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KEIKO FUJITA and MITSUO MORIMOTO

Conical Fourier transform of Hardy space of harmonic functions on the Lie ball

ABSTRACT. This paper is an extended version of a talk entitled "Hardy spaces of harmonic functions related to the complex sphere" and given at the 12-th Conference on Analytic Functions. The authors consider Hardy space of complex harmonic functions on the Lie ball with an inner product given by an integral on a part of the boundary of the Lie ball. They determine the image of the space under conical Fourier transformations.

1. Introduction. We denote $\mathbb{R}^{n+1}$ by $\mathbb{E}$ and $\mathbb{C}^{n+1}$ by $\tilde{\mathbb{E}}$. Let $z \cdot w = z_1 w_1 + \cdots + z_{n+1} w_{n+1}$, $z^2 = z \cdot z$, and $||z||^2 = z \cdot \overline{z}$. The open and the closed Lie balls of radius r with center at 0 are defined by

$$\tilde{B}(r) = \{z \in \tilde{\mathbb{E}} : L(z) < r\}, \quad 0 < r < \infty,$$
$$\tilde{B}[r] = \{z \in \tilde{\mathbb{E}} : L(z) \le r\}, \quad 0 \le r < \infty,$$

where $L(z) = \{||z||^2 + (||z||^4 - |z^2|^2)^{1/2}\}^{1/2}$ is the Lie norm. Note that $\tilde{B}(\infty) = \tilde{\mathbb{E}}$.

We denote by $\mathcal{O}(\tilde{B}(r))$ the space of holomorphic functions on $\tilde{B}(r)$ equipped with the topology of uniform convergence on compact sets and denote by

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$\mathcal{O}(\tilde{B}[r]) = \lim\operatorname{ind}_{r'>r}\mathcal{O}(\tilde{B}(r'))$ the space of germs of holomorphic functions on $\tilde{B}[r]$. Put

$$\mathcal{O}_\Delta(\tilde{B}(r)) = \{f \in \mathcal{O}(\tilde{B}(r)) : \Delta_z f(z) = 0\},$$
$$\mathcal{O}_\Delta(\tilde{B}[r]) = \lim\operatorname{ind}_{r'>r}\mathcal{O}_\Delta(\tilde{B}(r')),$$

where $\Delta_z = \partial^2/\partial z_1^2 + \partial^2/\partial z_2^2 + \cdots + \partial^2/\partial z_{n+1}^2$ is the complex Laplacian. We call an element of $\mathcal{O}_\Delta(\tilde{B}(r))$ a complex harmonic function on $\tilde{B}(r)$.

Let $n \geq 2$. We define the complex sphere with radius $\lambda \in \mathbb{C}$ by $\tilde{S}_\lambda = \{z \in \tilde{\mathbb{E}} : z^2 = \lambda^2\}$. If $\lambda = 0$, then $\tilde{S}_0$ is called the complex light cone (or the complex isotropic cone). Put

$$\tilde{S}_\lambda(r) = \tilde{S}_\lambda \cap \tilde{B}(r), \quad |\lambda| < r \leq \infty,$$
$$\tilde{S}_\lambda[r] = \tilde{S}_\lambda \cap \tilde{B}[r], \quad |\lambda| \leq r < \infty,$$
$$\tilde{S}_{\lambda,r} = \partial\tilde{S}_\lambda[r], \qquad |\lambda| \leq r < \infty.$$

If $|\lambda| < r$, then $\tilde{S}_{\lambda,r}$ is a $(2n-1)$-dimensional compact manifold on which the orthogonal group $SO(n+1)$ acts transitively. If $|\lambda| = r > 0$, then it reduces to the n-dimensional compact manifold $\tilde{S}_{\lambda,r} = \tilde{S}_\lambda[r] = \lambda S_1$, where $S_1 = \{x \in \mathbb{E} : x^2 = 1\}$ is the real unit sphere.

For $f, g \in \mathcal{O}_\Delta(\tilde{B}[r])$ we put

$$(f,g)_{\tilde{S}_{\lambda,r}} = \int_{\tilde{S}_{\lambda,r}} f(z)\overline{g(z)}\,\dot{d}z,$$

where $\dot{d}z$ is the normalized invariant measure on $\tilde{S}_{\lambda,r}$.

After some necessary preparation in Section 2 we show in Section 3 that $(f,g)_{\tilde{S}_{\lambda,r}}$ is an inner product on $\mathcal{O}_\Delta(\tilde{B}[r])$ and denote by $h^2_\lambda(\tilde{B}(r))$ the completion of $\mathcal{O}_\Delta(\tilde{B}[r])$ with respect to the inner product $(f,g)_{\tilde{S}_{\lambda,r}}$. We can see that $h^2_\lambda(\tilde{B}(r))$ is isomorphic to a Hardy space of harmonic functions on the Lie ball.

In Section 4, we define the conical Fourier transformation $\mathcal{F}^\Delta_{\mu,r}$ for $f \in \mathcal{O}_\Delta(\tilde{B}(r))$, where μ is another complex number with $|\mu| \leq r$. Then the conical Fourier transform $\mathcal{F}^\Delta_{\mu,r}f$ is given by

$$\mathcal{F}^\Delta_{\mu,r}f(\zeta) = \int_{\tilde{S}_{\mu,r}} \exp(sz\cdot\zeta)\overline{f(z/s)}\,\dot{d}z, \quad \zeta \in \tilde{S}_0,$$

which does not depend on $s > 1$ sufficiently close to 1.

Then in Section 5, by introducing a Radon measure on $\tilde{S}_0$, we construct the inverse mapping $\mathcal{M}_{\mu,r}$ of the conical Fourier transformation $\mathcal{F}^\Delta_{\mu,r}$.

We also study a Hilbert space $\mathcal{E}^2(\bar{S}_0;\mu,\lambda,r)$ of entire functions on $\bar{S}_0$ which are square integrable with respect to the Radon measure.

Finally, in Section 6, we show that the image of $h^2_\lambda(\bar{B}(r))$ under the conical Fourier transformation $\mathcal{F}^{\Delta}_{\mu,r}$ s isomorphic to $\mathcal{E}^2(\bar{S}_0;\mu,\lambda,r)$ and we study a reproducing kernel for $\mathcal{E}^2(\bar{S}_0;\mu,\lambda,r)$.

2. Homogeneous harmonic polynomials. We denote by $\mathcal{P}^k_\Delta(\bar{\mathbb{E}})$ the space of k-homogeneous harmonic polynomials on $\bar{\mathbb{E}}$. The dimension of $\mathcal{P}^k_\Delta(\bar{\mathbb{E}})$ is given by

$$N(k,n)=\frac{(2k+n-1)(k+n-2)!}{k!(n-1)!}=O(k^{n-1}).$$

Let $P_{k,n}(t)$ be the Legendre polynomial of degree k and of dimension $n+1$. The coefficient $\gamma_{k,n}$ of the highest power of $P_{k,n}(t)$ is known as

$$\gamma_{k,n}=\frac{\Gamma(k+(n+1)/2)2^k}{N(k,n)\Gamma((n+1)/2)k!}$$

and $\overline{P_{k,n}(t)}=P_{k,n}(\bar{t})$. The harmonic extension $\tilde{P}_{k,n}(z,w)$ of $P_{k,n}(z\cdot w)$ is defined by

$$\tilde{P}_{k,n}(z,w)=\left(\sqrt{z^2}\right)^k\left(\sqrt{w^2}\right)^k P_{k,n}\left(\frac{z}{\sqrt{z^2}}\cdot\frac{w}{\sqrt{w^2}}\right).$$

Then $\tilde{P}_{k,n}(z,w)$ is a symmetric k-homogeneous harmonic polynomial in z and in w. If $z^2=0$ or $w^2=0$, then $\tilde{P}_{k,n}(z,w)=\gamma_{k,n}(z\cdot w)^k$.

Theorem 2.1 ([6, Theorem 5.2]). *Define the k-harmonic component f_k of $f\in\mathcal{O}_\Delta(\bar{B}(r))$ by*

$$f_k(z)=N(k,n)\int_{S_1} f(\rho\omega)P_{k,n}(z/\rho,\omega)\,\dot{d\omega}\,,\quad 0<\rho<r\,,$$

where $\dot{d\omega}$ is the normalized invariant measure on S_1.

Then $f_k\in\mathcal{P}^k_\Delta(\bar{\mathbb{E}})$ and $\sum_{k=0}^\infty f_k(z)$ converges to $f(z)$ in the topology of $\mathcal{O}_\Delta(\bar{B}(r))$. Moreover, we have

$$f=\sum_{k=0}^\infty f_k(z)\in\mathcal{O}_\Delta(\bar{B}(r))\Longleftrightarrow\limsup_{k\to\infty}\|f_k\|^{1/k}_{C(S_1)}\le 1/r\,,$$

$$f=\sum_{k=0}^\infty f_k(z)\in\mathcal{O}_\Delta(\bar{B}[r])\Longleftrightarrow\limsup_{k\to\infty}\|f_k\|^{1/k}_{C(S_1)}<1/r\,,$$

where $\|f_k\|_{C(S_1)}=\sup\{|f_k(x)|:x\in S_1\}$.

For $f, g \in \mathcal{O}_\Delta(\tilde{B}[r])$ we define the sesquilinear form $(\cdot, \cdot)_{\tilde{S}_{\lambda,r}}$ by

$$(f,g)_{\tilde{S}_{\lambda,r}} = \int_{\tilde{S}_{\lambda,r}} f(z)\overline{g(z)}\,\dot{d}z\,,\ |\lambda| \leq r\,,$$

where $\dot{d}z$ is the normalized invariant measure on $\tilde{S}_{\lambda,r}$.

For $f_k \in \mathcal{P}^k_\Delta(\mathbb{E})$, $g_l \in \mathcal{P}^l_\Delta(\mathbb{E})$, R. Wada [13] proved the relation

$$(1)\quad \int_{\tilde{S}_{\lambda,r}} f_k(z)\overline{g_l(z)}\,\dot{d}z = L_{k,\lambda,r}\int_{S_1} f_k(x)\overline{g_l(x)}\,\dot{d}x = \begin{cases} L_{k,\lambda,r}\int_{S_1} f_k(x)\overline{g_l(x)}\,\dot{d}x \\ 0\,, \qquad (k \neq l), \end{cases}$$

where

$$L_{k,\lambda,r} \equiv \begin{cases} |\lambda|^2 P_{k,n}\left(\frac{1}{2}\left(\frac{r^2}{|\lambda|^2} + \frac{|\lambda|^2}{r^2}\right)\right), & \lambda \neq 0\,, \\ \frac{\gamma_{k,n}}{2^k}\, r^{2k}\,, & \lambda = 0\,. \end{cases}$$

Note that $L_{k,0,r} = \lim_{\lambda\to 0} L_{k,\lambda,r}$.

Lemma 2.2 ([3, Lemma 7.2]). *$L_{k,\lambda,r}$ is a monotone increasing function in $|\lambda|$; that is, for $0 < |\lambda| < |\mu| < r$ and $k \neq 0$, we have*

$$2^{-k}\gamma_{k,n} r^{2k} = L_{k,0,r} < L_{k,\lambda,r} < L_{k,\mu,r} < L_{k,r,r} = r^{2k}\,.$$

By Lemma 2.2 , Theorem 2.1 , and (1) we have

$$(f,g)_{\tilde{S}_{\lambda,r}} = \sum_{k=0}^{\infty}\int_{\tilde{S}_{\lambda,r}} f_k(z)\overline{g_k(z)}\,\dot{d}z = \sum_{k=0}^{\infty}\int_{S_1} f_k(x)\overline{g_k(x)}\,\dot{d}x < \infty\,.$$

Thus $(\cdot, \cdot)_{\tilde{S}_{\lambda,r}}$ is an inner product on $\mathcal{O}_\Delta(\tilde{B}[r])$.

The sesquilinear form $(f,g)_{\tilde{S}_{\lambda,r}} = \sum_{k=0}^{\infty}\int_{\tilde{S}_{\lambda,r}} f_k(z)\overline{g_k(z)}\,\dot{d}z$ was defined for $f, g \in \mathcal{O}_\Delta(\tilde{B}[r])$. However, by Theorem 2.1, for $f \in \mathcal{O}_\Delta(\tilde{B}[r])$ and $g \in \mathcal{O}_\Delta(\tilde{B}(r))$

$$\int_{\tilde{S}_{\lambda,r}} f(sz)\overline{g(z/s)}\,\dot{d}z = \sum_{k=0}^{\infty}\int_{\tilde{S}_{\lambda,r}} f_k(z)\overline{g_k(z)}\,\dot{d}z$$

is well-defined for $s > 1$ sufficiently close to 1 and does not depend on s. Sometime we set s. $\int_{\tilde{S}_{\lambda,r}} f(z)\overline{g(z)}\,\dot{d}z = \int_{\tilde{S}_{\lambda,r}} f(sz)\overline{g(z/s)}\,\dot{d}z$ and call it the symbolic integral over $\tilde{S}_{\lambda,r}$. Thus we can extend $(f,g)_{\tilde{S}_{\lambda,r}}$ to a separately continuous sesquilinear form on $\mathcal{O}_\Delta(\tilde{B}[r]) \times \mathcal{O}_\Delta(\tilde{B}(r))$ by the symbolic integral. Similarly we can extend $(f,g)_{\tilde{S}_{\lambda,r}}$ to a separately continuous sesquilinear form on $\mathcal{O}_\Delta(\tilde{B}(r)) \times \mathcal{O}_\Delta(\tilde{B}[r])$. Therefore, we still have

$$\overline{(f,g)_{\tilde{S}_{\lambda,r}}} = (g,f)_{\tilde{S}_{\lambda,r}}$$

for $f \in \mathcal{O}_\Delta(\tilde{B}[r])$ and $g \in \mathcal{O}_\Delta(\tilde{B}(r))$ or for $f \in \mathcal{O}_\Delta(\tilde{B}(r))$ and $g \in \mathcal{O}_\Delta(\tilde{B}[r])$.

3. Hardy spaces of harmonic functions on the Lie ball. Let $|\lambda| \le r$. We denote by $h^2_\lambda(\tilde{B}(r))$ the completion of $\mathcal{O}_\Delta(\tilde{B}[r])$ with respect to the inner product $(\cdot,\cdot)_{\tilde{S}_{\lambda,r}}$. By the definition,

$$h^2_\lambda(\tilde{B}(r)) = \left\{ f = \sum_{k=0}^{\infty} f_k : f_k \in \mathcal{P}^k_\Delta(\tilde{\mathbb{E}}),\ \sum_{k=0}^{\infty} \|f_k\|^2_{\tilde{S}_{\lambda,r}} < \infty \right\}.$$

Further, as in the proof of Lemma 3.2 in [2], we can see that $h^2_\lambda(\tilde{B}(r))$ is isomorphic to the Hardy space:

$$h^2_\lambda(\tilde{B}(r)) = \left\{ f \in \mathcal{O}_\Delta(\tilde{B}(r)) : \sup_{0<t<1} \int_{\tilde{S}_{\lambda,r}} |f(tz)|^2 \dot{d}z < \infty \right\}.$$

Proposition 3.1 ([4, Theorem 1.5]). *The Hardy space $h^2_\lambda(\tilde{B}(r))$ is a Hilbert space being a direct sum of the finite dimensional subspaces $\mathcal{P}^k_\Delta(\tilde{\mathbb{E}})$:*

$$h^2_\lambda(\tilde{B}(r)) = \bigoplus_{k=0}^{\infty} {}^k_\Delta(\tilde{\mathbb{E}}).$$

By using Lemma 2.2, we can prove the following

Theorem 3.2 ([4, Theorem 1.5]). *For $0 < |\lambda| < |\mu| < r$, we have*

$$\mathcal{O}_\Delta(\tilde{B}[r]) \subset h^2_r(\tilde{B}(r)) \subset h^2_\mu(\tilde{B}(r)) \subset h^2_\lambda(\tilde{B}(r)) \subset h^2_0(\tilde{B}(r)) \subset \mathcal{O}_\Delta(\tilde{B}(r)).$$

Now we consider the reproducing kernel. Since $|P_{k,n}(z,w)| \le L(z)^k L(w)^k$ and $\lim_{k\to\infty}(L_{k,\lambda,r})^{1/k} = r^2$ for $|\lambda| \le r$, the Poisson kernel

$$K_{\lambda,r}(z,w) = \sum_{k=0}^{\infty} \frac{N(k,n)}{L_{k,\lambda,r}} \bar{P}_{k,n}(z,\overline{w})$$

is a function on $\{(z,w) \in \tilde{\mathbb{E}} \times \tilde{\mathbb{E}} : L(z)L(w) < r^2\}$ and complex harmonic in z. It satisfies $\overline{K_{\lambda,r}(z,w)} = K_{\lambda,r}(w,z)$. In particular, $K_{r,r}(z,w)$ is the classical Poisson kernel and the restriction of $K_{0,r}(z,w)$ on $\tilde{S}_0 \times \tilde{\mathbb{E}}$ is called the Cauchy kernel on $\tilde{S}_0$ in [8]:

$$\begin{aligned} K_{r,r}(z,w) &= K_{1,1}(z/r, w/r)\,, \\ K_{1,1}(z,\overline{w}) &= \frac{1 - z^2w^2}{(1 + z^2w^2 - 2z\cdot w)^{(n+1)/2}}\,, \\ K_{0,r}(z,w) &= K_{0,1}(z/r, w/r)\,, \\ K_{0,1}(z,\overline{w})|_{\tilde{S}_0\times\tilde{\mathbb{E}}} &= \frac{1+2zw}{(1-2zw)^n}\,. \end{aligned}$$

Using the Poisson kernel, we have the following integral representation for $f \in \mathcal{O}_\Delta(\tilde{B}(r))$ (Theorem 3 in [7], see also [10] and [11]):

$$f(z) = s.\int_{\tilde{S}_{\lambda,r}} f(w)K_{\lambda,r}(z,w)\dot{d}w, \quad z \in \tilde{B}(r)\,.$$

For $f \in h^2_\lambda(\tilde{B}(r))$ we have

Theorem 3.3 ([4, Theorem 1.5]). *The Poisson kernel $K_{\lambda,r}(z,w)$ is a reproducing kernel of $h^2_\lambda(\tilde{B}(r))$ which means that for $f \in h^2_\lambda(\tilde{B}(r))$ we have the following integral representation:*

$$f(z) = (f(w), K_{\lambda,r}(w,z))_{\tilde{S}_{\lambda,r}} = \int_{\tilde{S}_{\lambda,r}} f(w)K_{\lambda,r}(z,w)\dot{d}w\,, \quad z \in \tilde{B}(r).$$

We denote by $L^2\mathcal{O}(\tilde{S}_{\lambda,r})$ the closed subspace of the space of square integrable functions on $\tilde{S}_{\lambda,r}$ generated by $\mathcal{H}^k(\tilde{S}_{\lambda,r}) = \mathcal{P}^k_\Delta(\tilde{\mathbb{E}})|_{\tilde{S}_{\lambda,r}}$, $k = 0,1,2,\ldots,$. Then as a corollary of Theorem 3 in [7] and Theorem 3. 3 we have

Corollary 3.4. *The restriction mapping α_λ gives the following linear topological isomorphisms:*

$$\begin{aligned} \alpha_\lambda &: h^2_\lambda(\tilde{B}(r)) \xrightarrow{\sim} \mathcal{O}_\Delta(\tilde{B}(r))\,, \\ \alpha_\lambda &: \mathcal{O}_\Delta(\tilde{\mathbb{E}}) \xrightarrow{\sim} \mathcal{O}(\tilde{S}_\lambda)\,, \end{aligned}$$

where $\mathcal{O}(\tilde{S}_\lambda)$ is the space of holomorphic functions on $\tilde{S}_\lambda$ equipped with the topology of uniform convergence on compact sets.

For related topics see [3].

4. Conical Fourier transformation. Let $\mathcal{O}'_\Delta(\tilde{B}[r])$ (resp., $\mathcal{O}'_\Delta(\tilde{B}(r))$ be the dual space of $\mathcal{O}_\Delta(\tilde{B}[r])$ (resp., $\mathcal{O}_\Delta(\tilde{B}(r))$). For $T \in \mathcal{O}'_\Delta(\tilde{B}[r])$, we define the Poisson transformation $\mathcal{P}_{\mu,r}$ by $\mathcal{P}_{\mu,r} : T \mapsto \mathcal{P}_{\mu,r}T(w) = \overline{\langle T_{z,\mu,r}(z,w)\rangle}$. Then we have the following

Theorem 4.1. *Let $0 < r < \infty$. The Poisson transformation establishes the following antilinear topological isomorphisms:*

$$\mathcal{P}_{\mu,r} : \mathcal{O}'_\Delta(\tilde{B}[r]) \xrightarrow{\sim} \mathcal{O}_\Delta(\tilde{B}(r)),$$
$$\mathcal{P}_{\mu,r} : \mathcal{O}'_\Delta(\tilde{B}(r)) \xrightarrow{\sim} \mathcal{O}_\Delta(\tilde{B}[r]).$$

Further, for $T \in \mathcal{O}'_\Delta(\tilde{B}(r))$ and $f \in \mathcal{O}_\Delta(\tilde{B}(r))$, or for $T \in \mathcal{O}'_\Delta(\tilde{B}[r])$ and $f \in \mathcal{O}_\Delta(\tilde{B}[r])$, we have

$$\langle T, f\rangle = (f, \mathcal{P}_{\mu,r}T)_{\tilde{S}_{\mu,r}}. \tag{2}$$

This can be proved similarly as Theorem 15 in [10].

Since $\Delta_z \exp(z\cdot\zeta) = 0$ for $\zeta \in \tilde{S}_0$, we can define the conical Fourier-Borel transformation for $T \in \mathcal{O}'_\Delta(\tilde{B}[r])$ by

$$\mathcal{F}^\Delta_r : T \mapsto \mathcal{F}^\Delta_r T(\zeta) = \langle T_z, \exp(z\zeta)\rangle, \quad \zeta \in \tilde{S}_0. \tag{3}$$

Put

$$\mathrm{Exp}(\tilde{S}_0;(r)) = \left\{ f \in \mathcal{O}(\tilde{S}_0) : \forall_{r'>r},\ \exists_{C>0} \text{ s.t. } |f(\zeta)| \le C\exp(r'L^*(\zeta)) \right\},$$

$$\mathrm{Exp}(\tilde{S}_0;[r]) = \left\{ f \in \mathcal{O}(\tilde{S}_0) : \forall_{r',r},\ \exists_{C>0} \text{ s.t. } |f(\zeta)| \le C\exp(r'L^*(\zeta)) \right\},$$

where

$$L^*(\zeta) = \sup\{|z\zeta| : L(z) \le 1\} = \left\{\left(\|\zeta\|^2 + |\zeta|^2\right)/2\right\}^{1/2}$$

is the dual Lie norm. Then we have the following

Theorem 4.2. *The conical Fourier-Borel transformation $\mathcal{F}^\Delta_r$ gives the following linear topological isomorphisms:*

(i) $$\mathcal{F}^\Delta_r : \mathcal{O}'_\Delta(\tilde{B}[r]) \xrightarrow{\sim} \mathrm{Exp}(\tilde{S}_0;(r)), \quad 0 \le r < \infty,$$

(ii) $$\mathcal{F}^\Delta_r : \mathcal{O}'_\Delta(\tilde{B}(r)) \xrightarrow{\sim} \mathrm{Exp}(\tilde{S}_0;[r]), \quad 0 < r \le \infty.$$

(cf. Theorem 18 in [9]).

Now we define the conical Fourier transformation $\mathcal{F}^{\Delta}_{\mu,r}$ on $\mathcal{O}_{\Delta}(\tilde{B}(r))$ by

$$\mathcal{F}^{\Delta}_{\mu,r} = \mathcal{F}^{\Delta}_{r} \circ (\mathcal{P}_{\mu,r})^{-1}.$$

Then for $f \in \mathcal{O}_{\Delta}(\tilde{B}(r))$, by (2) and (3), we have

$$\mathcal{F}^{\Delta}_{\mu,r} f(\zeta) = (\exp(z\zeta), f(z))_{\tilde{S}_{\mu,r}}, \ \zeta \in \tilde{S}_0.$$

Lemma 4.3. *For $f = \sum_{k=0}^{\infty} f_k$, $f \in \mathcal{O}_{\Delta}(\tilde{B}(r))$ and $f_k \in \mathcal{P}^k_{\Delta}(\tilde{\mathbb{E}})$, we have*

$$\mathcal{F}^{\Delta}_{\mu,r} f(\zeta) = \sum_{k=0}^{\infty} \frac{L_{k,\mu,r}}{N(k,n)k!\gamma_{k,n}} \overline{f_k}(\zeta), \tag{4}$$

where we put $\overline{f_k}(\zeta) = \overline{f_k(\bar{\zeta})}$ for $f_k \in \mathcal{H}^k(\tilde{S}_0) \equiv \mathcal{P}^k_{\Delta}(\tilde{\mathbb{E}})|_{\tilde{S}_0}$.

Proof. We have

$$\exp(z\zeta) = \sum_{k=0}^{\infty} \frac{1}{k!\gamma_{k,n}} \tilde{j}_k\left(i\sqrt{z^2}\sqrt{\zeta^2}\right) \tilde{P}_{k,n}(z,\zeta), \tag{5}$$

where

$$\tilde{j}_k(t) = \Gamma\left(k + \frac{n+1}{2}\right)(t?2)^{-\left(k+\frac{n-1}{2}\right)} J_{k+\frac{n-1}{2}}(t)$$
$$= \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma\left(k+\frac{n+1}{2}\right)}{\Gamma\left(k+\frac{n+1}{2}+l\right) l!} (t/2)^{2l}$$

is the entire Bessel function (see [6]). Thus by Theorem 2.1 and (1), we get (4). □

Theorems 4.1 and 4.2 imply the following

Theorem 4.4. *Let $0 < r < \infty$. The conical Fourier transformation $\mathcal{F}^{\Delta}_{\mu,r}$ gives following antilinear topological isomorphisms:*

$$\mathcal{F}^{\Delta}_{\mu,r} : \mathcal{O}_{\Delta}(\tilde{B}(r)) \xrightarrow{\sim} \mathrm{Exp}(\tilde{S}_0; (r)),$$
$$\mathcal{F}^{\Delta}_{\mu,r} : \mathcal{O}_{\Delta}(\tilde{B}[r]) \xrightarrow{\sim} \mathrm{Exp}(\tilde{S}_0; [r]).$$

By Corollary 3.4 we may assume $f_k \in \mathcal{P}^k_{\Delta}(\tilde{\mathbb{E}})$. Therefore by (4) and Theorem 2.1 we obtain the following proposition (see also [8, Thm. 12]):

Proposition 4.5. *Let* $f = \sum_{k=0}^{\infty} f_k \in \mathrm{Exp}(\tilde{S}_0;(r))$ *and* $f_k \in \mathcal{H}^k(\tilde{S}_0)$. *Then we have*

$$f = \sum_{k=0}^{\infty} f_k \in \mathrm{Exp}(\tilde{S}_0;(r)) \Longleftrightarrow \limsup_{k\to\infty} \|k! f_k\|_{C(\tilde{S}_{0,1})}^{1/k} \leq r/2\,,$$

$$f = \sum_{k=0}^{\infty} f_k \in \mathrm{Exp}(\tilde{S}_0;[r]) \Longleftrightarrow \limsup_{k\to\infty} \|k! f_k\|_{C(\tilde{S}_{0,1})}^{1/k} < r/2\,,$$

where $\|f_k\|_{C(\tilde{S}_{0,1})} = \sup\{|f_k(z)| : z \in \tilde{S}_{0,1}\}$.

5. Radon measures on $\tilde{S}_0$. Let $\rho_{\mu,r}(t)$ be a function on $[0,\infty)$ satisfying

$$(6) \qquad \int_0^\infty t^{2k} \rho_{\mu,r}(t)dt = \frac{(N(k,n)k!)^2 \gamma_{k,n} 2^k}{L_{k,\mu,r}}\,, \; k = 0, 1, \ldots\,,$$

and define the Radon measure $d\tilde{S}_{0(\mu,r)}$ on $\tilde{S}_0$ by

$$\int_{\tilde{S}_0} f(\zeta) d\tilde{S}_{0(\mu,r)}(\zeta) \equiv \int_0^\infty \int_{\tilde{S}_{0,1}} F(t\zeta')\dot{d}\zeta' \rho_{\mu,r}(t)dt.$$

Such a function$\rho_{\mu,r}$ does exist by a theorem of A. Duran [1]. In case of $|\mu| = r$, K. Ii [5] and R. Wada [12] constructed such a function $\rho_r(t)$ of exponential type $-r$ by means of the modified Bessel functions.

By Corollary 4.5 and $\lim_{k\to\infty}(L_{k,\lambda,r})^{1/k} = r^2$, for $F \in \mathrm{Exp}(\tilde{S}_0;[r])$ and $G \in \mathrm{Exp}(\tilde{S}_0;(r))$ (resp., $F \in \mathrm{Exp}(\tilde{S}_0;(r))$ and $G \in \mathrm{Exp}(\tilde{S}_0;[r])$) the integral

$$\int_{\tilde{S}_0} F(\zeta)\overline{G(\zeta)}\, d\tilde{S}_{0(\mu,r)}(\zeta)$$

is well-defined and it defines a separately continuous sesquilinear form on $\mathrm{Exp}(\tilde{S}_0;[r]) \times \mathrm{Exp}(\tilde{S}_0;(r))$ (resp., $\mathrm{Exp}(\tilde{S}_0;(r)) \times \mathrm{Exp}(\tilde{S}_0;[r])$). If $w \in \tilde{S}_0$ and $z \in B(r)$, then the function $w \mapsto \exp(z \cdot w)$ belongs to $\mathrm{Exp}(\tilde{S}_0;[r])$. Therefore, for $F \in \mathrm{Exp}(\tilde{S}_0;(r))$ we can define $\mathcal{M}_{\mu,r}F(z)$ by

$$(7) \qquad \mathcal{M}_{\mu,r}F(z) = \int_{\tilde{S}_0} \exp(z\zeta)\overline{F(\zeta)}\, d\tilde{S}_{0(\mu,r)}(\zeta)\,, \quad z \in \tilde{B}(r)\,.$$

We denote by $\mathcal{M}_{\mu,r}$ the transformation $F \mapsto \mathcal{M}_{\mu,r}F$. By Theorem 2.1, (5) and (1) we have the following

Lemma 5.1. *For $F=\sum_{k=0}^{\infty}F_k\in \mathrm{Exp}(\tilde{S}_0;(r))$ and $F_k\in\mathcal{H}^k(\tilde{S}_0)$, we have*

$$\mathcal{M}_{\mu,r}F(w)=\sum_{k=0}^{\infty}\frac{N(k,n)k!\gamma_{k,n}}{L_{k,\mu,r}}\overline{F_k}(w)\,.$$

Theorem 5.2. *The mapping $\mathcal{M}_{\mu,r}$ gives following antilinear topological isomorphisms and is inverse to the conical Fourier transformation $\mathcal{F}^{\Delta}_{\mu,r}$:*

$$\mathcal{M}_{\mu,r}:\mathrm{Exp}(\tilde{S}_0;(r))\xrightarrow{\sim}\mathcal{O}_{\Delta}(\tilde{B}(r))\,,$$
$$\mathcal{M}_{\mu,r}:\mathrm{Exp}(\tilde{S}_0;[r])\xrightarrow{\sim}\mathcal{O}_{\Delta}(\tilde{B}[r])\,.$$

Proof. By Lemmas 4.3 and 5.1 we have $\mathcal{M}_{\mu,r}\circ\mathcal{F}^{\Delta}_{\mu,r}f(z)=f(z)$ for $f\in\mathcal{O}_{\Delta}(\tilde{B}(r))$ and $\mathcal{F}^{\Delta}_{\mu,r}\circ\mathcal{M}_{\mu,r}=F(z)$ for $F\in\mathrm{Exp}(\tilde{S}_0;(r))$. Thus $\mathcal{M}_{\mu,r}$ is bijective, whereas $\mathcal{M}_{\mu,r}$ and $\mathcal{F}^{\Delta}_{\mu,r}$ are inverse to each other. □

For $\zeta,\xi\in\tilde{S}_0$ we put

$$E_{\mu,r}(\zeta,\xi)=\int_{\tilde{S}_0}\exp(z\zeta)\overline{\exp(z\xi)}\dot{d}z.$$

Proposition 5.3. *For $F\in\mathrm{Exp}(\tilde{S}_0;(r))$ we have*

$$F(\xi)=\int_{\tilde{S}_0}F(\zeta)\overline{E_{\mu,r}(\zeta,\xi)}\,d\tilde{S}_{0_{(\mu,r)}}(\zeta). \tag{8}$$

Proof. Let $F=\sum_{k=0}^{\infty}F_k\in\mathrm{Exp}(\tilde{S}_0;(r))$ and $F_k\in\mathcal{H}^k(\tilde{S}_0)$. Then

$$\begin{aligned}F(\xi)=\mathcal{F}^{\Delta}_{\mu,r}\circ\mathcal{M}_{\mu,r}F(\xi)&=\left(\exp(z\xi),\int_{\tilde{S}_0}\exp(z\zeta)\overline{F(\zeta)}\,d\tilde{S}_{0(\mu,r)}(\zeta)\right)_{\tilde{S}_{\mu,r}}\\&=\int_{\tilde{S}_{\mu,r}}\exp(sz\cdot\xi)\overline{\int_{\tilde{S}_0}\exp(z/s\cdot\zeta)\overline{F(\zeta)}\,d\tilde{S}_{0(\mu,r)}(\zeta)}\dot{d}z\\&=\int_{\tilde{S}_0}\int_{\tilde{S}_{\mu,r}}\exp(z\xi)\overline{\exp(z\zeta)}\dot{d}zF(\zeta)\,d\tilde{S}_{0(\mu,r)}(\zeta)\\&=\int_{\tilde{S}_0}F(\zeta)\overline{E_{\mu,r}(\zeta,\xi)}\,d\tilde{S}_{0(\mu,r)}(\zeta)\,,\end{aligned}$$

where $s>1$ is sufficiently close to 1. □

Now we employ the theorem of A. Duran ([1]) again, and there is a C^∞ function $\rho_{\mu,\lambda,r}(t)$ which satisfies

$$(9) \quad \int_0^\infty t^{2k}\rho_{\mu,\lambda,r}(t)dt = \frac{(N(k,n)k!)^2\gamma_{k,n}2^k L_{k,\lambda,r}}{(L_{k,\mu,r})^2}, \quad k = 0, 1, \ldots.$$

Define the Radon measure $d\tilde{S}_{0(\mu,\lambda,r)}$ on $\tilde{S}_0$ by

$$\int_{\tilde{S}_0} F(\zeta)d\tilde{S}_{0(\mu,\lambda,r)}(\zeta) = \int_0^\infty \left(\int_{\tilde{S}_{0,1}} F(t\zeta')\dot{\zeta}'\right)\rho_{\mu,\lambda,r}(t)dt.$$

When $|\mu| = |\lambda|$, (9) reduces to (6), $\rho_{\mu,\lambda,r}(t)$ to $\rho_{\mu,r}(t)$ and $d\tilde{S}_{0(\mu,\lambda,r)}$ to $d\tilde{S}_{0(\mu,r)}$. Put

$$\mathcal{E}^2(\tilde{S}_0;\mu,\lambda,r) = \left\{F \in \mathcal{O}(\tilde{S}_0) : \int_{\tilde{S}_0} |F(\zeta)|^2 d\tilde{S}_{0(\mu,\lambda,r)}(\zeta) < \infty\right\}.$$

When $|\mu| = |\lambda|$, we denote $\mathcal{E}^2(\tilde{S}_0;\mu,\lambda,r)$ by $\mathcal{E}^2(\tilde{S}_0;\mu,r)$.

Theorem 5.4. *The Hilbert space $\mathcal{E}^2(\tilde{S}_0;\mu,\lambda,r)$ is a Hilbert space being a direct sum of the finite dimensional subspaces $\mathcal{H}^h(\tilde{S}_0)$:*

$$\mathcal{E}^2(\tilde{S}_0;\mu,\lambda,r) = \bigoplus_{k=0}^{\infty} \mathcal{H}^k(\tilde{S}_0).$$

Proof. Let $F = \sum_{k=0}^\infty F_k(\zeta) \in \mathcal{E}^2(\tilde{S}_0;\mu,\lambda,r)$ and $F_k \in \mathcal{H}^k(\tilde{S}_0)$. By the definition of the Radon measure $d\tilde{S}_{0(\mu,\lambda,r)}$, we have

$$\begin{aligned} \int_{\tilde{S}_0} |F(\zeta)|^2 d\tilde{S}_{0(\mu,\lambda,r)}(\zeta) &= \int_0^\infty \left(\int_{\tilde{S}_{0,1}} F(t\zeta')\dot{\zeta}'\right)\rho_{\mu,\lambda,r}(t)dt \\ (10) \qquad &= \int_0^\infty \sum_{k=0}^\infty t^{2k}(F_k,F_k)_{\tilde{S}_{0,1}}\rho_{\mu,\lambda,r}(t)dt \\ &= \sum_{k=0}^\infty \frac{(N(k,n)k!)^2\gamma_{k,n}2^k L_{k,\lambda,r}}{(L_{k,\mu,r})^2}\|F_k\|^2_{\tilde{S}_{0,1}}. \end{aligned}$$

This completes the proof. □

By (10) and Lemma 2.2 we have the following

Corollary 5.5. *If* $|\mu_1| < |\mu_2| \le r$, *then*

$$\mathcal{E}^2(\tilde{S}_0; \mu_1, \lambda, r) \subset \mathcal{E}^2(\tilde{S}_0; \mu_2, \lambda, r).$$

If $|\lambda_1| < |\lambda_2| \le r$, *then*

$$\mathcal{E}^2(\tilde{S}_0; \mu, \lambda_1, r) \supset \mathcal{E}^2(\tilde{S}_0; \mu, \lambda_2, r).$$

If $|\mu_1| = |\lambda_1|, |\mu_2| = |\lambda_2|$ *and* $|\mu_1| < |\mu_2|$, *then*

$$\mathcal{E}^2(\tilde{S}_0; \mu_1, r) \subset \mathcal{E}^2(\tilde{S}_0; \mu_2, r).$$

6. The $\mathcal{F}^{\Delta}_{\mu,r}$ image of $h^2_\lambda(\tilde{B}(r))$. Now we consider the image of the Hardy space $h^2_\lambda(\tilde{B}(r))$ under the conical Fourier transformation $\mathcal{F}^{\Delta}_{\mu,r}$.

Let $f \in h^2_\lambda(\tilde{B}(r))$. Since $h^2_\lambda(\tilde{B}(r)) \subset \mathcal{O}_\Delta(\tilde{B}(r))$ and

$$\exp(z\zeta) = \sum_{k=0}^{\infty} \frac{(z\zeta)^k}{k!} = \sum_{k=0}^{\infty} \frac{\tilde{P}_{k,n}(z,\zeta)}{k!\gamma_{k,n}} \tag{11}$$

for $z \in \tilde{\mathbb{E}}$ and $\zeta \in \tilde{S}_0$, we have

$$\begin{aligned}
\mathcal{F}^{\Delta}_{\mu,r} f(\zeta) &= (\exp(z\zeta), f(z))_{\tilde{S}_{\mu,r}} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!\gamma_{k,n}} \left(\tilde{P}_{k,n}(z,\zeta), f(z)\right)_{\tilde{S}_{\mu,r}} \\
&= \sum_{k=0}^{\infty} \frac{L_{k,\mu,r}}{k!\gamma_{k,n} N(k,n)} \tilde{f}_k(\zeta) \qquad (13)\\
&= \sum_{k=0}^{\infty} \frac{L_{k,\mu,r}}{k!\gamma_{k,n} L_{k,\lambda,r}} \left(\tilde{P}_{k,n}(z,\zeta), f(z)\right)_{\tilde{S}_{\mu,r}}.
\end{aligned}$$

For $z \in \tilde{\mathbb{E}}$ and $\zeta \in \tilde{S}_0$, put

$$e^{\mu}_{\lambda}(z,\zeta) = \sum_{k=0}^{\infty} \frac{L_{k,\mu,r}(z\zeta)^k}{L_{k,\lambda,r} k!} = \sum_{k=0}^{\infty} \frac{L_{k,\mu,r}\tilde{P}_{k,n}(z,\zeta)}{L_{k,\lambda,r} k!\gamma_{k,n}}. \tag{13}$$

If $\zeta \in \tilde{S}_0$ is fixed, then $e^{\mu}_{\lambda}(\cdot,\zeta)$ is a complex harmonic function on $\tilde{\mathbb{E}}$. Hence $e^{\mu}_{\lambda}(\cdot,\zeta) \in h^2_\lambda(\tilde{B}(r))$. Therefore by (12), for $f \in h^2_\lambda(\tilde{B}(r))$, we have

$$\mathcal{F}^{\Delta}_{\mu,r} f(\zeta) = (\exp(z\zeta), f(z))_{\tilde{S}_{\mu,r}} = (e^{\mu}_{\lambda}(z,\zeta), f(z))_{\tilde{S}_{\lambda,r}}. \tag{14}$$

Theorem 6.1. *The conical Fourier transformation $\mathcal{F}^{\Delta}_{\mu,r}$ gives the antilinear unitary isomorphism:*

$$\mathcal{F}^{\Delta}_{\mu,r} : h^2_{\lambda}(\tilde{B}(r)) \xrightarrow{\sim} \mathcal{E}^2(\tilde{S}_0; \mu, \lambda, r). \tag{15}$$

Proof. Let $f = \sum_{k=0}^{\infty} f_k \in h^2_{\lambda}(\tilde{B}(r)), f_k \in \mathcal{P}^k_{\Delta}(\tilde{\mathbb{E}})$ and put $F(\zeta) = \mathcal{F}^{\Delta}_{\mu,r} f(\zeta)$. Then by Lemma 4.3, (9) and (1), we have

$$\begin{aligned}
\int_{\tilde{S}_0} |F(\zeta)|^2 d\tilde{S}_{0(\mu,\lambda,r)}(\zeta) &= \int_{\tilde{S}_0} |\mathcal{F}^{\Delta}_{\mu,r} f(\zeta)|^2 d\tilde{S}_{0(\mu,\lambda,r)}(\zeta) \\
&= \sum_{k=0}^{\infty} \frac{(N(k,n)k!)^2 \gamma_{k,n} 2^k L_{k,\lambda,r}}{(L_{k,\mu,r})^2} \left(\frac{L_{k,\mu,r}}{N(k,n)k!\gamma_{k,n}} \right)^2 ||f_k||^2_{\tilde{S}_{0,1}} \\
&= \sum_{k=0}^{\infty} \frac{2^k L_{k,\lambda,r}}{\gamma_{k,n}} ||f_k||^2_{\tilde{S}_{0,1}} = \sum_{k=0}^{\infty} \frac{L_{k,\lambda,r}}{L_{k,0,1}} ||f_k||^2_{\tilde{S}_{0,1}} \\
&= \sum_{k=0}^{\infty} L_{k,\lambda,r} ||f_k||^2_{\tilde{S}_1} = \sum_{k=0}^{\infty} ||f_k||^2_{\tilde{S}_{\lambda,r}} \\
&= ||f||^2_{\tilde{S}_{\lambda,r}} = \int_{\tilde{S}_{\lambda,r}} |f(z)|^2 \dot{z} < \infty .
\end{aligned}$$

□

Combining Theorem 6.1 with Theorems 3.2 and 4.4 , we obtain

Proposition 6.2. *Let $|\lambda| \leq r$ and $|\mu| \leq r$. Then we have*

$$\mathrm{Exp}(\tilde{S}_0; [r]) \subset \mathcal{E}^2(\tilde{S}_0; \mu, \lambda, r) \subset \mathrm{Exp}(\tilde{S}_0; (r)).$$

Since $\mathcal{E}^2(\tilde{S}_0; \mu, \lambda, r) \subset \mathrm{Exp}(\tilde{S}_0; (r))$, the inverse mapping of (15) is given by (7) and every $F \in \mathcal{E}^2(\tilde{S}_0; \mu, \lambda, r)$ is represented by the integral formula (8). But we also have formulas corresponding to (7) and (8) in terms of the function $e^{\mu}_{\lambda}(z, \zeta)$ and the measure $d\tilde{S}_{0(\mu,\lambda,r)}$:

Proposition 6.3. *Let $F \in \mathrm{Exp}(\tilde{S}_0; (r))$. Then we have*

$$\mathcal{M}_{\mu,r} F(z) = \int_{\tilde{S}_0} e^{\mu}_{\lambda}(z, \zeta) \overline{F(\zeta)} d\tilde{S}_{0(\lambda,\mu,r)}(\zeta).$$

Proof. By (7), (6), (9), (11) and (13), the statement easily follows. □

Theorem 6.4. *The function*

(i) $$E_{\mu,\lambda,r}(\zeta,\xi) = (e^{\mu}_{\lambda}(z,\zeta), e^{\mu}_{\lambda}(z,\xi))_{\tilde{S}_{\lambda,r}}$$

is a reproducing kernel for $\mathcal{E}^2(\tilde{S}_0;\mu,\lambda,r)$; *that is, for* $F \in \mathcal{E}^2(\tilde{S}_0;\mu,\lambda,r)$ *we have the following integral representation:*

$$F(\xi) = \int_{\tilde{S}_0} F(\zeta)\overline{E_{\mu,\lambda,r}(\zeta,\xi)}d\tilde{S}_{0(\lambda,\mu,r)}(\zeta).$$

We have

(ii) $$E_{\mu,\lambda,r}(\zeta,\xi) = \sum_{k=0}^{\infty} \frac{(L_{k,\mu,r})^2}{N(k,n)(k!\gamma_{k,n})^2 L_{k,\lambda,r}} \tilde{P}_{k,n}(\zeta,\overline{\xi}).$$

The Poisson kernel $K_{\lambda,r}(z,w)$ *can be given as follows:*

(iii) $$K_{\lambda,r}(z,w) = \int_{\tilde{S}_0} e^{\mu}_{\lambda}(z,\zeta)\overline{e^{\mu}_{\lambda}(w,\zeta)}d\tilde{S}_{0(\lambda,\mu,r)}(\zeta).$$

Proof. If we write down the formula $F(\xi) = \mathcal{F}^{\Delta}_{\mu,r} \circ \mathcal{M}_{\mu,r}F(\zeta)$ using the function $e^{\mu}_{\lambda}(z,\zeta)$ and (14), we get the reproducing formula (ii):

$$\begin{aligned} E_{\mu,\lambda,r}(\zeta,\xi) &= (e^{\mu}_{\lambda}(z,\zeta), e^{\mu}_{\lambda}(z,\xi))_{\tilde{S}_{\lambda,r}} \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{k!\gamma_{k,n}}\frac{L_{k,\mu,r}}{L_{k,\lambda,r}}\right)^2 \left(\tilde{P}_{k,n}(z,\zeta), \tilde{P}_{k,n}(z,\xi)\right)_{\tilde{S}_{\lambda,r}} \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{k!\gamma_{k,n}}\frac{L_{k,\mu,r}}{L_{k,\lambda,r}}\right)^2 \frac{L_{k,\lambda,r}}{N(k,n)} \tilde{P}_{k,n}(\zeta,\overline{\xi}) \\ &= \sum_{k=0}^{\infty} \frac{(L_{k,\mu,r})^2}{N(k,n)(k!\gamma_{k,n})^2 L_{k,\lambda,r}} \tilde{P}_{k,n}(\zeta,\overline{\xi}). \end{aligned}$$

(iii)

$$\begin{aligned} &\int_{\tilde{S}_0} e^{\mu}_{\lambda}(z,\zeta)\overline{e^{\mu}_{\lambda}(w,\zeta)}d\tilde{S}_{0(\lambda,\mu,r)}(\zeta) \\ &= \sum_{k=0}^{\infty} \frac{(N(k,n)k!)^2\gamma_{k,n}2^k L_{k,\lambda,r}}{(L_{k,\mu,r})^2} \left(\frac{L_{k,\mu,r}}{k!\gamma_{k,n}L_{k,\lambda,r}}\right)^2 \left(\tilde{P}_{k,n}(z,\zeta), \tilde{P}_{k,n}(w,\zeta)\right)_{\tilde{S}_{0,1}} \\ &= \sum_{k=0}^{\infty} \frac{N(k,n)^2 2^k}{\gamma_{k,n}L_{k,\lambda,r}} \frac{L_{k,0,1}}{N(k,n)} \tilde{P}_{k,n}(z,\overline{w}) = \sum_{k=0}^{\infty} \frac{N(k,n)}{L_{k,\lambda,r}} \tilde{P}_{k,n}(z,\overline{w} = K_{\lambda,r}(z,w). \end{aligned}$$

□

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Faculty of Culture and Education
Saga University, Saga 840-8502, Japan

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Department of Mathematics
International Christian University
3-10-2 Osawa, Mitaka-shi, Tokyo 181-8585, Japan