g_{|\mathbb{R}} \in \mathrm{QS}(\mathbb{R})$. Then the Beurling–Ahlfors extensions of g to $\mathbb{C}_+$ are qc. mappings; cf. [BA]. Moreover, Lehtinen's estimate [L, Thm. 1] shows that

$$g \in \mathrm{Q}\left(\overline{\mathbb{R}}; \min\{M^{3/2}, 2M-1\}\right)\ . \tag{2.6}$$

If $G \in \mathrm{QC}(\mathbb{C}_+)$ is a qc. extension of g to $\mathbb{C}_+$, then clearly

$$F := H_2^{-1} \circ G \circ H_1^{-1} \in \mathrm{QC}(\Omega)$$

is a qc. extension of f to Ω. Thus $f \in \mathrm{Q}(\Gamma)$. Moreover, by (2.6) we obtain (2.2). □

For a homeomorphism $f \in \mathrm{Hom}^+(\Gamma)$ we define

$$\delta(f;Q) := \max\left\{\mathrm{m}(f*Q)\,, \frac{1}{\mathrm{m}(f*Q)}\right\}\ , \quad Q \in \mathrm{HS}(\Omega)\ ;$$
$$\delta(f;z) := \sup\{\delta(f;Q) : Q \in \mathrm{HS}_z(\Omega)\}\ , \quad z \in \Gamma\ ;$$
$$\delta(f) := \sup\{\delta(f;Q) : Q \in \mathrm{HS}(\Omega)\} = \sup\{\delta(f;z) : z \in \Gamma\}\ .$$

We call $\delta(f)$ the *generalized quasisymmetric dilatation* of a homeomorphism $f \in \mathrm{Hom}^+(\Gamma)$. Write

$$\mathrm{GQS}(\Gamma; M) := \{f \in \mathrm{Hom}^+(\Gamma) : \delta(f) \le M\}\ , \quad M \ge 1\ ;$$
$$\mathrm{GQS}(\Gamma) := \{f \in \mathrm{Hom}^+(\Gamma) : \delta(f) < \infty\} = \bigcup_{M\ge 1} \mathrm{GQS}(\Gamma; M)\ .$$

In other words, $f \in \mathrm{GQS}(\Gamma; M)$ iff f satisfies (2.1) with M, $M \ge 1$.

Definition 2.3. Given $M \ge 1$ we call $f \in \mathrm{GQS}(\Gamma; M)$ a *generalized M–quasisymmetric homeomorphism of* Γ. A mapping f is said to be a *generalized quasisymmetric homeomorphism of* Γ if $f \in \mathrm{GQS}(\Gamma)$.

Remark 2.4. By Theorem 2.2 we have

$$\mathrm{Q}(\Gamma) = \mathrm{GQS}(\Gamma)\ ;$$
$$\mathrm{Q}(\Gamma; K) \subset \mathrm{GQS}(\Gamma; \lambda(K))\ , \quad K \ge 1\ ;$$
$$\mathrm{GQS}(\Gamma; M) \subset \mathrm{Q}\left(\Gamma; \min\{M^{3/2}, 2M-1\}\right)\ , \quad M \ge 1\ .$$

As shown in the proof of Theorem 2.2, the last inclusion can be improved as follows

$$M = \inf_{z\in\Gamma} \delta(f;z) \implies f \in \mathrm{Q}\left(\Gamma; \min\{M^{3/2}, 2M-1\}\right)\ , \quad f \in \mathrm{GQS}(\Gamma)\ .$$

Corollary 2.5. *The generalized quasisymmetric dilatation δ is conformally invariant, i.e. for every $f \in \mathrm{Hom}^+(\Gamma)$,*

$$\delta(h_1 \circ f \circ h_2) = \delta(f)\,, \quad h_1, h_2 \in \mathrm{Q}(\Gamma; 1)\,. \tag{2.7}$$

Moreover,

$$\mathrm{GQS}(\Gamma; 1) = \mathrm{Q}(\Gamma; 1)\,. \tag{2.8}$$

Proof. For every $Q \in \mathrm{HS}(\Omega)$ we have

$$\mathrm{m}((h_1 \circ f \circ h_2) * Q) = \mathrm{m}(h_1 * (f * (h_2 * Q))) = \mathrm{m}(f * (h_2 * Q))\,.$$

Since $Q \in \mathrm{HS}(\Omega)$ iff $h_2 * Q \in \mathrm{HS}(\Omega)$, $\delta(h_1 \circ f \circ h_2; Q) = \delta(f; h_2 * Q)$ and hence (2.7) follows.

Let id_Γ denote the identity self-mapping of Γ. Evidently, $\delta(\mathrm{id}_\Gamma) = 1$. Thus by (2.7), $\delta(f) = 1$ for all $f \in \mathrm{Q}(\Gamma; 1)$. Hence $\mathrm{Q}(\Gamma; 1) \subset \mathrm{GQS}(\Gamma; 1)$.

Conversely, assume that $f \in \mathrm{GQS}(\Gamma; 1)$. Then (2.2) in Theorem 2.2 shows that $f \in \mathrm{Q}(\Gamma; 1)$, and hence $\mathrm{GQS}(\Gamma; 1) \subset \mathrm{Q}(\Gamma; 1)$. The above inclusions yield (2.8). □

Remark 2.6. Let $z_1, z_2, z_3 \in \Gamma$ be a triple of points ordered according to the positive orientation of Γ with respect to Ω and let φ be the mapping from Lemma 1.1. Set $z_4 := \varphi^{-1}(-i)$ and $z := \varphi^{-1}(0)$. Since $Q := \mathbb{D}(1, i, -1, -i) \in \mathrm{HS}(\mathbb{D})$ and $\mathrm{c}(Q) = 0$, we see that $\Omega(z_1, z_2, z_3, z_4) = \varphi^{-1} * Q \in \mathrm{HS}(\Omega)$ and $\mathrm{c}(\Omega(z_1, z_2, z_3, z_4)) = z$ and that z_4, z are unique such points. Thus the points z_1, z_2, z_3 determine uniquely the hyperbolic square $\Omega(z_1, z_2, z_3, z_4)$ and its hyperbolic center. Similarly, given $z_1 \in \Gamma$ and $z \in \Omega$ we can uniquely determine $Q := \Omega(z_1, z_2, z_3, z_4) \in \mathrm{HS}(\Omega)$ such that $\mathrm{c}(Q) = z$. Therefore the generalized quasisymmetric dilatation δ gives a three real parameters description of the class $\mathrm{Q}(\Gamma)$ which is, by Corollary 2.5, conformally invariant.

3. The case of the real axis or the unit circle.

In this section we assume that $\Gamma := \mathbb{T}$ and $\Omega := \mathbb{D}$, or $\Gamma := \mathbb{R}$ and $\Omega := \mathbb{C}_+$.

Lemma 3.1. *For every quadrilateral $Q := \Omega(z_1, z_2, z_3, z_4)$,*

$$\mathrm{m}(Q) = \frac{[z_2, z_3, z_4, z_1]}{[z_1, z_2, z_3, z_4]} = \frac{1}{[z_1, z_2, z_3, z_4]} - 1\,. \tag{3.1}$$

In particular, $Q \in \mathrm{HS}(\Omega)$ iff $[z_1, z_2, z_3, z_4] = 1/2$.

Proof. Since the second module is conformally invariant, we may restrict ourselves to the case where $\Gamma := \overline{\mathbb{R}}$ and $Q := \mathbb{C}_+(x_1, x_2, x_3, \infty)$. Then by (1.4),

$$\mathrm{m}(Q) = \frac{x_2 - x_1}{x_3 - x_2} = \frac{[x_2, x_3, \infty, x_1]}{[x_1, x_2, x_3, \infty]}$$

which combined with the identity

$$[z_1, z_2, z_3, z_4] + [z_2, z_3, z_4, z_1] = 1$$

shows (3.1). The latter part of the lemma follows easily from (3.1). □

Corollary 3.2. *Given a triple of points $z_1, z_2, z_3 \in \Gamma$ ordered according to the positive orientation of Γ with respect to Ω, there exist unique points $z_4 \in \Gamma$ and $z \in \Omega$ such that $Q := \Omega(z_1, z_2, z_3, z_4) \in \mathrm{HS}(\Omega)$ and $\mathrm{c}(Q) = z$. Moreover, the following equalities hold:*

(3.2) $$z_4 = \frac{(z_3 - z_2)z_1 - (z_2 - z_1)z_3}{(z_3 - z_2) - (z_2 - z_1)}$$

and

$$\mathrm{c}(Q) = \frac{(z_3 - z_2)z_1 + i(z_2 - z_1)z_3}{(z_3 - z_2) + i(z_2 - z_1)} \ .tag3.3$$

Proof. The equality (3.2) follows directly from the equality $[z_1, z_2, z_3, z_4] = 1/2$. By the equality (1.3) we have

(3.4) $$\mathrm{c}(\mathbb{C}_+(-t, 0, t, \infty)) = it\,, \quad t > 0\,.$$

There exists a unique conformal self-mapping h of $\hat{\mathbb{C}}$ satisfying

$$h(-t) = z_1\,, \quad h(0) = z_2\,, \quad h(t) = z_3\,.$$

Since $h(\mathbb{C}_+) = \Omega$ and since hyperbolic center is conformally invariant, we have $\mathrm{c}(Q) = h(it)$ by (3.4). Then (3.3) follows from the equality

$$[z_1, z_2, z_3, h(it)] = [-t, 0, t, it]\,. \quad \square$$

By (3.1) we obtain

Corollary 3.3. *If $f \in \mathrm{Hom}^+(\Gamma)$ and if $M \geq 1$, then $f \in \mathrm{GQS}(\Gamma; M)$ iff the inequality*

$$\frac{1}{M+1} \leq [f(z_1), f(z_2), f(z_3), f(z_4)] \leq \frac{M}{M+1}$$

holds for all $\Omega(z_1, z_2, z_3, z_4) \in \mathrm{HS}(\Omega)$.

Combining Corollary 3.3 with the first inclusion in Remark 2.4 we obtain

Corollary 3.4. *If $K \geq 1$ and if $F \in \mathrm{QC}(\Omega; K)$, then the mapping $f := \mathrm{Tr}[F]$ satisfies the inequality*

$$\frac{1}{\lambda(K)+1} \leq [f(z_1), f(z_2), f(z_3), f(z_4)] \leq \frac{\lambda(K)}{\lambda(K)+1}$$

for all $\Omega(z_1, z_2, z_3, z_4) \in \mathrm{HS}(\Omega)$.

4. Applications. In this section we give some results that are obtained by using the generalized quasisymmetry. Applying (1.8) and the identity [Z, (2.4)]

$$\mathrm{M}(\mathbb{C}_+(x_1, x_2, x_3, x_4)) = \frac{2}{\pi}\mu\left(\sqrt{[x_1, x_2, x_3, x_4]}\right)$$

for all positively ordered quadruples of points $x_1, x_2, x_3, x_4 \in \overline{\mathbb{R}}$, we can easily show that for every $K \geq 1$,

$$G \in \mathrm{QC}(\mathbb{C}_+; K) \implies \mathrm{Tr}[G] \in \mathrm{QH}(\overline{\mathbb{R}}; K)\,; \tag{4.1}$$

cf. [Z, Thm. 2.1]. We use (4.1) to prove Theorem 4.1 which is a generalization of the result by Krzyż [K, Thm. 1]. For $K \geq 1$ and $0 < \rho \leq 1$ set

$$A(K,\rho) := (1+\lambda(K))\Phi_{1/K}\left(\sqrt{\frac{2\rho}{1+\rho}}\right)^{-2} - 1$$

and

$$B(K,\rho) := \frac{1+\lambda(K)}{\lambda(K)}\Phi_K\left(\sqrt{\frac{2\rho}{1+\rho}}\right)^{-2} - 1\,.$$

It is easy to check that for all $K \geq 1$ and $0 < \rho \leq 1$,

$$B(K,\rho)^{-1} \leq \lambda(K) \leq A(K,\rho)$$

and $B(K,\rho)^{-1} = \lambda(K) = A(K,\rho)$ iff $\rho = 1$.

Theorem 4.1. *Suppose that $K \geq 1$ and that a mapping $F \in \mathrm{QC}(\mathbb{D}, K)$ satisfies $F(0) = 0$. If $I_1, I_2 \subset \mathbb{T}$ are adjacent arcs of positive length satisfying $\rho := |I_2|_1/|I_1|_1 \leq 1$, then*

$$A(K,\rho)^{-1} \leq B(K,\rho) \leq \frac{|F^*(I_1)|_1}{|F^*(I_2)|_1} \leq A(K,\rho)\,. \tag{4.2}$$

Proof. Assume first that $|I_1|_1 > |I_2|_1$ and that adjacent arcs $I_1, I_2 \subset \mathbb{T}$ are ordered according to the positive orientation of $\mathbb{T}$, i.e. $\{e^{it} : t_1 \le t \le t_2\} = I_1$ and $\{e^{it} : t_2 \le t \le t_3\} = I_2$ for some $t_1, t_2, t_3 \in \mathbb{R}$ satisfying $0 \le t_1 < 2\pi$, $t_1 < t_2 < t_3 \le t_1 + 2\pi$. Following Krzyż [K] we can assign to F a K-qc. self-mapping G of $\mathbb{C}_+$ satisfying the identity $F\left(e^{iz}\right) = e^{iG(z)}$, $z \in \mathbb{C}_+$. The mapping G is uniquely determined if we assume $0 \le G^*(0) < 2\pi$. Then $G^*(\infty) = \infty$ and Corollary 3.4 says that the inequality

$$(4.3) \qquad \frac{1}{\lambda(K)+1} \le [G^*(z_1), G^*(z_2), G^*(z_3), G^*(z_4)] \le \frac{\lambda(K)}{\lambda(K)+1}$$

holds for all $z_1, z_2, z_3 \in \mathbb{R}$, $z_1 < z_2 < z_3$, where z_4 is given by (3.2). Assume now that the points $z_1, z_2, z_3 \in \mathbb{R}$ are chosen such that $z_l = t_l$ for $l = 1, 2, 3$. Then $\{e^{it} : z_1 \le t \le z_2\} = I_1$ and $\{e^{it} : z_2 \le t \le z_3\} = I_2$. Hence

$$(4.4) \qquad |I_1|_1 = z_2 - z_1 \quad \text{and} \quad |I_2|_1 = z_3 - z_2 \ .$$

From (3.2) and (4.4) it follows that

$$z_4 = z_2 + 2\left(\frac{1}{|I_2|_1} - \frac{1}{|I_1|_1}\right)^{-1} > z_3 = z_2 + |I_2|_1$$

and consequently

$$(4.5) \qquad [z_1, z_2, z_4, \infty] = \frac{z_2 - z_4}{z_1 - z_4} = \frac{2\rho}{1+\rho} .$$

Note that

$$(4.6) \qquad \begin{aligned} &\frac{|F^*(I_2)|_1}{|F^*(I_1)|_1 + |F^*(I_2)|_1} \cdot \frac{1}{[G^*(z_1), G^*(z_2), G^*(z_3), G^*(z_4)]} \\ &= \frac{G^*(z_2) - G^*(z_4)}{G^*(z_1) - G^*(z_4)} = [G^*(z_1), G^*(z_2), G^*(z_4), G^*(\infty)] . \end{aligned}$$

We conclude from (4.1), (4.5) and (4.6) that

$$(4.7) \qquad \Phi_{1/K}\left(\sqrt{\frac{2\rho}{1+\rho}}\right)^2 \le \frac{G^*(z_2) - G^*(z_4)}{G^*(z_1) - G^*(z_4)} \le \Phi_K\left(\sqrt{\frac{2\rho}{1+\rho}}\right)^2 .$$

Combining (4.3) with (4.6) and (4.7) we obtain (4.2). In the case where $|I_1|_1 > |I_2|_1$ and I_1 and I_2 are not ordered according to the positive orientation of $\mathbb{T}$, we apply the above reasoning again, with F replaced by the function $\mathbb{D} \ni z \mapsto \overline{F(\overline{z})} \in \mathbb{D}$, to obtain (4.2). If $|I_1|_1 = |I_2|_1$, then $\rho = 1$, $z_4 = \infty$ and (4.2) follows directly from (4.3). □

By Theorem 4.1 we immediately obtain

Corollary 4.2. *Suppose that $K \geq 1$ and that a mapping $F \in \mathrm{QC}(\mathbb{D}, K)$ satisfies $F(0) = 0$. If $M \geq 1$ and if $f \in \mathrm{QS}(\mathbb{T}; M)$, then*

$$F^* \circ f \in \mathrm{QS}(\mathbb{T}; A(K, 1/M)) . \tag{4.8}$$

Given $M \geq 1$ and $f \in \mathrm{QS}(\mathbb{T}; M)$ we can apply Lehtinen's estimate (2.6) to show that $f = \mathrm{Tr}[F]$ for some $F \in \mathrm{QC}\left(\mathbb{D}; \min\{M^{3/2}, 2M - 1\}\right)$ satisfying $F(0) = 0$; see the discussion in [P, p. 68]. Then Corollary 4.2 yields

Corollary 4.3. *If $M_1, M_2 \geq 1$, if $f_1 \in \mathrm{QS}(\mathbb{T}; M_1)$ and if $f_2 \in \mathrm{QS}(\mathbb{T}; M_2)$, then*

$$f_2 \circ f_1 \in \mathrm{QS}\left(\mathbb{T}; A(\min\{M_2^{3/2}, 2M_2 - 1\}, 1/M_1)\right) . \tag{4.9}$$

Analyzing the proof of Theorem 4.1 we additionally obtain

Corollary 4.4. *If $K, M \geq 1$, $f \in \mathrm{QS}(\mathbb{R}; M)$ and if $g \in \mathrm{QH}(\overline{\mathbb{R}}; K)$ satisfies $g(\infty) = \infty$, then* (4.8) *holds with F^* and $\mathbb{T}$ replaced by g and $\overline{\mathbb{R}}$, respectively.*

Applying again Lehtinen's estimate (2.6) we deduce from Corollary 4.4 the following counterpart of Corollary 4.3.

Corollary 4.5. *If $M_1, M_2 \geq 1$, $f_1 \in \mathrm{QS}(\mathbb{R}; M_1)$ and if $f_2 \in \mathrm{QS}(\mathbb{R}; M_2)$, then* (4.9) *holds with $\mathbb{T}$ replaced by $\mathbb{R}$.*

References

[AVV] Anderson, G. D., M. K. Vamanamurthy and M. Vuorinen, *Distortion function for plane quasiconformal mappings*, Israel J. Math. **62** (1988), 1-16.

[BA] Beurling, A. and L. V. Ahlfors, *The boundary correspondence under quasiconformal mappings*, Acta Math. **96** (1956), 125-142.

[G] Garnett, J. B., *Bounded Analytic Functions*, Academic Press, New York, 1981.

[HP] Hersch, J. and A. Pfluger, *Généralisation du lemme de Schwarz et du principe de la mesure harmonique pour les fonctions pseudo-analytiques*, C. R. Acad. Sci. Paris. **234** (1952), 43-45.

[K] Krzyż, J. G., *Quasicircles and harmonic measure*, Ann. Acad. Sci. Fenn. Ser. A I. Math. **12** (1987), 19-24.

[L] Lehtinen, M., *Remarks on the maximal dilatation of the Beurling-Ahlfors extension*, Ann. Acad. Sci. Fenn. Ser. A. I. Math. **9** (1984), 133-139.

[Le] Lehto, O., *Univalent Functions and Teichmüller Spaces*, Graduate Texts in Math. 109, Springer, New York, 1987.

[LV] Lehto, O. and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, Grundlehren 126, 2nd., Springer, Berlin, 1973.

[LVV] Lehto, O., K. I. Virtanen and J. Väisälä, *Contributions to the distortion theory of quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A. I. Math. **273** (1959), 1-14.

[P] Partyka, D., *The generalized Neumann-Poincaré operator and its spectrum*, Dissertationes Math. No. 366, Institute of Mathematics, Polish Academy of Sciences, Warszawa, 1997.

[Z] Zając, J., *Quasihomographies in the theory of Teichmüller spaces*, Dissertationes Math. No. 357, Institute of Mathematics, Polish Academy of Sciences, Warszawa, 1996.

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