ANNALES
UNIVERSITATIS MARIAE CURIE – SKLODOWSKA
LUBLIN – POLONIA
VOL. LIII, 15 SECTIO A 1999

MARCELINA MOCANU

Some sufficient conditions for the convergence of the derivatives of weakly quasiregular mappings

ABSTRACT. The aim of this paper is to prove n - dimensional generalizations of two theorems from the book of Lehto and Virtanen [8], which deal with the convergence of the derivatives for a sequence of plane K-quasiconformal mappings. We use the methods and results of T. Iwaniec and G. Martin ([4], [5]).

1. Introduction. Developing the ideas from the paper of S. Donaldson and D. Sullivan [3], T. Iwaniec and G. Martin introduced in [4] a new approach in the theory of quasiregular mappings, in the case when n is even. This approach has a strong analogy with the two-dimensional case and relies on the study of the Hodge theory on L^p - spaces and of a new Beurling- Ahlfors type singular operator.

The Beltrami equation plays an important role in the theory of planar quasiconformal mappings.

The basic idea is to consider a weakly quasiregular mapping $f : \Omega \to \mathbb{R}^n$ as a solution of an n - dimensional Beltrami system and then to lift this Beltrami system to the exterior bundle $\Lambda^l(\Omega)$.

If n is even and $l = n/2$ they consider in this way the Beltrami equation in even dimensions, which gives the possibility to apply the Beurling- Ahlfors operator. Using the L^p - norm of the Beurling-Ahlfors operator Iwaniec and Martin have proved in even dimensions a regularity theorem which generalizes a well-known theorem of Bojarski [1].

They also proved, for the first time, a Caccioppoli type L^p - estimate, with $p < n$, for quasiregular mappings in even dimensions.

In [5] Iwaniec has obtained similar results for all dimensions, by a different approach based on maximal inequalities.

Let Ω be a domain in $\mathbb{R}^n$ and let $f = (f^1, f^2, \dots, f^n) : \Omega \to \mathbb{R}^n$ be a mapping of Sobolev class $W^{1,p}_{\mathrm{loc}}(\Omega)$, $1 \le p < \infty$. The formal differential of f, denoted by $Df(x)$, is defined for almost every $x \in \Omega$ and belongs to $L^p_{\mathrm{loc}}(\Omega, GL(n))$. We denote by $J(x,f)$ the Jacobian determinant of f.

Definition 1. A mapping $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is said to be weakly K - quasiregular, $1 \le K < \infty$, if

(i) $J(x,f) \ge 0$ a.e.;

(ii) $\max_{|h|=1} |Df(x)h| \le K \min_{|h|=1} |Df(x)h|$ for almost every $x \in \Omega$.

If $p = n$ we say that f is K - quasiregular.

We notice that (i) and (ii) imply $|Df(x)|^n \le K^{n-1} J(x,f)$ a.e. The matrix dilatation of f is defined as $G(x) = J(x,f)^{-(2/n)} Df(x)^t Df(x)$ if $Df(x)$ exists and $J(x,f) \ne 0$, otherwise $G(x) = I$. In this way, f becomes a weak solution of the following n - dimensional Beltrami system

$$Df(x)^t Df(x) = J(x,f)^{2/n} G(x) .$$

In order to state the Beltrami equation in even dimension we need some auxiliary notation and terminology.

We denote by $\Lambda^k(\Omega)$ the space of k-differential forms $\omega = \sum_I \omega_I dx^I$ whose coefficients are complex valued distributions. Here $I = (i_1, i_2, \dots, i_k)$ runs over all ordered k-tuples of integers $1 \le i_1 < i_2 < \dots < i_k \le n$ and $dx^I = dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}$. In the sequel, to each space Φ of complex functions defined on Ω there corresponds the space $\Phi(\Omega, \Lambda^k)$ of k - differential forms with coefficients in Φ. Without saying so every time, we will sometimes assume that the distributions in question are represented by locally integrable functions. In such a case the pointwise inner product of two differential forms $\lambda, \mu \in \Lambda^k(\Omega)$ is a function denoted by

$$< \lambda, \mu > = \sum_I \lambda_I \overline{\mu}_I .$$

The Hodge star operator $* : \Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega)$ is defined by the rule

$$\overline{\mu} \wedge *\lambda = < \lambda, \mu > dx^1 \wedge \dots \wedge dx^n .$$

For $n = 2l$ we denote by $\Lambda^{\pm}$ the eigenspaces of $* : \Lambda^l(\Omega) \to \Lambda^l(\Omega)$, namely $\Lambda^{\pm} = \{\omega \in \Lambda^l(\Omega) : *\omega = \pm i^l \omega\}$. This gives an orthogonal decomposition $\Lambda^l(\Omega) = \Lambda^+ \oplus \Lambda^-$.

Let ω_+ and ω_- be the projections of $\omega \in \Lambda^l(\Omega)$ on Λ^+ and Λ^-, respectively. We make use of the exterior derivative operator $d : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$ and of its formal adjoint, the Hodge operator $\delta : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$, $\delta = (-1)^{n(n-k)+1} * d *$. Then the Laplace-Beltrami operator takes the form $\Delta = d\delta + \delta d : \Lambda^k(\Omega) \to \Lambda^k(\Omega)$. The Beurling-Ahlfors operator $S : L^p(\mathbb{R}^n, \Lambda^k) \to L^p(\mathbb{R}^n, \Lambda^k)$, which generalizes the complex Hilbert transform, has the formal expression $S = (d\delta - \delta d) \circ \Delta^{-1}$. It is known that S is bounded in all the spaces $L^p(\mathbb{R}^n, \Lambda^k)$ with $1 < p < \infty$ and $k = 0, 1, 2, \ldots, n$.

In dimension $2l$ this operator permutes the spaces $L^p(\mathbb{R}^{2l}, \Lambda^+)$, and $L^p(\mathbb{R}^{2l}, \Lambda^-)$, thus $S \circ d^{\pm} = d^{\pm}$, where $d^{\pm}\omega = (d\omega)_{\pm}$.

Let $f : \Omega \to \mathbb{R}^n$ be a mapping of Sobolev class $W^{1,p}_{\mathrm{loc}}(\Omega)$ and let $k \leq p$ be a positive integer. Then f induces a homomorphism $f^* : C^\infty(\mathbb{R}^n, \Lambda^k) \to L^{p/k}_{\mathrm{loc}}(\Omega, \Lambda^k)$, called the pullback, which is defined by

$$f^*\left(\sum_I \alpha_I dx^I\right)(x) = \sum_I \alpha_I(f(x))\, df^{i_1} \wedge df^{i_2} \wedge \ldots df^{i_k}.$$

Another pullback is that of l - covectors induced by a linear transformation. Let $T : E \to F$ be a linear operator between n - dimensional vector spaces and let $T' : F' \to E'$ be its dual. The pullback of l - covectors via T, which will be denoted by $T_\# : \Lambda^l F \to \Lambda^l E$, is defined as

$$T_\#(\zeta_1 \wedge \zeta_2 \wedge \ldots \wedge \zeta_n) = T'\zeta_1 \wedge T'\zeta_2 \wedge \ldots \wedge T'\zeta_n,$$

for $\zeta_1, \zeta_2, \ldots \zeta_n \in F'$ and then extended linearly to $\Lambda^l F$.

Now we may recall the definition of the Beltrami coefficient of a weakly quasiregular mapping, the notion which generalizes the complex dilatation of a planar quasiregular mapping. Let G be the distortion tensor of the weakly quasiregular mapping f.

The Beltrami coefficient of f, denoted by $\mu_f : \Lambda^l\Omega \to \Lambda^l\Omega$, is a bundle map such that

$$\mu_f(x) = \left[(G(x) - I)(G(x) + I)^{-1}\right]_\#.$$

We recall that the operator norm $|\mu_f(x)|$ satisfies for every $x \in \Omega$ the inequality

$$|\mu_f(x)| \leq \frac{K^l - 1}{K^l + 1}. \tag{1}$$

The Beltrami equation in even dimensions takes then the form

$$d^+(f^*\alpha) = \mu_f d^-(f^*\alpha) \tag{2}$$

where $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a weakly quasiregular mapping in a domain $\Omega \subset \mathbb{R}^{2l}$ with $l \le p < \infty$, and α is an $(l-1)$ - differential form with linear coefficients, such that $d^+\alpha = 0$.

We need the following regularity results ([7]).

Proposition 1 (The Caccioppoli type estimate). *For each dimension $n \ge 2$ and $K \ge 1$ there exist exponents $q(n,K) < n < p(n,K)$ such that if $f \in W^{1,r}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is weakly K - quasiregular mapping with $q(n,K) \le r \le p(n,K)$ then*

$$\big|\big|\, |\varphi Df|\, \big|\big|_r \le c(n,K,r) \big|\big|\, |\nabla\varphi|\, |f|\, \big|\big|_r \tag{3}$$

for all $\varphi \in C_0^\infty(\Omega)$.

An important consequence of the Cacciopolli type estimate is

Proposition 2 (The Regularity Theorem). *Every weakly K-quasiregular mapping of the class $W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ with $q = q(n,K) < n$ belongs to $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ where $p = p(n,K) > n$.*

In the case $n = 2l$ we can say more about these exponents and the constant $c(n,K,r)$. Suppose that $n = 2l$. Let f be a K - quasiregular mapping and let $p_0 = p_0(f) < 2 < q_0(f) = q_0$ be the critical exponents of f, defined by

$$|\mu_f|\, ||S||_{p_0} = |\mu_f|\, ||S||_{q_0} = 1\,. \tag{4}$$

Here we have used the norms of $S : L^p(\mathbb{R}^{2l}, \Lambda^l) \to L^p(\mathbb{R}^{2l}, \Lambda^l)$ for $p = p_0$ and $p = q_0$. Then $|\mu_f|\, ||S||_p < 1$ for every $p \in (q_0(f), p_0(f))$.

Now the Regularity Theorem can be given somewhat more precise formulation: If $f \in W^{sl}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2l})$ is weakly quasiregular with $s > q_0(f)$, then $f \in W^{1,pl}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2l})$ for every $p \in (q_0(f), p_0(f))$. Consequently f is quasiregular in the usual sense.

The Caccioppoli type estimate holds now for every $p \in (q_0(f), p_0(f))$ and takes the form

$$\big|\big|\, |\varphi Df|\, \big|\big|_{pl} \le \left(\frac{8lK^l ||S||_p}{1 - |\mu_f|\, ||S||_p} \right) \big|\big|\, |\nabla\varphi|\, |f|\, \big|\big|_{pl}\,.$$

Remark 1. A more careful analysis of the proof of Lemma 2.9 in [4] actually shows that it suffices to assume less, namely $s > [(2l-1)/(2l)]q_0(f)$, in the Regularity Theorem reformulated above.

In order to obtain an evaluation of the exponents $q(2l, K)$ and $p(2l, K)$ which does not depend on the particular choice of the K - quasiregular mapping f, we consider the exponents $q_0' = q_0'(l, K)$ and $p_0' = p_0'(l, K)$ defined by

$$(5) \qquad ||S||_{q_0'} = ||S||_{p_0'} = \frac{K^l + 1}{K^l - 1}.$$

By (4) and (5) it is clear that $q_0(f) < q_0'(l, K) < 2 < p_0'(l, K) < p_0(f)$ for each weakly K - quasiregular mapping f. Then we may take in both the Caccioppoli type estimate and the Regularity Theorem $q(2l, K)$ and $p(2l, K)$ to be any numbers such that

$$lq_0'(l, K) < q(2l, K) < 2l < p(2l, K) < lp_0'(l, K).$$

Moreover, an explicit form for the constant in (3) is also available,

$$c(2l, K, r) = \frac{8lK^l||S||_{r/l}}{1 - [(K^l - 1)/(K^l + 1)]||S||_{r/l}}.$$

2. Weak convergence of derivatives. Lehto and Virtanen have proved in [8] that the locally uniform convergence of a sequence $\{w_n\}$ of planar K-quasiregular mappings to a mapping w implies the weak convergence of the derivatives $\partial w_n/\partial z$ and $\partial w_n/\partial \bar{z}$ to $\partial w/\partial z$ and $\partial w/\partial \bar{z}$, respectively.

We will prove an analogous result for n - dimensional K - quasiregular mappings, which is a generalization of the lemma mentioned above, even for $n = 2$.

We recall the definitions of the weak convergence in L^p_{loc}, for functions and for differential forms, respectively, given in [9].

Definition 2. Let $U \subset \mathbb{R}^n$ be an open set and let $p \geq 1$. We say that the sequence $\{f_m\}$ of functions $f_m \in L^p_{\text{loc}}(U)$, $m = 1, 2, \ldots$, converges weakly to f_0 in $L^p_{\text{loc}}(U)$ if it is bounded in $L^p_{\text{loc}}(U)$ and for every function $\varphi \in C_0^\infty(U)$

$$\lim_{m\to\infty} \int_U f_m \varphi dx = \int_U f \varphi dx.$$

Let $\{\omega_m\}$ be a sequence of differential forms in $L^p_{\text{loc}}(U, \Lambda^k)$, where $1 \leq k \leq n$. Then we say that this sequence of forms converges to a form ω_0

weakly in $L^p_{\text{loc}}(U, \Lambda^k)$ if the coefficients of the forms ω_m converge weakly in $L^p_{\text{loc}}(U)$ to the corresponding coefficients of ω_0.

Remark 2. We notice that the definition of the weak convergence of functions in L^p_{loc} may be carried over to the class of mappings $f_m : U \to \mathbb{R}^k$, for any $k \geq 1$.

It is useful to recall that $\{\omega_m\}$ converges weakly in $L^p_{\text{loc}}(U, \Lambda^k)$ to ω_0 if and only if the sequence of norms $(|\omega_m|)$ is bounded in $L^p_{\text{loc}}(U)$ and

$$\lim_{m\to\infty} \int_U \omega_m \wedge \theta = \int_U \omega \wedge \theta \tag{6}$$

for every form $\theta \in C_0^\infty(U, \Lambda^{n-k})$.

The following two lemmas from [9] will play a key role in the proof of Theorem 1 in this paper.

The first lemma is a test for a function to belong to the class $W^p_{\text{loc}}(U)$, while the second provides sufficient conditions for the weak convergence of the pullbacks $f_m^*(dx^I)$, $m = 1, 2, \ldots$, to $f_0^*(dx^I)$.

Lemma 1. *Let u_0 be a function in $L^1_{\text{loc}}(U)$. Assume that there exists a sequence $u_m : U \to \mathbb{R}$ of functions in $W^{1,p}_{\text{loc}}(U)$, where $1 \leq p \leq \infty$, converging to u_0 in $L^1_{\text{loc}}(U)$ and bounded in $W^{1,p}_{\text{loc}}(U)$. Then $u_0 \in W^{1,p}_{\text{loc}}(U)$.*

Remark 3. An analogous statement is true for mappings.

Lemma 2. *Let U be an open set in $\mathbb{R}^n$, $1 \leq k \leq n$ and let*

$$g_m = (g_m^1, g_m^2, \ldots, g_m^k),\ m = 1, 2, \ldots,$$

be a sequence of mappings of class $W^{1,p}_{\text{loc}}(U)$ where $p \geq k$. Assume that g_m is bounded in $W^{1,p}_{\text{loc}}(U)$ and that $g_m \to g_0$ in $L^1_{\text{loc}}(U)$, where $g_0 = (g_0^1, g_0^2, \ldots, g_0^k)$. Then the sequence of forms $\{dg_m^1 \wedge dg_m^2 \wedge \ldots \wedge dg_m^k\}$ converges to $\{dg_0^1 \wedge dg_0^2 \wedge \ldots \wedge dg_0^k\}$ weakly in $L^{p/k}_{\text{loc}}(U, \Lambda^k)$.

Remark 4. For $k = n$ the weak convergence of the Jacobian determinants follows: $J(\,\cdot\,, g_m\,) \to J(\,\cdot\,, g_0\,)$ weakly in $L^{p/n}_{\text{loc}}(U)$.

Here is our first main result.

Theorem 1. *Let Ω be a domain in $\mathbb{R}^n$ and let $K \geq 1$. Suppose that $s \in [q(n,K), p(n,K)]$ and h is an integer in the interval $[1,s]$.*
If $\{f_m\}$ is a sequence of weakly K - quasiregular mappings in $W^{1,s}_{\mathrm{loc}}(\Omega)$ such that $f_m \to f_0$ in $L^s_{\mathrm{loc}}(\Omega)$ for some $f_0 : \Omega \to \mathbb{R}^n$, then:

(i) $f_0 \in W^{1,s}_{\mathrm{loc}}(\Omega)$;

(ii) *f_0 is quasiregular in the usual sense if $s \geq n$.*

(iii) *For every $(h-1)$ - form α with linear coefficients the sequence of forms $\{df_m^* \alpha\}$ converges to $df_0^* \alpha$ weakly in $L^{s/h}_{\mathrm{loc}}(\Omega, \Lambda^h)$.*

Proof. We notice that f_m are actually quasiregular and $f_m \in W^{1,q}_{\mathrm{loc}}(\Omega)$ for every $q \in [q(n,K), p(n,K)]$. Obviously, $f_m \to f_0$ in $L^s_{\mathrm{loc}}(\Omega)$.

In order to verify that $\{f_m\}$ is bounded in $W^{1,s}_{\mathrm{loc}}(\Omega)$ it suffices to prove that the sequence of norms $\{|Df_m|\}$ is bounded in $L^s_{\mathrm{loc}}(\Omega)$.

Let F be a compact subset of Ω and let $\varphi \in C_0^\infty(\Omega)$ be such that $\varphi = 1$ on F. We denote the support of φ by F'. Applying the Caccioppoli type estimate (3), we obtain

$$
\begin{aligned}
\| |Df_m| \|_{s,F} &\leq \| |\varphi Df_m| \|_s \leq c(n,K,S) \| |\nabla\varphi| |f_m| \|_{s,F'} \\
&\leq c(n,K,s) \sup_{F'} |\nabla \varphi| \, \|f_m\|_{s,F'} \, .
\end{aligned}
\tag{7}
$$

Since $\{f_m\}$ converges in $L^s(F')$ and therefore is bounded therein, (7) implies that $\{|Df_m|\}$ is bounded in $L^s(F)$. Now, by Lemma 1, $f_0 \in W^{1,s}_{\mathrm{loc}}(\Omega)$.

We will prove (ii) under the assumption that $s \geq n$. Applying Lemma 2, we have for every $i = 1, 2, \ldots, n$, $df_m^i \to df_0^i$ weakly in $L^s_{\mathrm{loc}}(\Omega, \Lambda^l)$, hence $\nabla f_m^i \to \nabla f_0^i$ weakly in $L^n_{\mathrm{loc}}(\Omega)$.

Let $V \subset\subset \Omega$ be a ball. By the lower semicontinuity property of the Dirichlet integral (Proposition VI. 7.10, [10]) we have

$$
\int_V |\nabla f_0^i|^n dx \leq \liminf_{m\to\infty} \int_V |\nabla f_m^i|^n dx \tag{8}
$$

for $i = 1, 2, \ldots, n$. Let $\psi \in C_0^\infty(\Omega)$ with $\psi = 1$ on V. Then, applying the K - quasiregularity of f_m and Remark 4, we get

$$
\begin{aligned}
\liminf_{m\to\infty} \int_v |\nabla f_m^i|^n dx &\leq \liminf_{m\to\infty} \int_\Omega \psi \|Df_m(x)\|^n dx \\
&\leq K^{n-1} \liminf_{m\to\infty} \int_\Omega \psi J(x, f_m) dx \\
&= K^{n-1} \lim_{m\to\infty} \int_\Omega \psi J(x, f_m) dx \\
&= K^{n-1} \int_\Omega \psi J(x, f_0) dx \, .
\end{aligned}
\tag{9}
$$

For every $i = 1, 2, \ldots, n$, it follows by (8) and (9) that

$$\int_V |\nabla f_0^i|^n dx \leq K^{n-1} \int_\Omega \psi J(x, f_0) dx. \tag{10}$$

By an elementary calculation

$$||Df_0(x)||^n \leq \left(\sum_{i=1}^n |\nabla f_0^i| \right)^n \leq n^{n-1} \sum_{i=1}^n |\nabla f_0^i|^n. \tag{11}$$

Then (10) and (11) imply

$$\int_V ||Df_0(x)||^n dx \leq (nK)^{n-1} \int_\Omega \psi J(x, f_0) dx.$$

Using the Sobolev averaging kernel we build a sequence of test functions $\psi_k \in C_0^\infty(\Omega)$ such that $0 \leq \psi_k \leq 1$ and $\psi_k(x) \to \chi_V(x)$ for almost every $x \in \Omega$ (see [9], p.11). Then we apply the latter inequality to $\psi = \psi_k$, for every $k \geq 1$, and letting $k \to \infty$ we obtain by Lebesgue theorem

$$\int_V ||Df_0(x)||^n dx \leq (nK)^{n-1} \int_V J(x, f_0) dx.$$

Let $g(x) = ||Df_0(x)||^{n-1} - (nK)^{n-1} J(x, f_0)$, defined for almost every $x \in \Omega$. We define also the set function ν as $\nu(E) = \int_E g(x) dx$ for every Borel set $E \subset V$. The latter inequality shows that $\nu(B) \leq 0$ for every ball $B \subset V$. Then the derivative of ν with respect to the Lebesgue measure, which equals g a.e. in V, is non-positive. This proves the inequality

$$||Df_0(x)||^n dx \leq (nK)^{n-1} J(x, f_0) \tag{12}$$

a.e. in V, for every ball V compactly contained in Ω. Since we may find a sequence of balls compactly contained in Ω, which almost cover Ω, (12) holds a.e. in Ω. We conclude that f_0 is $(nK)^{n-1}$ - quasiregular.

Let α be a $(h-1)$ - form with linear coefficients. Then $df_m^* \alpha = f_m^*(d\alpha)$ for every $m \geq 0$. It suffices to prove that condition (iii) is fulfilled if $d\alpha = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$. Taking into account that (f_m) is bounded in $W_{\mathrm{loc}}^{1,s}(\Omega)$ and that $f_m \to f_0$ in $L_{\mathrm{loc}}^s(\Omega)$, the same is true for $g_m = (f_m^1, f_m^2, \ldots, f_m^h)$, $m = 0, 1, \ldots$.

Now by Lemma 2 we get

$$f_m^*(d\alpha) = df_m^1 \wedge df_m^2 \wedge \ldots \wedge df_m^h \to df_0^1 \wedge df_0^2 \wedge \ldots \wedge df_0^h = f_0^*(d\alpha)$$

weakly in $L_{\mathrm{loc}}^{s/h}(\Omega, \Lambda^h)$.

Next we check that Theorem 1 implies Lemma IV.5.1 from [8]. If $\omega_m \to \omega_0$ weakly in $L^p_{\mathrm{loc}}(U, \Lambda^k)$ then $*\omega_m \to *\omega_0$ weakly in $L^p_{\mathrm{loc}}(U, \Lambda^{n-k})$. Let $n = 2l$. Applying Theorem 1 for $h = l$ and $s \in [q(2l, K), p(2l, K)], s \geq l$, we have, for every $(h-1)$ - form with linear coefficients

$$d^{\pm} f_m^* \alpha \to d^{\pm} f_0^* \alpha \quad \text{weakly in} \quad L^{s/l}_{\mathrm{loc}}(\Omega, \Lambda^l)\,. \tag{13}$$

Assume now that $n = s = 2$ and let $\alpha = x + iy$. Thus $d^+ f_m^* \alpha = (\partial f_m / \partial \bar{z}) d\bar{z}$ and $d^- f_m^* \alpha = (\partial f_m / \partial z) dz$. Then by (6) and (13) we get

$$\lim_{m\to\infty} \int_\Omega \left(\frac{\partial f_m}{\partial \bar{z}} d\bar{z} \right) \wedge (\varphi dz) = \int_\Omega \left(\frac{\partial f_0}{\partial \bar{z}} d\bar{z} \right) \wedge (\varphi dz)$$
$$\lim_{m\to\infty} \int_\Omega \left(\frac{\partial f_m}{\partial z} dz \right) \wedge (\varphi d\bar{z}) = \int_\Omega \left(\frac{\partial f_0}{\partial z} dz \right) \wedge (\varphi d\bar{z})$$

for every $\varphi \in C_0^\infty(\Omega)$.

Let $R \subset\subset \Omega$ be a horizontal rectangle. Using a standard approximation of χ_R by functions in $C_0^\infty(\Omega)$ we get the conclusion of Lemma 5.1 ([8]), namely

$$\lim_{m\to\infty} \int_R \frac{\partial f_m}{\partial \bar{z}} dx \wedge dy = \int_R \frac{\partial f_0}{\partial \bar{z}} dx \wedge dy$$

and

$$\lim_{m\to\infty} \int_R \frac{\partial f_m}{\partial z} dx \wedge dy = \int_R \frac{\partial f_0}{\partial z} dx \wedge dy.$$

3. L^p - convergence of derivatives. In the proof of our main result we need the following estimates.

Lemma 3. *Let Ω be a domain in $\mathbb{R}^{2l}$. Consider the exponents $p, q \in (1, \infty)$ and an integer $k \in \{1, 2, \ldots, 2l\}$. Let $f, g : \Omega \to \mathbb{R}^{2l}$ be mappings in $W^{1,pl}_{\mathrm{loc}}(\Omega)$. For each ordered k - tuple $I = (i_1, i_2, \ldots, i_k)$ write $df^I = df^{i_1} \wedge df^{i_2} \wedge \ldots \wedge df^{i_k}$. Then, for almost every $x \in \Omega$, the following pointwise estimates hold:*

$$|df^I(x) - dg^I(x)| \leq \frac{k}{2} |Df(x) - Dg(x)| \Big(|Df(x)|^{k-1} + |Dg(x)|^{k-1} \Big) \tag{14}$$

(15)
$$|df^I(x) - dg^I(x)| \leq \frac{k^q}{2} |Df(x) - Dg(x)|^q \Big(|Df(x)|^{q(k-1)} + |Dg(x)|^{q(k-1)} \Big)\,.$$

Proof. First we mention the following inequality whose proof is elementary:

$$
(16)\qquad
\begin{aligned}
|a_1a_2\dots a_m - b_1b_2\dots b_m| &\le \max_{j=\overline{1,m}} |a_j - b_j| \sum_{j=0}^{m-1} |a|^{m-1-j}|b|^j \\
&\le \frac{m}{2} \max_{j=\overline{1,m}} |a_j - b_j| \Big(|a|^{m-1} + |b|^{m-1} \Big).
\end{aligned}
$$

Here m is a positive integer, $a_j, b_j, j = 1, 2, \dots, m$, are real numbers and we set $|a| = \max_{j=\overline{1,m}} |a_j|$ and $|b| = \max_{j=\overline{1,m}} |b_j|$.

Let $\Omega_1 = \{x \in \Omega : Df(x)$ and $Dg(x)$ exist$\}$. Then $\Omega \setminus \Omega_1$ has Lebesgue measure zero. For every $x \in \Omega_1$ we make use of (16) and we get

$$
\begin{aligned}
|df^I(x) - dg^I(x)| &\le \frac{k}{2} \max_{j=\overline{1,k}} |df^j(x) - dg^j(x)| \\
&\quad \times \left(\max_{j=\overline{1,k}} |df^j(x)|^{k-1} + \max_{j=\overline{1,k}} |dg^j(x)|^{k-1} \right) \\
&\le \frac{k}{2} |Df(x) - Dg(x)| \left(|Df(x)|^{k-1} + |Dg(x)|^{k-1} \right).
\end{aligned}
$$

(15) is a straightforward consequence of (14) via Hölder's inequality.

Theorem 2. *Let Ω be a domain in $\mathbb{R}^{2l}$. Fix exponents $p > [(2l-1)/(2l)]q_0'$, and $q_0' < q < p_0'$. Suppose that $f_m : \Omega \to \mathbb{R}^{2l}, m = 0, 1, 2\dots$, are weakly K-quasiregular mappings of the class $W^{1,pl}_{\mathrm{loc}}(\Omega)$ such that*

(1) *The sequence $(f_m), m = 1, 2, \dots$, converges to f_0 uniformly on compact subsets of Ω;*
(2) *$|\mu_m(x) - \mu(x)| \to 0$ for almost every $x \in \Omega$;*
(3) *$|Df_m(\cdot) - Df_0(\cdot)| \to 0$ in $L^{(ql)/2}_{\mathrm{loc}}(\Omega)$.*

Then $df_m^*\alpha \to df_0^*\alpha$ in $L^q_{\mathrm{loc}}(\Omega)$ for every $(l-1)$ - form α with linear coefficients, such that $d^+\alpha = 0$.

Proof. We notice that, for every $m = 0, 1, \dots$, f_m are actually K - quasiregular and $f_m \in W^{1,sl}_{\mathrm{loc}}(\Omega)$ for every $s \in (q_0', p_0')$. Then $df^J \in L^{s/h}_{\mathrm{loc}}(\Omega, \Lambda^h)$ for each h - tuple J. Moreover, f_m has continuous representative on Ω (see [10], VII. 3.9). In what follows $s \in (q_0', p_0')$. Let α be a $(l-1)$ - form with linear coefficients, $\alpha(x) = \sum_I \sum_{i=1}^{2l} c_I^i x^i dx^I$, where in the first sum I runs over all ordered $(l-1)$-tuples. Then $f_m^*\alpha = \sum_I \sum_{i=1}^{2l} c_I^i f_m^i df_m^I$ and

$$
df_m^*\alpha = f_m^*(d\alpha) = \sum_I \sum_{i=1}^{2l} c_I^i df_m^i \wedge df_m^I,
$$

hence $f_m^*\alpha \in L^{(sl)/(l-1)}(\Omega, \Lambda^{l-1})$ and $df_m^*\alpha \in L^s_{\mathrm{loc}}(\Omega, \Lambda^l)$.

Let $\eta \in C_0^\infty(\Omega)$. Then

$$d(\eta f_m^*\alpha) = d\eta \wedge f_m^*\alpha + \eta df_m^*\alpha \tag{17}$$

belongs to $L^s(\mathbb{R}^{2l}, \Lambda^l)$. We recall for each $f_m, m = 0, 1, \dots$, the Beltrami equation in even dimension:

$$d^+ f_m^*\alpha = \mu_m d^- f_m^*\alpha. \tag{18}$$

The Ahlfors-Beurling operator S permutes $d^+\beta$ and $d^-\beta$, for all forms β such that, for some $\varepsilon > 0, d\beta \in L^{1+\varepsilon}(\mathbb{R}^{2l}, \Lambda^l)$. Then it follows that

$$d^-(\eta f_m^*\alpha) = S \circ d^+(\eta f_m^*\alpha). \tag{19}$$

Let F be a compact subset of Ω. Now we assume that $\eta = 1$ on F and we denote the support of φ by F'. It suffices to prove that

$$d^+(\eta f_m^*\alpha) \to d^+(\eta f_0^*\alpha) \text{ in } L^q(\mathbb{R}^{2l}, \Lambda^l). \tag{20}$$

Indeed, since the operator $S : L^q(\mathbb{R}^{2l}, \Lambda^l) \to L^q(\mathbb{R}^{2l}, \Lambda^l)$ is bounded, (20) still holds after replacing $+$ by $-$. But $d = d^+ + d^-$, thus

$$d(\eta f_m^*\alpha) \to d(\eta f_0^*\alpha) \text{ in } L^q(\mathbb{R}^{2l}, \Lambda^l). \tag{21}$$

By (17) and (18) we have

$$d^+(\eta f_m^*\alpha) - \mu_m d^-(\eta f_m^*\alpha) = (d\eta \wedge f_m^*\alpha)_+ - \mu_m(d\eta \wedge f_m^*\alpha)_-.$$

Applying the relation above and (19) we get for every $m = 0, 1, 2, \dots$

$$(I - \mu_m S)d^+(\eta f_m^*\alpha) = (d\eta \wedge f_m^*\alpha)_+ - \mu_m(d\eta \wedge f_m^*\alpha).$$

Then, for $m = 1, 2, \dots$,

$$(I - \mu_m S)\left[d^+(\eta f_m^*\alpha) - d^+(\eta f_0^*\alpha)\right] = (\mu_m - \mu_0)d^-(\eta f_m^*\alpha) + \omega_m - \omega_0. \tag{22}$$

Here we write $\omega_m = (d\eta \wedge f_m^*\alpha)_+ - \mu_m(d\eta \wedge f_m^*\alpha)_-$. According to our hypothesis

$$|\mu_m|\,||S||_q \le \frac{K^l - 1}{K^l + 1}||S||_q < 1,$$

and, therefore, $(I - \mu_m S)$ is an invertible operator in $L^q(\mathbb{R}^{2l}, \Lambda^l)$ and the norm of its inverse satisfies:

$$\begin{aligned} ||(I - \mu_m S)^{-1}||_q &\le \frac{1}{1 - ||\mu_m S||_q} \le \frac{1}{1 - |\mu_m|\,||S||_q} \\ &\le \frac{1}{1 - [(K^l - 1)/(K^l + 1)]||S||_q} = N(l, K, q). \end{aligned} \tag{23}$$

Then (22) and (23) imply

$$(24)\qquad \begin{aligned} &||d^+(\eta f_m^*\alpha) - d^+(\eta f_0^*\alpha)||_q \le N(l,K,q) \\ &\qquad \times \Big[||(\mu_m - \mu_0)d^-(\eta f_0^*\alpha)||_q + ||\omega_m - \omega_0||_q\Big]. \end{aligned}$$

Applying Lebesgue Dominated Convergence Theorem we obtain

$$(25)\qquad \lim_{m\to\infty} \big|\big|(\mu_m - \mu_0)d^-(\eta f_0^*\alpha)\big|\big|_q = 0,$$

as $(\mu_m - \mu_0)d^-(\eta f_0^*\alpha) \to 0$ almost everywhere in Ω and

$$|(\mu_m - \mu_0)d^-(\eta f_0^*\alpha)| \le 2\frac{K^l - 1}{K^l + 1}\,|d(\eta f_0^*\alpha)|,$$

pointwise. The proof will be completed once we check that

$$(26)\qquad \lim_{m\to\infty} ||\omega_m - \omega_0||_q = 0.$$

From the identity

$$\begin{aligned} \omega_m - \omega_0 &= (d\eta\wedge f_m^*\alpha - d\eta\wedge f_0^*\alpha)_+ - \mu_m(d\eta\wedge f_m^*\alpha - d\eta\wedge f_0^*\alpha)_- \\ &\quad - (\mu_m - \mu_0)(d\eta\wedge f_0^*\alpha)_-, \end{aligned}$$

we have the following pointwise estimate almost everywhere:

$$(27)\qquad |\omega_m - \omega_0| \le \frac{K^l}{K^l + 1}|\nabla\eta|\,|f_m^*\alpha - f_0^*\alpha| + |\mu_m - \mu_0|\,|\nabla\eta|\,|f_0^*\alpha|.$$

Again, by Lebesgue Dominated Convergence Theorem

$$(28)\qquad \lim_{m\to\infty} \big|\big|\,|\mu_m - \mu_0|\,|\nabla\eta|\,|f_0^*\alpha|\,\big|\big|_q = 0.$$

It suffices to prove for a fixed $(l-1)$ - form α with $d^+\alpha = 0$ that

$$(29)\qquad \lim_{m\to\infty} \big|\big|\,|\nabla\eta|\,|f_m^*\alpha - f_0^*\alpha|\,\big|\big|_q = 0.$$

Then, by using (27), (28) and (29), (26) follows. We return to the proof of (29). If $l = 1$, we take $\alpha = x^1 + ix^2$. Thus

$$|\nabla\eta|\,|f_m^*\alpha - f_0^*\alpha| \le 2|\nabla\eta|\,|f_m - f_0|.$$

Then, in the case $n = 2$, the conclusion of Theorem 2 follows from Condition (1) and (2), Condition (3) being superfluous.

Suppose that $l \geq 2$ and let $\alpha = x^1 dx^2 \wedge \ldots \wedge dx^l - (-i)^l x^{l+1} dx^{l+2} \wedge \ldots \wedge dx^{2l}$. Then we can write

$$|f_m^* \alpha - f_0^* \alpha| \leq |f_m - f_0| \left(|df_m^I| + |df_m^J|\right) + |f_0| \left(|df_m^I - df_0^I| + |df_m^J - df_0^J|\right).$$

Moreover, we have

$$\tag{30} \begin{aligned} &|\nabla \eta|^q |f_m^* \alpha - f_0^* \alpha|^q \leq 2^{2q-1} |\nabla \eta|^q |f_m - f_0|^q |Df_m|^{q(l-1)} \\ &\quad + 2^{q-1} |\nabla \eta|^q ||f_0||_{\infty, F'} \left(|df_m^I - df_0^I|^q + |df_m^J - df_0^J|^q\right). \end{aligned}$$

We may take $\eta = \varphi^l$, with $\varphi \in C_0^\infty(\Omega)$, in the relation above. Then by Hölder's inequality:

(31)

$$\begin{aligned} &\int_\Omega |\nabla \eta|^q |f_m - f_0|^q |Df_m|^{q(l-1)} dx = l^q \int_\Omega |\varphi Df_m|^{q(l-1)} \Big(|\nabla \varphi|\, |f_m - f_0|\Big)^q dx \\ &\leq l^q \left[\int_\Omega (|\nabla \varphi|\, |f_m - f_0|)^{ql} dx\right]^{1/l} \left[\int_\Omega |\varphi Df_m|^{ql} dx\right]^{(l-1)/l}. \end{aligned}$$

It is clear that

$$\tag{32} \left[\int_\Omega (|\nabla \varphi|\, |f_m - f_0|)^{ql} dx\right]^{1/l} \leq m(F')^{1/l} \left(\sup_{F'} |\nabla \varphi| \sup_{F'} |f_m - f_0|\right)^q.$$

By the Caccioppoli type estimate (3) we have

$$\tag{33} \begin{aligned} &\int_\Omega |\varphi Df_m|^{ql} \leq c(2l, K, ql)^{q^l} \int_\Omega |\nabla \varphi|^{ql} |f_m|^{ql} \\ &\leq c(2l, K, ql)^{ql}\, m(F') \left(\sup_{F'} |\nabla \varphi|\right)^{ql} \left(\sup_{F'} |f_m - f_0| + ||f_0||_{\infty, F'}\right)^{ql}. \end{aligned}$$

The latter three estimates (31), (32) and (33) imply

$$\tag{34} \begin{aligned} &||\, |\nabla \eta|\, |f_m - f_0|\, |Df_m|^{l-1} ||_q \leq l c(2l, K, ql)^{(l-1)} m(F')^{1/q} \\ &\quad \times \sup_{F'} |\nabla \varphi|^l \sup_{F'} |f_m - f_0| \left(\sup_{F'} |f_m - f_0| + ||f_0||_{\infty, F'}\right)^{l-1}. \end{aligned}$$

Thus, by Condition (1) and by continuity of f_0 this relation yields

$$\tag{35} \lim_{m \to \infty} ||\, |\nabla \eta|\, |f_m - f_0|\, |Df_m|^{l-1} ||_q = 0.$$

Having disposed of this fact we are going to prove that

$$\tag{36} \lim_{m \to \infty} ||\, |\nabla \eta|\, |f_0|\, |df_m^I - df_0^I|\, ||_q = 0.$$

The same calculation shall apply with J in place of I. By Lemma 3

$$|df_m^I - df_0^I|^q \leq \frac{(l-1)^q}{2} |Df_m - Df_0|^q \left(|Df_m|^{q(l-2)} + |Df_0|^{q(l-2)}\right).$$

Now we take $\eta = \psi^{l-1}$ with $\psi \in C_0^\infty(\Omega)$. Then

$$\begin{aligned} \| \, |\nabla\eta| \, |f_0| \, |df_m^I - df_0^I| \, \|_q^q &\leq \frac{(l-1)^{2q}}{2} \int_\Omega (|\nabla\psi| \, |Df_m - Df_0|)^q \\ &\quad \times \left[|\psi Df_m|^{q(l-2)} + |\psi Df_0|^{q(l-2)} \right] dx \\ &\leq \left(\int_\Omega (|\nabla\varphi|^{ql/2} \, |Df_m - Df_0|)^{ql/2} dx \right)^{2/l} \\ \times \Bigg[\left(\int_\Omega |\psi Df_m|^{ql} dx \right)^{(l-2)/2} &+ \left(\int_\Omega |\psi Df_0|^{ql} dx \right)^{(l-2)/l} \Bigg]. \end{aligned}$$

Applying again the Caccioppoli type estimate we conclude that

$$\begin{aligned} \| \, |\nabla\eta| \, |f_0| \, |df_m^I - df_0^I| \, \|_q &\leq \frac{(l-1)^2}{2^{1/q}} c(2l, K, ql)^{(l-2)} m(F')^{(l-2)/ql} \\ &\quad \times \sup_{F'} |\nabla\psi|^{l-1} \| Df_m - Df_0 \|_{ql/2, F'} T_m \, , \end{aligned} \tag{37}$$

where $T_m = (\sup_{F'} |f_m - f_0| + \|f_0\|_{\infty,F'})^{l-2} + \|f_0\|_{\infty,F'}^{l-2}$.

Taking into account Conditions (3) and (1) this estimate implies (36), as required. Finally, we return to the estimate (30) and with the help of (35) and (36) we obtain (29), which completes the proof of Theorem 2.

Finally, let us show that Theorem 2 indeed generalizes Theorem V.5.3 of Lehto and Virtanen [8].

To this end we use the complex variable $z = x^1 + ix^2$. As we have already seen, Condition (3) is not necessary for $n = 2$.

Let f be a weakly quasiregular mapping with the distortion tensor G. We denote by $G_j^i(z)$ the entries of the matrix representation of $G(z)$ with respect to the standard basis of $\mathbb{R}^2$. Then the Beltrami coefficient $\mu(z)$ of f at z, can be identified with the matrix

$$\mu(z) = \frac{1}{G_1^1(z) + G_2^2(z) + 2} \begin{pmatrix} G_1^1(z) - G_2^2(z) & 2G_2^1(z) \\ 2G_2^1(z) & G_2^2(z) - G_1^1(z) \end{pmatrix},$$

which corresponds to the complex number

$$\frac{1}{G_1^1(z) + G_2^2(z) + 2} \left(G_1^1(z) - G_2^2(z) + 2iG_2^1(z) \right) = \left(\frac{\partial f}{\partial \bar{z}}(z) \right) \Big/ \left(\frac{\partial f}{\partial z}(z) \right).$$

This is customarily called the complex dilatation of f at z. The above identification yields the same norm; the operator norm of the matrix and the modulus of the complex number identified with $\mu(z)$ are equal.

Condition (2) simply means that the sequence of the complex dilatations of f_m converges a.e. in Ω. Note that $\dim \Lambda^{\pm} = 1$ and $\alpha = cz$ for some complex number c. The assertion of Theorem 2 can now be written as $\frac{\partial f_m}{\partial \overline{z}} d\overline{z} \to \frac{\partial f_0}{\partial \overline{z}} d\overline{z}$ and $\frac{\partial f_m}{\partial z} dz \to \frac{\partial f_0}{\partial z} dz$ in $L^q_{\text{loc}}(\Omega)$, for all $q \in (q_0'(2,K), p_0'(2,K)$, which is nothing else than the convergence in $L^q_{\text{loc}}(\Omega)$ of the derivatives $\partial f_m/\partial \overline{z}$ and $\partial f_m/\partial z$ to $\partial f_0/\partial \overline{z}$ and $\partial f_0/\partial z$, respectively.

Acknowledgement. I wish to express my deepest appreciation to the referee for carefully reading and correcting an earlier draft of this paper.

References

[1] Bojarski, B., *Homeomorphic solutions of Beltrami systems*, Dokl. Akad. Nauk **102** (1955), 661-664.

[2] ———, T. Iwaniec, *Analytical foundations of the theory of quasiconformal mappings in $\mathbb{R}^n$*, Ann. Acad. Sci. Fenn. Math. **8** (1983), 257-324.

[3] Donaldson, S. K., D. Sullivan, *Quasiconformal 4-manifolds*, Acta Math. **163** (1989), 181-252..

[4] Iwaniec, T., G. Martin, *Quasiconformal mappings in even dimensions*, Acta Math. **170** (1993), 29-81.

[5] Iwaniec, T., *p-harmonic tensors and quasiregular mappings*, Ann. of Math. **136** (1992), 589-624.

[6] ———, *Geometric function theory and partial differential equations*, Lectures in Seillac (France), May 27- June 2, 1995.

[7] ———, *Current advances in quasiconformal geometry and nonlinear analysis*, 1995, Preprint.

[8] Lehto, O., K. L. Virtanen, *Quasiconformal Mappings in the Plane*, Grundlehren der mathematischen Wissenschaften, vol. 126, Springer Berlin-Heidelberg- New York, 1973.

[9] Reshetnyak, Yu. G., *Space Mappings with Bounded Distortion*. Transl. of Math. Monographs,, vol. 73, Amer. Math. Soc., Providence, R.I., 1989.

[10] Rickman, S., *Quasiregular Mappings*, Springer -Verlag, Berlin -Heidelberg, 1993.

Department of Mathematics and Physics
University of Bacau
Spiru Haret 8, Bacau 5500, Romania
e-mail: marcim@ub.ro

received November 20, 1998