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Variations on a theorem of Fejér and Riesz

This lecture is dedicated to Jan Krzyż
on the occasion of his 75^{th} birthday

ABSTRACT. This lecture concerns variants of a pair of inequalities due to L. Fejér and F. Riesz which are related to hyperbolic geometry, Carleson measures, the level set problem, the higher variation of a function and the one-dimensional heat equation.

1. Introduction. I will describe here several results which are related to the following two attractive theorems due to L. Fejér and F. Riesz [6] and to F. Riesz [23]. Throughout this lecture D will denote a simply connected proper subdomain of the plane $\mathbb{R}^2$, B the open unit disk, H the upper half plane and L the real axis.

Theorem 1.1 (Fejér-Riesz). *If g is analytic in B and continuous in $\overline{B}$, then*

$$\int_{L\cap B} |g|^p\, ds \le \tfrac{1}{2} \int_{\partial B} |g|^p\, ds$$

for $0 < p < \infty$.

Theorem 1.1 is closely related to the following inequality.

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Theorem 1.2 (Riesz). *If u is harmonic in B and continuous in $\overline{B}$, then*

$$\text{variation}_{L\cap B}(u) \leq \tfrac{1}{2}\,\text{variation}_{\partial B}(u).$$

The following inequality is an immediate consequence of the above two theorems.

Corollary 1.3. *If f is conformal in B and continuous in $\overline{B}$, then*

$$\text{length}(f(L\cap B)) \leq \tfrac{1}{2}\,\text{length}(f(\partial B)).$$

Proof. Let $g = f'$ and $p = 1$ in the Fejér-Riesz Theorem or let $u = f$ in the Riesz Theorem. □

Remark. By the Riemann mapping theorem, for each $0 < a < \infty$ there exists a conformal mapping $f : B \to D$ where

$$D = \{z = x + i\,y : |x|/a + |y| < 1\},$$

such that $L\cap B$ corresponds to $L \cap D$. The Carathéodory extension theorem then implies that f is continuous in $\overline{B}$ and hence that

$$\frac{\text{length}(f(L\cap B))}{\text{length}(f(\partial B))} = \frac{2\,a}{4\sqrt{a^2+1}} \to \frac{1}{2}$$

as $a \to \infty$. Thus the constant $\frac{1}{2}$ is sharp in Corollary 1.3 and hence also in the Theorems of Fejér-Riesz and Riesz.

In what follows I will give five variants of the inequalities of Fejér-Riesz and Riesz which are connected with

1. hyperbolic geometry,
2. Carleson measures,
3. the level set problem,
4. the higher variation of a function,
5. the one-dimensional heat equation.

2. Hyperbolic geometry. The following is a variant of Corollary 1.3 which was first conjectured by Piranian and later established by Gehring and Hayman in [15].

Theorem 2.1. *If f is conformal in B and continuous in $\overline{B} \cap H$, then*

$$\text{length}(f(L \cap B)) \leq c \, \text{length}(f(\partial B \cap H))$$

where c is an absolute constant.

Remark. The sharp value of c in Theorem 2.1 is not known. The proof given in [15] yields the bounds $\pi \leq c < 74$. Jaenisch showed later in [18] that Theorem 2.1 holds with $4.5 \leq c \leq 17.5$.

Theorem 2.1 has an interesting interpretation in terms of the hyperbolic geometry. If $g : D \to B$ is conformal, then

$$\rho_D(z) = \frac{2|g'(z)|}{1 - |g(z)|^2}$$

is independent of choice of g and the *hyperbolic distance* h_D in D is given by

$$h_D(z_1, z_2) = \inf_{\alpha} \int_{\alpha} \rho \, ds$$

where α is any arc joining z_1, z_2 in D. The unique arc β for which this infimum is attained is said to be a *hyperbolic geodesic.*

Corollary 2.2. *If β is a hyperbolic geodesic in D and if α is an arc which joins the endpoints of β in D, then*

$$l(\beta) \leq c \, l(\alpha)$$

where c is the constant in Theorem 2.1.

Proof. Suppose that α meets the hyperbolic line containing β only at the endpoints of β. Then we can choose a conformal mapping $g : D \to B$ so that $g(\alpha) \cup g(\beta)$ bounds a Jordan domain $D' \subset B \cap H$ and $g(\beta) \subset L$.

Let h map D' conformally onto $B \cap H$ so that $g(\beta) = L \cap B$ and reflect in L. Then

$$f = (h \circ g)^{-1}$$

is conformal in B, continuous in $\overline{B}$ and

$$l(\beta) = \int_{L \cap B} |f'| \, ds \leq c \int_{\partial B \cap H} |f'| \, ds = c \, l(\alpha)$$

by Theorem 2.1. The general case then follows easily from this special case. □

Remark. Corollary 2.2 says that in a simply connected domain, a hyperbolic geodesic β minimizes up to a fixed multiplicative constant the euclidean as well as the hyperbolic length of all arcs α joining its endpoints. This is not the case in a multiply connected domain [1]. See [17] and [22] for other developments concerning Corollary 2.2.

3. Carleson measures

Definition 3.1 A non-negative measure μ in B is a Carleson measure if there exists a constant b such that

$$\mu(U \cap B) \leq b \operatorname{rad}(U)$$

for each disk U with center on ∂B.

The following theorem due to Carleson [5] illustrates why this particular class of measures is important.

Theorem 3.2. *A non-negative measure μ in B is a Carleson measure if and only if there is a constant c such that for each function g analytic in B and continuous in $\overline{B}$,*

$$\int_B |g|^p \, d\mu \leq c \int_{\partial B} |g|^p \, ds$$

for $0 < p < \infty$.

Example 3.3. For each Borel set $E \subset B$ let $\mu(E) = \operatorname{length}(E \cap L)$. Then

$$\mu(U \cap B) \leq 2 \operatorname{rad}(U)$$

for each disk U with center on ∂B and hence μ is a Carleson measure.

Remark. If μ is the measure in Example 3.3, then by Theorem 3.2 there is constant c such that

$$\int_{L \cap B} |g|^p \, ds = \int_B |g|^p \, d\mu \leq c \int_{\partial B} |g|^p \, ds$$

for g analytic in B and continuous in $\overline{B}$ and $0 < p < \infty$. Thus Theorem 3.2 is a far reaching extension of the Fejér-Riesz theorem.

The following lemma yields another useful characterization for Carleson measures. See Lemma 3.3 in Chapter 6 of [9].

Lemma 3.4. *A non-negative measure μ in B is a Carleson measure if and only if there is a constant b such that*

$$\int_B |h'|\, d\mu \le b$$

for all conformal $h : B \to B$.

4. Level set problem. Corollary 1.3 implies that

$$\text{length}(f(L \cap B)) \le \tfrac{1}{2}\, \text{length}(f(\partial B))$$

whenever $f : B \to D$ is conformal in B and continuous in $\overline{B}$. It is reasonable to ask if one can reverse the roles of B and D in this inequality. That is, does there exist a constant a such that

$$\text{length}(f(L \cap D)) \le a\, \text{length}(f(\partial D))$$

whenever $f : D \to B$ is conformal in D and continuous in $\overline{D}$. This question was answered in the affirmative by Hayman and Wu who established the following result [16].

Theorem 4.1. *If $f : D \to B$ is conformal, then*

$$\text{length}(f(L \cap D)) \le b,$$

where b is an absolute constant.

Remarks. Piranian and Weitsman were the first to conjecture that Theorem 4.1 holds and the proof in [16] yields the result with $b = 10^{37}$. A different argument with additional consequences was later given by Garnett, Gehring and Jones in [10]; see Theorem 4.3 below. The value of the constant b has been studied by several people.

1. Flinn: $7.4 \le b$. In addition $b \le \pi^2$ if $H \subset D$ [8].
2. Fernández, Heinonen and Martio: $b \le 4\pi^2$ [7].
3. Øyma: $\pi^2 \le b \le 4\pi$ in [20] and [21].
4. Rohde: $b < 4\pi$ [24].

The following consequence of Theorem 4.1 in [10] allows one to replace the unit disk B in the Fejér-Riesz Theorem by a Jordan domain D with a rectifiable boundary.

Lemma 4.2. *If $f : D \to B$ is conformal, then $\mu(E) = \text{length}(E \cap f(L \cap D))$ is a Carleson measure.*

Proof. Suppose that $h : B \to B$ is conformal. Then $g = h \circ f : D \to B$ is conformal and

$$\int_B |h'|\, d\mu = \int_{f(L\cap D)} |h'|\, ds = \text{length}(g(L \cap D)) \leq b$$

by Theorem 4.1. Hence μ is a Carleson measure by Lemma 3.4. □

If we now combine Theorem 3.2 and Lemma 4.2 we obtain the following versions of the Fejér-Riesz Theorem and Corollary 1.3 [10].

Theorem 4.3. *If ∂D is a rectifiable Jordan curve and if g is analytic in D and continuous in $\overline{D}$, then*

$$\int_{L\cap D} |g|^p\, ds \leq c \int_{\partial D} |g|^p\, ds$$

for $0 < p < \infty$ where c is an absolute constant.

Proof. Suppose that $f : D \to B$ is conformal and let

$$\mu(E) = \text{length}(E \cap f(L \cap D)).$$

Then μ is a Carleson measure by Lemma 4.2. Next choose h analytic in B so that

$$g(z)^p = h(f(z))^p f'(z).$$

Then Theorem 3.2 implies that

$$\int_{L\cap D} |g|^p\, ds = \int_{f(L\cap D)} |h|^p\, ds = \int_B |h|^p\, d\mu \leq c \int_{\partial B} |h|^p\, ds = c \int_{\partial D} |g|^p\, ds.$$

□

Corollary 4.4. *If f is conformal in D and continuous in $\overline{D}$, then*

$$\text{length}(f(L \cap D)) \leq c\, \text{length}(f(\partial D))$$

where c is an absolute constant.

Remark. The disk B of the Fejér-Riesz Theorem has disappeared in Theorem 4.3 and its Corollary. What about the line L? The answer, given by Bishop and Jones in [2], depends on the notion of a regular curve due to Ahlfors.

Definition 4.5. An arc C is regular if there is a constant a such that

$$\text{length}(C \cap U) \leq a\, \text{rad}(U)$$

for each disk U.

Theorem 4.6. Theorem 4.3 *holds with* C *in place of* L *if and only if* C *is regular.*

Remark. The disk B and the line L are now both gone from the original Fejér-Riesz Theorem! What about the analytic function g or the conformal mapping f?

Program of Bonk-Koskela-Rohde [3].

1. The goal is to characterize the densities $\sigma > 0$ in B for which there exist analogues of the results for the case where $\sigma = |f'|$ and f is conformal in B.
2. Two properties:
   a. Harnack type inequality,
   b. Growth rate inequality.
3. Many results of function theory follow if σ satisfies the above properties in B.
4. Example: If β is a hyperbolic geodesic in B, then

$$\int_\beta \sigma\, ds \leq c \int_\alpha \sigma\, ds$$

   for all α joining the endpoints of β in B where c is an absolute constant. This is the inequality in Corollary 2.2 when $\sigma = |f'|$.

Problem. What are the analogues of Theorem 4.3 and Corollary 4.4 for such a density σ?

5. Higher variation of a function. If f is defined over an interval I, then for $1 \leq p < \infty$ we can define the p^{th} power variation of f over I by

$$p \text{ variation}_I(f) = \sup_\tau \left(\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|^p \right)^{1/p}$$

where the supremum is taken over all subdivisions $\tau = \{x_0 < x_1 < \dots < x_n\}$ of I. The p^{th} power variation of f interpolates between the usual variation and the oscillation of f as p varies between 1 and ∞. See, for example, [4], [12], [19], [26] and [27].

We have the corresponding extension of the Riesz Theorem [11].

Theorem 5.1. *If u is harmonic in B and continuous in $\overline{B}$, then*

$$p \text{ variation }_{L\cap B}(u) \le \tfrac{1}{2}\ p \text{ variation }_{\partial B}(u)$$

for $1 \le p < \infty$.

6. One-dimensional heat equation. The Riesz Theorem takes the following form when $D = H$.

Theorem 6.1. *If u is harmonic in H and continuous in $\overline{H}$, then for $|a| < \infty$ and $0 < b < \infty$*

$$\int_b^\infty |u_y(a,y)|\, dy \le \tfrac{1}{2} \int_{-\infty}^\infty |u_x(x,b)|\, dx.$$

Proof. If h maps B conformally onto $\{z = x+iy : b < y < \infty\}$ and $L \cap B$ onto $\{z = a+iy : b < y < \infty\}$, then $v = u \circ h$ is harmonic in B, continuous in $\overline{B}$ and

$$\begin{aligned}\int_b^\infty |u_y(a,y)|\, dy &= \text{variation }_{L\cap B}(v)\\ &\le \frac{1}{2} \text{ variation }_{\partial B}(v) = \frac{1}{2}\int_{-\infty}^\infty |u_x(x,b)|\, dx.\end{aligned}$$ □

For $|x| < \infty$ and $t > 0$ let $u = u(x,t)$ denote the absolute temperature in an infinite insulated rod with unit thermal conductivity and unit cross-section spread along the x-axis. Then

$$u_t = u_{xx} \text{ and } u > 0$$

for $(x,t) \in H$. Temperature functions behave in many ways like positive harmonic functions. See Widder [25] and [13], [14].

The following is an analogue of Theorem 6.1 for temperature functions [14].

Theorem 6.2. *If u is a temperature function in H, then for $|a| < \infty$ and $0 < b < \infty$*

$$\int_b^\infty |u_x(a,t)|\, dt \le \frac{1}{2}\int_{-\infty}^\infty u(x,b)\, dx, \tag{6.3}$$

$$\int_b^\infty |u_t(a,t)|\, dt \le \frac{1}{2}\int_{-\infty}^\infty |u_x(x,b)|\, dx. \tag{6.4}$$

Remarks. The following physical interpretations of Theorem 6.2 yield an interesting way of viewing the Riesz Theorem.

1. Suppose that the heat in the rod at time $t = b$ is equal to $A < \infty$, that is,

$$\int_{-\infty}^{\infty} u(x,b)\,dx = A.$$

Then inequality (6.3) says that the total heat flow across each fixed section of the rod in the time interval $b \leq t < \infty$ never exceeds $A/2$.

2. Suppose next that the variation of temperature along the rod at time $t = b$ is equal to $V < \infty$. Then inequality (6.4) says that at each section of the rod the temperature variation in time for $b \leq t < \infty$ never exceeds $V/2$.

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