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On uniform approximations by polyanalytic polynomials on compact subsets of the plane

Abstract. Let L be a fixed homogeneous elliptic operator in $\mathbb{R}^2$ and let P_L be the space of all polynomials in $\mathbb{R}^2$ (so called L-polynomials) which are annihilated by L. For any set $E \subset \mathbb{R}^2$ let $L(E)$ denote the set of all functions u which satisfy $Lu = 0$ in some open set $E' \supseteq E$ depending on u.

In this paper the author deals with the following Problem A: Under what conditions on a compact set $X \subset \mathbb{R}^2$ each function $f \in C(X)$ with $Lf = 0$ on the interior X° of X can be uniformly approximated on X by L-polynomials.

1. Introduction. Let $C(X)$ be the space of all continuous complex-valued functions on a compact set X with the uniform norm. The following theorem of S. N. Mergelyan [7] is well known.

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Theorem M. *Let X be a compact set in $\mathbb{C}$. Then each function $f \in C(X)$, holomorphic on X° (the interior of X), can be uniformly approximated on X by polynomials of a complex variable if and only if $\mathbb{C} \setminus X$ is connected.*

We are interested in analogous results for approximation by bianalytic polynomials and by polynomial solutions of other elliptic equations.

Let L be any fixed homogeneous elliptic operator in $\mathbb{R}^2$ of order $n \geq 2$ with constant complex coefficients. In this paper our special attention will be given to the case $L = L_n = \overline{\partial}^n = \partial^n/\partial\overline{z}^n$ (recall that $\overline{\partial} = \partial/\partial\overline{z}$ is the Cauchy-Riemann operator) and to the case $L = L_n^\lambda = (\partial/\partial x_1 - \lambda\partial/\partial x_2)^n$ (where $\lambda \in \mathbb{C} \setminus \mathbb{R}$). In other words we will consider the approximation by n-analytic polynomials (for $n \geq 2$) and by solutions of the elliptic equations (of order $n \geq 2$) with constant complex coefficients and equal characteristic roots.

Let P_L be the space of all polynomials in $\mathbb{R}^2$ which are annihilated by L (these polynomials are called L-polynomials).

For any set $E \subset \mathbb{R}^2$ let us denote by $L(E)$ the set of all functions u, which are defined and satisfy the equation $Lu = 0$ in some open set $E' \supseteq E$, (depending on u). Denote by $P_L(X)$ the closure in $C(X)$ of the set $\{p|_X \mid p \in P_L\}$.

We will be interested in the approximation problem for "*classes of functions*", which can be formulated as follows.

Problem A. *Under what conditions on a compact set $X \subset \mathbb{R}^2$ each function $f \in C(X)$ with $Lf = 0$ on the interior X° of X (a necessary approximability condition) can be uniformly on X (with arbitrary accuracy) approximated by L-polynomials (or, in another words, when $P_L(X) = C(X) \cap L(X^\circ)$)?*

Recall that a function f is said to be n-analytic in a domain $D \subset \mathbb{C}$ ($n \geq 1$) if $L_n f = 0$ everywhere in D in the classical sense. Note that if D is a domain in $\mathbb{C}$ then any function f which is n-analytic in D has the form $f(z) = \sum_{k=0}^{n-1} \overline{z}^k f_k(z)$, where f_k ($k = 0, \ldots, n-1$) are analytic in D.

Respectively, a polynomial p is said to be n-analytic if $L_n p = 0$ everywhere in $\mathbb{C}$. It is clear that any n-analytic polynomial p has the form $p(z) = \sum_{k=0}^{n-1} \overline{z}^k p_k(z)$, where p_k ($k = 0, \ldots, n-1$) are polynomials of a complex variable.

In what follows, 2-analytic functions and 2-analytic polynomials are called bianalytic functions and bianalytic polynomials, respectively.

Set $P_n = P_{L_n}$ and $P_n(X) = P_{L_n}(X)$.

In the case $n = 1$ the approximation problem stated above was completely solved by S.N.Mergelyan in pure topological terms (see [7] and Theorem M above).

The sufficient condition for uniform approximability of functions by n-analytic polynomials ($n \geq 2$), similar to the assumption of the Mergelyan theorem, was obtained by J. J. Carmona [1]:

Theorem C. *If the complement $\mathbb{C} \setminus X$ of the compact set $X \subset \mathbb{C}$ is connected then $P_n(X) = C(X) \cap L_n(X^\circ)$ for any integer $n \geq 2$.*

An approximability criterion in Problem A for $L = L_n$ ($n \geq 2$) and for closed rectifiable Jordan curves X was obtained in [3]. This criterion was formulated in terms of special analytic characteristic of the curve under consideration. So, the sufficiency condition in Theorem C is not necessary and there are no topological criteria for n-analytic ($n \geq 2$) polynomial approximations.

In what follows we will use the following notation. A *contour* means a closed Jordan curve. If Γ is a contour then $D(\Gamma)$ denotes a domain bounded by Γ and not containing ∞. If it is clear from the context then we will write D instead of $D(\Gamma)$.

For the readers convenience we recall the main definition and state the main result of the paper [3].

Definition F1. A rectifiable contour Γ is said to be a Nevanlinna contour if $\bar{\zeta} = G(\zeta)/F(\zeta)$ on Γ, where G and F ($F \not\equiv 0$) are bounded analytic functions in $D(\Gamma)$ and equality is understood in the sense of (angular) boundary values almost everywhere with respect to the length differential on Γ.

Note that by the boundary uniqueness theorem [5, Chapt. X, §2, Th. 3] the function G/F in $D(\Gamma)$ is uniquely determined.

Simple calculations show that the circle is a Nevanlinna contour but the boundary of an arbitrary polygon and the boundary of an arbitrary ellipse which is not a circle are not Nevanlinna contours.

Theorem F2. *Let Γ be a rectifiable contour in $\mathbb{C}$ and let $n \geq 2$ be an integer. The following conditions are equivalent:*
(a) $P_n(\Gamma) \neq C(\Gamma)$,
(b) Γ *is a Nevanlinna contour.*

In this paper we shall consider the mentioned approximation problem for n-analytic functions ($n > 1$) in a more general setting. Our first main result (Theorem 2.3 in §2) is a generalization of Theorem F2 to arbitrary (not necessary rectifiable) contours in $\mathbb{C}$. Problem A for $L = L_n$ ($n \geq 2$) and for compact sets X of special type, which are not contours is considered in §3 (see Proposition 3.2). In particular, Proposition 3.2 points out essential differences between the cases of uniform approximation by bianalytic and harmonic polynomials.

Problem A is closely related to the Dirichlet problem for the operator L. In §4 this relation is studied for $L = L_n$ $(n > 1)$. It is proved (see Proposition 4.1), that if Γ is a contour which contains some analytic arc, then the classical Dirichlet problem for bianalytic functions is (in general) unsolvable in the domain $D(\Gamma)$. It follows from Theorem 4.2 that there exists a contour Γ in $\mathbb{C}$ such that Problem A with $X = \Gamma$ and $L = L_2$ and the Dirichlet problem in the domain $D(\Gamma)$ for bianalytic functions are not equivalent. One unsolved problem related to the Dirichlet problem for bianalytic functions is stated in §4.

In §5 all mentioned results are generalized to elliptic operators with equal characteristic roots.

In what follows the signs □ and ■ denote the beginning and the end of the proof, resp.

2. Uniform approximations by n-analytic polynomials on contours in $\mathbb{C}$. Let Γ be a contour (not necessary rectifiable) in $\mathbb{C}$ and let $D = D(\Gamma)$ be a domain, bounded by Γ and not containing ∞. Denote by $B = \{|w| < 1\}$ the unit disk and denote by $\gamma = \{\eta \in \mathbb{C} : |\eta| = 1\}$ the unit circle.

Fix a conformal map h of the unit disk B onto D which is extended to a homeomorphism of $\overline{B}$ onto $\overline{D}$ by Carathéodory's theorem and let $\tau = h^{-1}$ be an inverse mapping.

In what follows all measures are finite, complex and Borel.

Let ν be a measure on γ. Define a measure $h(\nu)$ on Γ by setting $h(\nu)(S) = \nu(\tau(S))$ where S is a Borel subset of Γ. For the measure μ on Γ we define the measure $\tau(\mu)$ on γ by analogy. Put $\mathfrak{L} = h(\sigma)$, where $d\sigma(\eta) = d\eta$ on γ.

For the case of non-rectifiable contours we need the following modification of the notion of a Nevanlinna contour.

Definition 2.1. A contour Γ is said to be a Nevanlinna-type contour if $\overline{\zeta} = u(\tau(\zeta))/v(\tau(\zeta))$ almost everywhere on Γ with respect to the measure $\mathfrak{L}$, where $u(\eta)$ and $v(\eta)$ $(v \not\equiv 0)$ are boundary values of some bounded functions $u(w)$ and $v(w)$ holomorphic in the unit disk.

It is easy to see that the definition of a Nevanlinna-type contour is independent of the map h and so the notion of a Nevanlinna-type contour is well defined.

Note that for rectifiable contours the notions of a Nevanlinna contour and a Nevanlinna-type contour are the same. The author doesn't know any example of the Nevanlinna-type contour Γ such that Γ is not a Nevanlinna contour. However, the following statement holds:

Remark 2.2. If Γ is an arbitrary closed Jordan curve, such that Γ contains two analytically independent analytic arcs then Γ is not a Nevanlinna-type contour (see the proof of the Corollary 5.5).

The main result of the present paper is the following theorem.

Theorem 2.3. *Let Γ be a contour in $\mathbb{C}$ and let $n \geq 2$ be an integer. The following conditions are equivalent:*
(a) $P_n(\Gamma) \neq C(\Gamma)$,
(b) Γ *is a Nevanlinna-type contour.*

Note that if Γ is an arbitrary closed Jordan curve in $\mathbb{R}^2$, then the conditions for identity of the spaces $P_n(\Gamma)$ and $C(\Gamma)$ are the same for all $n \geq 2$.

Before proving the Theorem 2.3 we will formulate and prove some technical propositions. Recall that γ is the unit circle.

Proposition 2.4. *Let μ be a measure on Γ. Then μ is orthogonal to the system $\{z^k\}_{k=0}^{\infty}$ if and only if the measure $\nu = \tau(\mu)$ on γ is orthogonal to the system $\{w^k\}_{k=0}^{\infty}$.*

□ Suppose that μ is orthogonal to the system $\{z^k\}_{k=0}^{\infty}$ on Γ.

By the Walsh theorem [10, Chapter 2, §2.4] $P_1(\overline{D}) = C(\overline{D}) \cap L_1(D)$. Hence, since $\tau^k \in C(\overline{D}) \cap L_1(D)$, we have

$$\int_{\gamma} \eta^k d\nu(\eta) = \int_{\Gamma} [\tau(\zeta)]^k d\mu(\zeta) = 0$$

for any $k = 0, 1, \ldots$. Hence, ν is orthogonal to the system $\{w^k\}_{k=0}^{\infty}$ on γ.

Conversely, suppose that ν is orthogonal to the system $\{w^k\}_{k=0}^{\infty}$ on γ. But $\mu = h(\nu)$ and the orthogonality of μ to the system $\{z^k\}_{k=0}^{\infty}$ on Γ may be verified by the same way. ■

Let ν be a measure on γ such that ν is orthogonal to the system $\{w^k\}_{k=0}^{\infty}$. By the F.Riesz-M.Riesz theorem [4, Chapt. 2, §7] it follows that ν is absolutely continuous with respect to the measure σ on γ, that is, there exists such a σ-integrable function $\varphi(\cdot)$ on γ that $d\nu(\eta) = \varphi(\eta)d\eta$. It is well known that $\varphi(\eta)$ are angular boundary values of a function $\varphi(w)$ that belongs to the class H_1 (recall that H_1 is the Hardy class in the unit disk). We have

Proposition 2.5. *Suppose that the measure ν on γ is orthogonal to the system $\{w^k\}_{k=0}^{\infty}$ and let $\mu = h(\nu)$. Then $d\mu(\zeta) = \varphi(\tau(\zeta))d\mathfrak{L}(\zeta)$, where $\varphi = d\nu/d\sigma$.*

□ Suppose that ν is orthogonal to the system $\{w^k\}_{k=0}^{\infty}$ on γ and let $\varphi(\cdot)$ be the density of ν with respect to the measure σ, that is $\nu(S_1) = \int_{S_1} \varphi(\eta)d\eta$.

Let $S \subset \Gamma$. Then we have

$$\mu(S) = \nu(\tau(S)) = \int_{\tau(S)} \varphi(\eta)d\eta = \int_S \varphi(\tau(\zeta))d\mathfrak{L}(\zeta)$$

by definition of the measure $\mathfrak{L}$. ■

□ **Proof of Theorem 2.3.** Let $n \geq 2$ be an integer.

(a) $\Longrightarrow$ (b) Suppose that $P_n(\Gamma) \neq C(\Gamma)$. Since $P_2(\Gamma) \subset P_n(\Gamma)$, it follows that $P_2(\Gamma) \neq C(\Gamma)$. Then there exists a non-zero measure μ_1 on Γ such that

$$\int_\Gamma \zeta^m \, d\mu_1(\zeta) = 0,$$

$$\int_\Gamma \overline{\zeta}\zeta^m \, d\mu_1(\zeta) = 0, \tag{2.1}$$

holds for all $m \in \mathbb{Z}_+ = \{0, 1, \dots\}$.

Define the measure μ_2 on Γ by setting:

$$d\mu_2(\zeta) = \overline{\zeta}\, d\mu_1(\zeta). \tag{2.2}$$

It follows from (2.1), (2.2) that $\int_\Gamma \zeta^m \, d\mu_2(\zeta) = 0$, for all $m \in \mathbb{Z}_+$. Then the measures μ_1 and μ_2 are orthogonal to all complex polynomials on Γ.

Put $\nu_s = \tau(\mu_s)$, $s = 1$ and 2. According to the Proposition 2.4 the measures ν_1 and ν_2 are orthogonal to the system $\{w^k\}_{k=0}^{\infty}$ on γ, that is $d\nu_s(\eta) = u_s(\eta)d\eta$, $s = 1, 2$. Here $u_1(\eta)$ and $u_2(\eta)$ are the boundary values of some functions $u_1(w)$ and $u_2(w)$ that belong to the class H_1.

According to Proposition 2.4 we have

$$d\mu_s(\zeta) = u_s(\tau(\zeta))d\mathfrak{L}(\zeta), \qquad s = 1, 2. \tag{2.3}$$

By (2.2) and (2.3) $d\mu_2(\zeta) = u_2(\tau(\zeta))d\mathfrak{L}(\zeta) = \overline{\zeta}d\mu_1(\zeta) = \overline{\zeta}u_1(\tau(\zeta))d\mathfrak{L}(\zeta)$, or, in other words, we have $\overline{\zeta} = u_2(\tau(\zeta))/u_1(\tau(\zeta))$ $\mathfrak{L}$-almost everywhere on Γ. According to [6, Th. 6.11] $H_1 \subset N$ (where N is the Nevanlinna class) and so we can replace the ratio $u_2(w)/u_1(w)$ by the ratio of two bounded analytic functions u and v in the unit disk $u_2(w)/u_1(w) = u(w)/v(w)$. Hence the contour Γ is a Nevanlinna-type contour.

(b) $\Longrightarrow$ (a) Let Γ be a Nevanlinna-type contour. We need to show that $P_n(\Gamma) \neq C(\Gamma)$). Define a measure μ on Γ by setting

$$d\mu(\zeta) = v^{n-1}(\tau(\zeta))d\mathfrak{L}(\zeta).$$

Then, according to Cauchy theorem, we have by definition of $\mathfrak{L}$:

$$\begin{aligned}\int_\Gamma \overline{\zeta}^k \zeta^m \, d\mu(\zeta) &= \int_\Gamma \overline{\zeta}^k \zeta^m v^{n-1}(\tau(\zeta))\, d\mathfrak{L}(\zeta) \\ &= \int_\Gamma u^k(\tau(\zeta)) v^{n-k-1}(\tau(\zeta))\zeta^m \, d\mathfrak{L}(\zeta) \\ &= \int_\gamma u^k(\eta) v^{n-k-1}(\eta) h^m(\eta)\, d\eta = 0\end{aligned}$$

for any $m \in \mathbb{Z}_+$ and for integer $k < n$. It is clear, that $\mu \not\equiv 0$. Theorem 2.3 is proved. ■

3. Uniform approximation by n-analytic polynomials on special compact sets in $\mathbb{C}$. Recall, that a contour Γ is said to be an *analytic contour* if it is an image of the circle under a map which is conformal in a neighborhood of this circle. Respectively, an *analytic arc* is a conformal image of a segment.

It is well known that if Γ is an analytic contour then there exists a function S holomorphic in some neighborhood U of Γ such that $\Gamma = \{z \in U : \overline{z} = S(z)\}$. Respectively, if γ is an analytic arc then there exists an analytic element (U, S) such that S is holomorphic in the neighborhood U of γ and $\overline{\zeta} = S(\zeta)$ on γ. This function S is called the *Schwarz function* of the contour Γ (or of the arc γ).

Let γ_1 and γ_2 be two analytic arcs. Denote by (U_1, S_1) and (U_2, S_2) the corresponding analytic elements. We say, that γ_1 and γ_2 are *analytically dependent* if the analytic elements (U_1, S_1) and (U_2, S_2) are analytic continuations of each other. Otherwise, we say that γ_1 and γ_2 are *analytically independent*.

We will use the following result of Davis (see [2, Chapter 14]).

Theorem D. *Let D be a Jordan domain whose boundary ∂D is analytic and has the Schwarz function $S(z)$. Assume that $0 \in D$ and that $z = h(w)$ ($h(0) = 0$) maps B conformally onto D. Then $S(z)$ is meromorphic in D if and only if $h(w)$ is a rational function of w.*

In other words, an analytic contour Γ is a Nevanlinna contour if and only if the conformal map of the unit disk onto the domain $D(\Gamma)$ is rational.

We need the following

Lemma 3.1. *Let Γ be a contour in $\mathbb{C}$. Then in any neighborhood of Γ there exists an analytic Nevanlinna contour.*

□ Let $D = D(\Gamma)$ and let h be a conformal map of the unit disk B onto D. For $\delta \in (0,1)$ we denote by B_δ the disk $\{|w| < \delta\}$ and set $h_\delta = h|_{B_\delta}$.

By Theorem M for any $\varepsilon > 0$ there exists a polynomial p_ε^δ of a complex variable such that $\|h_\delta - p_\varepsilon^\delta\|_{\overline{B_\delta}} < \varepsilon$.

Let V be a neighborhood of Γ and $U = V \cap D$. Then there exists such $\delta = \delta(U)$ and $\varepsilon = \varepsilon(U)$ that $\Gamma_\varepsilon = p_\varepsilon^\delta(\partial B_\delta)$ is the contour in U. By construction Γ_ε is an analytic contour and Γ_ε is a Nevanlinna contour by Theorem D. ■

Now we are going to prove the following conditions for uniform approximability of functions by n-analytic polynomials.

Proposition 3.2. *Let $n \geq 2$ be an integer.*

1. *Let Γ_1 and Γ_2 be two contours in $\mathbb{C}$ such that $\Gamma_2 \subset D(\Gamma_1)$. Set $X = \overline{D(\Gamma_1)} \setminus D(\Gamma_2)$. Then $P_n(X) \neq C(X) \cap L_n(X^\circ)$.*
2. *Let X be a compact set in $\mathbb{C}$ such that $X = \Gamma \cup \{a_1, \ldots, a_p\}$, where Γ is not a Nevanlinna contour and $a_1, \ldots, a_p$ are the points, such that $a_j \in D(\Gamma)$ ($j = 1, \ldots, p$). Then $P_n(X) = C(X)$.*

□ **1.** By Lemma 3.1 there exists an analytic Nevanlinna contour $\Gamma \subset X^\circ$. Let $S(\cdot)$ be a Schwarz function of Γ. Then S may be extended to a meromorphic function in $D(\Gamma)$. Let $a_1, \ldots, a_k$ be the poles of $S(\cdot)$ in $D(\Gamma)$. Put $F(z) = \prod_{j=1}^k (z - a_j)$ and $G(z) = S(z)F(z)$. Take a point $b \in D(\Gamma_2) \setminus \{a_1, \ldots, a_k\}$ such that $G(b) \neq 0$. Suppose that the function $f(z) = \overline{z}/(z-b)|_\Gamma$ may be uniformly approximated on X by a sequence of n-analytic polynomials. Then f may be uniformly on Γ approximated by the sequence of the n-analytic polynomials $\left\{q_j(z) = \sum_{k=0}^{n-1} \overline{z}^k p_k(z)\right\}$. Then the function $g(z) = G(z)F^{n-2}(z)/(z-b)$ may be uniformly approximated on Γ by the sequence of functions $\left\{g_n(z) = \sum_{k=0}^{n-1} G^k(z) F^{n-k-1}(z) p_k(z)\right\}$ holomorphic in $D(\Gamma)$. But g has a pole at the point b. This contradicts the maximum principle for holomorphic functions. Hence $P_n(X) \neq C(X) \cap L_n(X^\circ)$.

2. Assume the converse: $P_n(X) \neq C(X)$. Then $P_2(X) \neq C(X)$ and there exists a measure μ on X such that $\int_X z^k\, d\mu(z) = 0$ and $\int_X \overline{z} z^k\, d\mu(z) = 0$, where $k = 0, 1, 2, \ldots$. Define the measures μ_1 and μ_2 on Γ by:

$$d\mu_1(\zeta) = \left[\prod_{j=1}^{p}(a_j - \zeta)\right] d\mu(\zeta), \qquad d\mu_2(\zeta) = \left[(\overline{a_1} - \overline{\zeta}) \prod_{j=2}^{p}(a_j - \zeta)\right] d\mu(\zeta).$$

It is not difficult to verify that the measures μ_1 and μ_2 are orthogonal to all complex polynomials: $\int_\Gamma z^k\, d\mu_s(z) = 0$ (as $s = 1, 2$ and $k = 0, 1, 2, \ldots$). Then, we have $d\mu_s(\zeta) = f_s(\zeta) d\zeta$, $s = 1, 2$, where the functions $f_s(\zeta)$ (as $s = 1, 2$) are angular boundary values of $f_s(z)$ (as $s = 1, 2$) that belong to the class E_1 (with respect to $D(\Gamma)$).

Simple calculations show that

$$(\overline{a_1} - \overline{\zeta}) f_1(\zeta) \prod_{j=2}^{p}(a_j - \zeta) = f_2(\zeta) \prod_{j=1}^{p}(a_j - \zeta)$$

almost everywhere on Γ and according to this equality it is not difficult to show that the function $\overline{\zeta}$ on Γ can be represented as a quotient of functions that belong to the class E_1. It is well known that each function $f \in E_1$ in D may be represented as a quotient of two bounded functions holomorphic in D. Thus, the contour Γ is a Nevanlinna contour. This contradiction proves part 2. ∎

It is worth comparing the conditions for uniform approximability of functions by bianalytic and by harmonic polynomials.

The following Walsh–Lebesgue theorem (see [4, Chapt. II] and [9, p.503]) is a criterion of uniform approximability of functions by harmonic polynomials. We formulate it in such a form as in [8]. For a compact set X denote by $\widehat{X}$ the union of X and all bounded components of its complement $\mathbb{C} \setminus X$.

Theorem WL. *Let X be a compact set in $\mathbb{C}$. Then*

$$P_\Delta(X) = C(X) \cap \Delta(X^\circ)$$

if and only if $\partial X = \partial \widehat{X}$ (here Δ is the Laplace operator in $\mathbb{C}$).

So, Proposition 3.2 points out the essential difference between the cases of uniform approximation by bianalytic and harmonic polynomials. In fact, if the compact set X_1 satisfies the conditions of part 1 of Proposition 3.2, then $P_n(X_1) \neq C(X_1) \cap L_n(X_1^\circ)$ and if the compact set X_2 satisfies the conditions of part 2 of Proposition 3.2, then $P_n(X_2) = C(X_2)$ (here $X_2^\circ = \emptyset$). Clearly, $\partial X_s \neq \partial \widehat{X_s}$ for $s = 1$ and $s = 2$.

4. Relations with the Dirichlet problem. In this section we consider the Dirichlet problem for n-analytic functions (solvability and uniqueness) and discuss its relations with the uniform approximation problem stated above.

For a contour Γ in $\mathbb{R}^2$ and for the elliptic operator L of order $n \geq 2$ with constant complex coefficients we set

$$S_L(\Gamma) = \left\{ f \in C(\Gamma) \,\middle|\, \exists F \in C(\overline{D(\Gamma)}) \cap L(D(\Gamma)) \quad \text{such that} \quad F\Big|_\Gamma = f \right\},$$

and $S_{L_n}(\Gamma) = S_n(\Gamma)$. We will be interested in the following problem: Under what conditions on a contour Γ is $P_n(\Gamma)$ equal to $S_n(\Gamma)$?

It follows from Theorem C that $S_n(\Gamma) \subset P_n(\Gamma)$ for any integer n and it is easily seen from the maximum principle for holomorphic functions and for harmonic functions that $P_1(\Gamma) = S_1(\Gamma)$ and $P_\Delta(\Gamma) = S_\Delta(\Gamma)$, respectively.

Note that the maximum principle for n-analytic $(n > 1)$ functions is not true. In fact, the function $f(z) = 1 - z\overline{z}$ is equal to zero on the unit circle, but $|f| > 0$ in the interior of the unit disk.

Most likely, by this reason, the classes $P_n(\Gamma)$ and $S_n(\Gamma)$ are not equivalent for $n > 1$. Without loss of generality we consider only the case $n = 2$. We show, that $C(\Gamma) \neq S_2(\Gamma)$ even under very simple conditions for the contour Γ. The case of an arbitrary integer $n > 2$ can be considered analogously.

Proposition 4.1. *Let Γ be a contour containing an analytic arc γ. Then $C(\Gamma) \neq S_2(\Gamma)$.*

□ We construct a neighborhood U of the arc γ and a function φ (analytic in U) such that $\overline{\zeta} = \varphi(\zeta)$ on γ. Consider the function $f = 1/(z-a)|_\Gamma$, $a \in D \cap U$ and show that $f \notin S_2(\Gamma)$.

Suppose that $f \in S_2(\Gamma)$ and let F be a desired continuation. Then $F(z) = F_0(z) + \overline{z}F_1(z)$ where F_0 and F_1 are analytic in $D = D(\Gamma)$. According to [1, Lemma 3] we have

$$|F_1(z)| \leq A\frac{\omega(F\,,\,\mathrm{dist}(z\,,\Gamma))}{\mathrm{dist}(z\,,\Gamma)}, \quad \text{for} \quad z \in D. \tag{4.4}$$

Here $\omega(F, \delta)$ is the module of continuity of functions F on $\overline{D}$ and A is an absolute constant.

Consider the function $F_\Gamma(z) = F_0(z) + \varphi(z)F_1(z)$ which is defined in $D \cap U$ (recall that φ is analytic in $D \cap U$). According to (4.4) we have

$$F(z) - F_\Gamma(z) = F_1(z)(\overline{z} - \varphi(z)) \rightrightarrows 0$$

as $z \to \Gamma \cap U$, $z \in D \cap U$, (here $\rightrightarrows$ denotes the uniform convergence). Consequently, F_Γ is continuous in $\overline{D} \cap U$ and equal to f on Γ. According to the boundary uniqueness theorem [5, Theorem X.2.3] we have $F_\Gamma = f$ everywhere in $D \cap U$. Note that f has a pole at the point a and there are no singularities of F_Γ in $D \cap U$. This contradiction shows that $f \notin S_2(\Gamma)$. ■

The following result follows from Remark 2.2 and Proposition 4.1.

Theorem 4.2. *Let Γ be a contour containing two analytically independent analytic arcs. Then $P_2(\Gamma) = C(\Gamma) \neq S_2(\Gamma)$.*

The following problem remains open: Is it true that $C(\Gamma) \neq S_2(\Gamma)$ for any contour $\Gamma \subset \mathbb{C}$ (or even for a boundary Γ of an arbitrary Carathéodory domain)?

5. Generalization to elliptic operators with equal characteristic roots.

Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $L_n^\lambda u = (\partial/\partial x_1 - \lambda\partial/\partial x_2)^n u$. Then the operator L_n^λ is elliptic. Put $P_n^\lambda = P_{L_n^\lambda}$ and $P_n^\lambda(X) = P_{L_n^\lambda}(X)$.

Let $z = x_1 + ix_2$ and $x = (x_1, x_2)$. Set

$$\partial_1 = \partial/\partial x_1 - \lambda\partial/\partial x_2, \qquad \partial_2 = \partial/\partial x_1 + \lambda\partial/\partial x_2,$$

so that $L_n^\lambda u = \partial_1^n u$. Define the "new variables"

$$z_1 = \frac{1}{2}\left(x_1 - \frac{1}{\lambda}x_2\right), \quad z_2 = \frac{1}{2}\left(x_1 + \frac{1}{\lambda}x_2\right).$$

Then the following orthogonality property holds: $\partial_s z_t = \delta_{st}$ as $s, t = 1, 2$. For example, if $\lambda = -i$ then we have $\partial_1 = 2\overline{\partial}$ and $z_1 = \overline{z}/2$, $z_2 = z/2$.

Define the transformation $T_\lambda : \mathbb{C} \to \mathbb{C}$ by setting $T_\lambda z = z_2$. Then $T_\lambda \overline{z} = z_1$.

Proposition 5.1. *Let D be a domain in $\mathbb{R}^2_{(x_1,x_2)}$, $f \in L_n^\lambda(D)$ and $g(y) = f(T_\lambda^{-1} y)$. Then $\overline{\partial}^n g = 0$ in $T_\lambda D \subset \mathbb{R}^2_{(y_1,y_2)}$.*

□ Let $\lambda = a + ib$ and $1/\lambda = c + id$. Take a point $y \in T_\lambda D$ and set $x = T_\lambda^{-1} y$. One has

$$\overline{\partial} g(y) = \frac{1}{2}\left(\frac{\partial g}{\partial y_1} + i\frac{\partial g}{\partial y_2}\right)(y) = \frac{1}{d}\left((d - ic)\frac{\partial f}{\partial x_1} + i\frac{\partial f}{\partial x_2}\right)(x)$$

$$= -\frac{i}{d}\left(\frac{1}{\lambda}\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right)(x) = \frac{i\overline{\lambda}}{b}\partial_1 f(x).$$

By induction we can prove that $\overline{\partial}^n g(y) = [(i\overline{\lambda})^n / b^n] L_n^\lambda f(x)$ which ends the proof. ■

Corollary 5.2.

(1) *A polynomial solution p of the equation $L_n^\lambda u = 0$ has the form $p(z) = \sum_{k=0}^{n-1} z_1^k p_k(z_2)$, where $p_0(\cdot), p_1(\cdot), \ldots, p_{n-1}(\cdot)$ are complex polynomials.*

(2) *Let X be a compact set in $\mathbb{R}^2$ and let $Y = T_\lambda X$. Then the image of the class $P_n^\lambda(X)$ under the transformation T_λ (namely under the transformation $g(y) = f(T_\lambda^{-1} y))$ is the class $P_n(Y)$ and the image of the class $L_n^\lambda(X)$ is the class $L_n(Y)$.*

It is clear, that if X is a compact set in $\mathbb{R}^2_{(x_1,x_2)}$ then $Y = T_\lambda X$ is a compact set in $\mathbb{R}^2_{(y_1,y_2)}$ and all topological properties of X and Y are the same.

Note that for the operator L_n^λ the following statements are true and immediately follow from Theorem C and Theorem 2.3, respectively, after the change of variables (see Proposition 5.1 and Corollary 5.2 for justification of the mentioned change of variables).

Corollary 5.3. *Let $X \subset \mathbb{R}^2$ be a compact set with connected complement and let $n \geq 1$ be an integer. Then $P_n^\lambda(X) = C(X) \cap L_n^\lambda(X^\circ)$.*

Corollary 5.4. *Let Γ be a contour in $\mathbb{R}^2$ and $n \geq 2$ be an integer. Then $P_n^\lambda(\Gamma) \neq C(\Gamma)$ if and only if the contour $T_\lambda\Gamma$ is a Nevanlinna-type contour.*

The following corollary of Theorem 5.4 is useful for the analysis of the concrete examples.

Corollary 5.5. *Let Γ be a contour including two analytically independent analytic arcs, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and let $n \geq 2$ be an integer. Then $P_n^\lambda(\Gamma) = C(\Gamma)$.*

□ It follows from Proposition 2.3 that if Γ' is a contour and Γ' contains two analytically independent analytic arcs, then $P_n(\Gamma') = C(\Gamma')$.

In fact, let us show that if $P_n(\Gamma') \neq C(\Gamma')$ then any two analytic arcs $\gamma_1 \subset \Gamma'$ and $\gamma_2 \subset \Gamma'$ are analytically dependent. Since $P_n(\Gamma') \neq C(\Gamma')$, it follows from Theorem 2.3 that Γ' is a Nevanlinna-type contour. On the other hand, according to the definition of an analytic arc we have $\zeta = \varphi_s(\zeta)$ on γ_s, where φ_s is analytic in the neighborhood U_s of γ_s ($s = 1$ and 2). Assume that $U_1 \cap D(\Gamma')$ and $U_2 \cap D(\Gamma')$ are simply connected and $U_1 \cap U_2 = \emptyset$. Compare the representation of the function $\overline{\zeta}$ on Γ', using in the definition of a Nevanlinna-type contour with the above mentioned representations of such functions on γ_1 and γ_2. Then the analytic elements (φ_1, U_1) and (φ_2, U_2) can be analytically continued into each other so that γ_1 and γ_2 are analytically dependent.

Note that the transformation T_λ maps two analytically independent analytic arcs into two analytically independent analytic arcs and apply the part 2 of Corollary 5.2. ■

The following result is a consequence of Proposition 4.1 and Corollary 5.5.

Theorem 5.6. *There exists a contour* Γ *with* $P_2^\lambda(\Gamma) = C(\Gamma) \neq S_{L_2^\lambda}(\Gamma)$.

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