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ABDALLAH LYZZAIK

Univalent harmonic mappings and a conjecture of J. C. C. Nitsche

Abstract. This paper is a presentation of the author's talk delivered at the Conference and roughly resembles the introductory chapter of [4]. The main result in [4] is quoted here as Theorem 1 and the steps leading to its proof are indicated at the end of this presentation.

Let D be a domain of the complex plane $\mathbb{C}$. A *univalent harmonic mapping* f of D is an injective complex-valued function of the form $f(z) = u(z) + iv(z)$ that satisfies Laplace's equation $f_{z\bar{z}} = 0$ in D. Thus every conformal or anti-conformal function f of D is a univalent harmonic mapping. A result of H. Lewy, cf. e. g. [2], asserts that the Jacobian $J = |f_z|^2 - |f_{\bar{z}}|^2$ of f can never vanish. For convenience, we assume that J is always positive and consequently f is sense preserving. Then $|f_z| > 0$ in D, and a short computation yields that the *second dilatation* $\omega = \overline{f_{\bar{z}}}/f_z$ of f is indeed an analytic function of D into the open unit disc $\mathbf{D}$. If $|\omega(z)| < k < 1$ in D and $K = (1+k)/(1-k)$, then f is quasiconformal with *maximal dilatation* K, or simply K-quasiconformal.

Let $A(\rho, 1), 0 \leq \rho < 1$, be the annulus $\{z : \rho < |z| < 1\}$, and let f be a univalent harmonic mapping of $A(r, 1)$ onto $A(R, 1)$. If $f_{\bar{z}} = 0$, then f is a rotation and $r = R$.

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If f is K-quasiconformal, then $r^K \leq R \leq r^{1/K}$ [3, p. 38]. However, if f is neither conformal nor quasiconformal, then R is possibly zero as with the harmonic mapping

$$f(z) = (z - r^2/\overline{z})/(1 - r^2)$$

which can be easily shown to map $A(r,1)$ univalently onto the punctured disc $A(0,1)$. On the other hand, R admits a universal upper bound (less than 1) as was shown in 1962 by J. C. C. Nitsche [3]. To state this result, let $\mathcal{K}(r)$ be the class of univalent harmonic mappings of the annulus $A(r,1)$ onto some annulus $A(R,1)$, and let $\kappa(r)$ be the supremum of R as f ranges over all $f \in \mathcal{K}(r)$. Using Harnack's inequality [1, pp. 235-237], Nitsche proved the following interesting result [5]:

Theorem. *The value $\kappa(r)$ is less than* 1.

Consider now the class of harmonic mappings

(1) $$f_t(z) = tz + (1-t)/\overline{z} = [t\sigma + (1-t)/\sigma]e^{i\theta}, \ (z = \sigma e^{i\theta}).$$

Each f_t maps concentric circles onto concentric circles, and maps $A(r,1)$ univalently onto $A(R(t),1)$, $R(t) = tr + (1-t)/r$, if and only if, $1/(1+r^2) \leq t \leq 1/(1-r^2)$. Restricted to these values of t, Nitsche [5] observed that $R(t)$ admits its maximum value $2r/(r^2+1)$ at $t = 1/(1+r^2)$. This led him to suggest the following

Conjecture. $\kappa(r) \leq 2r/(1+r^2)$.

The conjecture was raised again in 1989 by G. Schober [6] as "an intriguing open problem", and subsequently in 1994 by D. Bshouty and W. Hengartner [2] as "open problem 3.1". Looking closer at Nitsche's proof of the above theorem, the latter authors observed that the proof also applies to the wider class of harmonic mappings of $A(r,1)$ that are not necessarily univalent and that admit a point in each of the vertical strips $\{w : R < \Re w < 1\}$ and $\{w : -1 < \Re w < -R\}$. Consequently, they remarked that $\kappa(r)$ is unlikely to be found by parlaying Nitsche's proof of his afore-mentioned theorem.

Apparently, no quantitative upper bound for $\kappa(r)$ is known, and personal communications with D. Bshouty, P. Duren, and W. Hengartner confirm this. The object of this talk is to give the first such bound. This is done in terms of the *Grötzsch's ring domain*, $B(s)$, $0 < s < 1$, which is the doubly-connected open subset of the unit disc whose boundary components are the unit circle and the segment $\{x : 0 \leq x \leq s\}$. Note that every annulus is conformally equivalent to a unique Grötzsch's domain.

We state our result as

Theorem 1 ([4]). *Let f be a univalent harmonic mapping of the annulus $A(r,1)$, $0<r<1$, onto the annulus $A(R,1)$, and let $B(s)$ be the Grötzsch's ring domain that is conformally equivalent to $A(r,1)$. Then $R<s$.*

It follows at once that $\kappa(r)\le s$. There is a good chance that this inequality is sharp as shall be seen later in [4, Remark 3]. Meanwhile however, this remains an open problem.

We verify the theorem by comparing its resulting bound s with the bound $2r/(1+r^2)$ that is suggested in Nitsche's conjecture [5]. To do so, we invoke the notion of the module M of a ring domain (μ in the case of a Grötzsch's ring domain), and the properties that M is conformally invariant and is strictly decreasing with respect to proper set-inclusion. With r and s being as in Theorem 1 we have:

$$\mu(s)=M(A(r,1))=\log(1/r). \tag{2}$$

Now the Möbius transformation $(z+r)/(1+rz)$ maps $A(r,1)$ conformally onto the ring domain B, bounded by the unit circle and the circle whose diameter is the segment joining the origin to the point $2r/(1+r^2)$. Obviously, B is a proper subset of the Grötzsch's domain $B(2r/(1+r^2))$. So

$$M(B)<\mu(2r/(1+r^2)),$$

which with the equality $M(B)=M(A(r,1))$ and (2) yield

$$\mu(s)<\mu(2r/(1+r^2)).$$

Hence $2r/(1+r^2)<s$.

As applications of Theorem 1 we have:

(i) If $r=e^{-\pi/2}$, then $\mu(s)=\pi/2$ which yields $s=1/\sqrt{2}$, [3, p. 61]. Then $R\le 1/\sqrt{2}\approx 0.707\ldots$ by Theorem 1, while Nitsche's bound for this case is $2e^{-\pi/2}/(1+e^{-\pi})\approx 0.398\ldots$.

(ii) By (2) and [3, p. 61] again, we conclude

$$\log(1/r)=\mu(s)<\log(4/s),$$

or $s<4r$. Then $R<4r$ by Theorem 1; a useful inequality whenever $r<1/4$. Here note that Nitsche's bound is of order $2r$ as $r\to 0$.

We conclude with a sketch of the proof of Theorem 1. With f as in the theorem, it is first shown that f is associated with an analytic function of $M(A(r,1))$ whose image surface embeds properly in a special smooth doubly-connected covering X of $\mathbb{C}$. This yields $\mu(s)<M(X)$. Next, it is shown that modules of X and $M(A(R,1))$ satisfy $M(X)\le M(R)$. Therefore, $\mu(s)<M(R)$ and consequently $R<s$.

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Department of Mathematics
American University of Beirut
Beirut, Lebanon

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