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A note on quasisymmetric functions and *BMO*

ABSTRACT. We present examples of quasisymmetric functions on the line that are absolutely continuous but for which the logarithm of the derivative is not in *BMO*.

Introduction. The motivation for the problem studied in this note is the following result of H. M. Reimann, [R]:

Theorem 1. *If f is a quasiconformal mapping of $\mathbb{R}^n (n \geq 2)$ onto itself with Jacobian determinant J_f, then $\log J_f \in BMO$.*

Since the analogue of quasiconformal mappings on $\mathbb{R}$ are the quasisymmetric mappings, it is natural to ask if $\log \varphi' \in$ BMO whenever φ is such a mapping. But A. Beurling and L. Ahlfors, ([B/A], p. 139), gave an example of a completely singular quasisymmetric mapping, and for this function φ, $\log \varphi'$ is not even locally integrable. More examples of quasisymmetric functions that are not absolutely continuous are known. (See footnote, p. 139 in [B/A].)

D. Partyka asked the following question during the Lublin Conference 8.31 – 9.4.1998: Suppose that $\varphi : \mathbb{R} \to \mathbb{R}$ is quasisymmetric and absolutely continuous. Is $\log \varphi'$ in BMO? We answer this question in the negative.

Main result. Our result is the following:

Theorem 2. *There exists an absolutely continuous quasisymmetric function* $\varphi: \mathbb{R} \to \mathbb{R}$ *with* $\log \varphi' \notin BMO$.

To prove this theorem we need:

Lemma 1. *If* ω *is defined on an interval (finite or infinite),* $\omega \geq 0$ *and* $\log \omega \in BMO$, *then there exist constants* $C > 0$ *and* $\alpha \in (0,1]$ *such that for all measurable sets* E *and all intervals* I *in the domain of* ω *such that* $E \subseteq I$, *we have:*

$$\frac{|E|}{|I|} \leq C \left(\int_E \omega(x)^\alpha dx \Big/ \int_I \omega(x)^\alpha dx \right)^{1/2}. \tag{1}$$

Here $|\;|$ *denotes the Lebesgue measure.*

This inequality is equivalent to the fact that $\omega^\alpha \in A_2$, the Muckenhoupt class. Proof of this Lemma follows from [C/F], [R/R] or [G], p. 258.

The next observation is due to P.W. Jones (private communication).

Lemma 2. *If* $f: [a,b] \to [c,d]$ *is a quasisymmetric homeomorphism that is not absolutely continuous, and if* i *is the identity function* $i(x) = x$, *then the function* $\Phi = f + i$ *is quasisymmetric and not absolutely continuous, with an inverse* $\varphi = \Phi^{-1}$ *that is quasisymmetric and absolutely continuous.*

Proof of Lemma 2. A ϱ-quasisymmetric function f defined on an interval I (finite or infinite), is a continuous, strictly increasing, real-valued function satisfying:

$$\varrho^{-1} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq \varrho$$

for $t > 0$ and for all $x, x+t$ and $x - t$ in I. (See also [Ke].)

It follows that if f is quasisymmetric in I, so is $f + i$ in the same interval. Since f is not absolutely continuous clearly the same is true for $f + i$. Next since $f + i$ increases distance, $\varphi = (f+i)^{-1}$ decreases distance, and φ is therefore absolutely continuous. That φ is also quasisymmetric follows from Theorem 9 in [Ke]. □

We shall also need the following generalization of Theorem [5] in [Ke].

Lemma 3. *If $\varphi: [a,b] \to [c,d]$ is a ϱ-quasisymmetric bijection, then φ has a $28\varrho^4$-quasisymmetric extension $\tilde{\varphi}$ to $\mathbb{R}$. Moreover, $\tilde{\varphi}$ is absolutely continuous if and only if φ is absolutely continuous.*

Proof of Lemma 3. It follows easily that if φ is ϱ-quasisymmetric then $S \circ \varphi \circ T$ is ϱ-quasisymmetric when S and T are linear mappings. The first claim of the lemma follows from this observation combined with Theorem 5 in [Ke]. The second claim follows from inspection of the proof in [Ke].

Proof of Theorem 2. Let $f: [0,2\pi] \to [0,2\pi]$ be the Beurling - Ahlfors function mentioned above. Then the function $\varphi: [0,4\pi] \to [0,2\pi]$ defined by:

$$\varphi = \Phi^{-1} = (f + i)^{-1}$$

is quasisymmetric and absolutely continuous with an inverse Φ which is not absolutely continuous. Assume for contradiction that $\log \varphi' \in$ BMO. Then we know from above that there exist constants $\alpha \in (0,1]$ and $C > 0$ such that (1) holds, i.e.:

$$\frac{|E|}{|I|} \le C \left(\int_E \varphi'(x)^\alpha dx \Big/ \int_I \varphi'(x)^\alpha dx \right)^{1/2} \tag{2}$$

for all measurable sets $E \subseteq I = [0,4\pi]$.

Since Φ is not absolutely continuous, there exist $\varepsilon > 0$ and open sets $E'_n \subseteq [0,2\pi]$ for each natural number n with $|E'_n| < 1/n$, but with $|\Phi(E'_n)| > \varepsilon$. The sets $E_n = \Phi(E'_n)$ are also open sets and $\varphi(E_n) = E'_n$. Hence our statement is equivalent to the following. There exist open sets $E_n \subseteq [0,4\pi]$ with $|E_n| > \varepsilon$ and such that

$$|\varphi(E_n)| = \int_{E_n} \varphi'(x) dx < \frac{1}{n}$$

since φ is absolutely continuous and E_n is a Borel set.

Since $0 < \alpha \le 1$, we have by Hölder's inequality:

$$\left(\frac{1}{|E_n|} \int_{E_n} \varphi'(x)^\alpha dx \right)^{1/\alpha} \le \frac{1}{|E_n|} \int_{E_n} \varphi'(x) dx,$$

and consequently

$$\int_{E_n} \varphi'(x)^\alpha dx \le |E_n|^{1-\alpha} \left(\int_{E_n} \varphi'(x) dx \right)^\alpha. \tag{3}$$

Now $|E_n| \le 4\pi$ and $1 - \alpha \ge 0$, and therefore $\lim_{n\to\infty} \int_{E_n} \varphi'(x)^\alpha dx = 0$. This is in contradiction to (2) since $|E_n| > \varepsilon$ for all n.

To extend our function φ to the whole line, we use Lemma 3. □

Remarks. In [R] the following result is proved:

Theorem 3. *Suppose that f is a self homeomorphism of $\hat{\mathbb{R}}^n$ which is ACL and differentiable a.e. Then f is quasiconformal if and only if the mapping $v \curvearrowright v \circ f$ is a bijective isomorphism of the space BMO for which $\| v \circ f \|_* \leq C \| v \|_*$.*

(See also [Ka])

Again, a natural question to ask is the following: Does this result hold for $n = 1$ with quasisymmetric functions instead of quasiconformal mappings? In [J] P. W. Jones has given a complete answer to this question. Theorem 3 is true for $n = 1$ if the function $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing homeomorphism with $f' \in A_\infty$, the Muckenhoupt class.

But one could ask the question: If $f : \mathbb{R} \to \mathbb{R}$ is an absolutely continuous quasisymmetric function, does it then follow that $f' \in A_\infty$?

Again our example above answers this question in the negative. Since $A_\infty = \bigcup_{1<p<\infty} A_p$, it follows easily that for our function $\bar{\varphi}$ above, $\bar{\varphi}' \notin A_\infty$.

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