ANNALES
UNIVERSITATIS MARIAE CURIE – SKŁODOWSKA
LUBLIN – POLONIA
VOL. LIII, 2 SECTIO A 1999

DAOUD BSHOUTY
WALTER HENGARTNER and M. NAGHIBI-BEIDOKHTI

P-valent harmonic mappings with finite Blaschke dilatations

ABSTRACT. In this paper we aim at studying p-valent harmonic mappings defined on a simply connected domain D in $\mathbb{C}$. In contrast to the analytic case, we show that if the number of zeros of a harmonic polynomial of degree n is not infinity, it lies between n and n^2. Another astonishing behaviour is the fact that there exists a two-valent entire harmonic mapping which is not a polynomial. However, a p-valent entire harmonic mapping whose second dilatation function a is a rational function satisfying $|a(\infty)| \neq 1$ is necessarily a harmonic polynomial. In the second part we provide necessary and sufficient conditions for the following problem: Given a simply connected Jordan domain Ω and a finite Blaschke product of degree n. Under what conditions there exists a sense-preserving continuous boundary correspondence $f*$ from the unit circle onto $\partial\Omega$ covering it p times such that its Poisson integral f is p-valent and has the second dilatation a.

1. Introduction. Let D be a domain of the complex plane $\mathbb{C}$. A complex valued function $w = f(z) = u(z) + iv(z), z = x + iy \in D$ is called a harmonic mapping on D, if its real part $u(z)$ and its imaginary part $v(z)$ are harmonic

1991 *Mathematics Subject Classification*. Primary 30C55, Secondary 31A05, 30C62, 49Q05.

Key words and phrases. Harmonic polynomials, p-valent harmonic mappings.

Supported by grants from the Promotion of Research Grant, Technion, the Wenger Grant, Technion Canadian Society, the NSERC of Canada and the FCAR of Quebec

functions on D. Using the standard notations of complex partial derivatives $f_z = (f_x - if_y)/2$ and $f_{\overline{z}} = (f_x + if_y)/2$, f is harmonic on D if, and only if

$$\Delta f = 4 f_{z\overline{z}} \equiv 0 \tag{1.1}$$

on D. From (1.1), we immediately deduce the representation

$$f(z) = h(z) + \overline{g(z)}, \tag{1.2}$$

where h and g are analytic multiforms on D. The Jacobian of f is

$$J_f = |f_z|^2 - |f_{\overline{z}}|^2 = |h'|^2 - |g'|^2.$$

Observe that h' and g' are analytic functions on D and that f is an open mapping which preserves the orientation if and only if f is not a constant and its second dilatation function

$$a(z) = g'(z)/h'(z) \tag{1.3}$$

is analytic on D and satisfies $|a(z)| < 1$ there . The following result strengthens Lemma 2.1 in [3].

Lemma 1.1. *Let $w = f(z)$ be a complex valued function in $C"(D)$ such that its image $f(D)$ is neither a linear segment nor a point. Then f is harmonic on D if, and only if either $\overline{f}$ is analytic on D or f is a solution of*

$$\overline{f_{\overline{z}}(z)} = a(z) f_z(z) \tag{1.4}$$

where $a(z)$ is a meromorphic function on D such that $a \not\equiv e^{i\alpha}$ for some real α.

Proof. If f is a harmonic mapping on D, then (1.2) implies that either $h' \equiv 0$ or else $a = g'/h'$ is meromorphic on D and f is a solution of (1.4). Suppose that $a \equiv e^{i\alpha}$ for some real α, then $g'(z) = e^{i\alpha}h'(z)$ and $g(z) = e^{i\alpha}h(z) + q$ where q is a constant. Hence,

$$f = h + e^{-i\alpha}\overline{h} + \overline{q} = 2e^{-i\alpha/2}\,\mathrm{Re}(e^{i\alpha/2}h) + \overline{q}$$

which implies that $f(D)$ is either a point or a linear segment. But this case is excluded and hence the necessity of the conditions follows.

For the sufficiency we first let $\overline{f}$ be analytic on D. Then $f(z) = \overline{g(z)}$ is harmonic. Suppose now that f is a solution of (1.4) and that a is meromorphic on D such that $|a|$ is not identically one. Differentiating (1.4) with respect to $\overline{z}$ yields

$$\overline{\Delta f} = 4\overline{(f_{\overline{z}})_z} = 4(\overline{f_{\overline{z}}})_{\overline{z}} = 4a f_{z\overline{z}} = a\Delta f.$$

Hence, $\Delta f(z) = 0$ whenever $|a(z)| \neq 1$. Since $|a|$ is not identically one, the level set $E = \{z : |a(z)| = 1\}$ consists of countably many analytic arcs. Finally, f being in $C"(D)$ implies that $\Delta f \equiv 0$ on D. This concludes the proof of the lemma. □

Remarks 1.2.

(1) It is sufficient to require in Lemma 1.1 that f belongs to the Sobolev space $W^{1,2}_{loc}$. The regularity of a then implies that f belongs to $C"$.

(2) If $|a| \equiv 1$, we cannot conclude that every solution of (1.4) is harmonic. Indeed, $f(z) = e^{i\alpha/2} z\overline{z}$ is a solution of (1.4) with $a(z) \equiv e^{i\alpha}$ that is not harmonic in D.

Definition 1.3.

(1) Let f be a function defined on D. We denote the set of zeros of f by $Z(f, D)$ and its cardinality by $NZ(f, D)$. The mapping f is called p-valent on D, if for all complex w, $NZ(f - w, D) \leq p$ and if there is a w_0 such that $NZ(f - w_0, D) = p$.

(2) A function f is called locally p-valent at $z_0 \in D$ if there is an $r_0 > 0$ such that f is p-valent on each disk $z : |z - z_0| < r,\ r < r_0$.

A. Lyzzaik [4] gave a complete description for the local valencies of a harmonic mapping.

2. p-valent harmonic polynomials. In contrast to the analytic case, there are nonconstant harmonic polynomials which do not vanish. For example $f(z) = 1 + z - \overline{z}$ is such a one. On the other hand, there are harmonic polynomials which are not p-valent for every $p > 0$. For example the mapping $f(z) = z + \overline{z}$ is such a one. Moreover, in both examples the limit $\lim_{z\to\infty} f(z)$ does not exist in $\overline{\mathbb{C}}$. We shall also show that there are harmonic mappings on $\mathbb{C}$ which are not polynomials such that $\lim_{z\to\infty} f(z) = \infty$. It then follows that such a mapping takes on each value w at most finitely many times. In other words, for each $w \in \mathbb{C}$, there is a natural number $p(w)$ such that $NZ(f - w, \mathbb{C}) = p(w)$.

Theorem 2.1. *There are harmonic mappings on $\mathbb{C}$ which are not polynomials and satisfy $\lim_{z\to\infty} f(z) = \infty$.*

Proof. Consider the closed set $F = \{z = x + iy : |x| \geq 1\} \cup \{z : x = 0\}$. Define $k(z) = z$ if $|x| \geq 1$ and $k(z) = 0$ if $x = 0$. By the theorem of Arakeljan in uniform approximation theory (see for example [2]) , there exists an entire function $G(z)$ such that $\sup_{z\in F} |G(z) - k(z)| < 1$. Since G

is bounded on the imaginary axis, we conclude that G is not a polynomial. Moreover, we have $|\operatorname{Re} G(z)| \geq |x| - 1$ on $\{z : |x| \geq 1\}$ and $|\operatorname{Re} G(z)| \geq 0 > |x| - 1$ on $\{z : |x| < 1\}$. Hence, $|\operatorname{Re} G(z)| \geq |x| - 1$ for all $z \in \mathbb{C}$. Define $f(z) = \operatorname{Re} G(z) + iy$. Then we have

$$|f(z)| \geq \max(|\operatorname{Re} G(z)|, |y|) \geq \frac{1}{2}(|\operatorname{Re} G(z)| + |y|) \geq \frac{1}{2}(|x| + |y| - 1)$$

for all $z \in \mathbb{C}$ which implies that $\lim_{z \to \infty} f(z) = \infty$. □

A p-valent entire (analytic) function is necessarily a polynomial. Let f be a harmonic p-valent mapping on $\mathbb{C}$. Is it true that f is a polynomial ? In general this is not the case as we shall see in Example 2.3. However, the answer is affirmative if $a(\infty)$ exists and $|a(\infty)| \neq 1$.

Theorem 2.2. *Let f be a harmonic p-valent mapping on C. If $a(\infty)$ exists and $|a(\infty)| \neq 1$, then f is a polynomial.*

Proof. Without loss of generality we may assume that $|a(\infty)| < 1$. Hence, $|a(z)| < 1$ in $|z| > r$ for some $r > 0$ and thus $f(z)$ is an orientation preserving harmonic mapping there. Then f admits the representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k + \sum_{k=-\infty}^{\infty} \overline{b_k z^k} + C \ln |z|.$$

Set $f_1(z) = f(1/z)$. Then $f_1(z)$ is a harmonic sense-preserving mapping on $D = \{z : 0 < z < 1/r\}$. By the similarity principle for quasiregular mappings, $f_1(z) = A \circ \chi(z)$, where χ is a homeomorphism from D onto D (keeping the origin fixed) and A is an analytic function on D. Since f is p-valent, A has a pole at $z = 0$ of some order m and by the distortion theorem for quasiconformal mappings in the unit disk we conclude that $|\chi(z)| = O(|z|^K)$ as z tends to zero for arbitrary $K > (1 + |a(\infty)|)/(1 - |a(\infty)|)$. It follows that the Fourier coefficients a_n and b_n are zero for all $n > K$. Furthermore, since f is harmonic in the plane, we conclude that $C = 0$. Therefore, the singular part of f at infinity is of the form

$$S(f, z) = \sum_{k=1}^{N} a_k z^k + \sum_{k=1}^{N} \overline{b_k z^k}$$

where $|b_N| < |a_N|$. Finally $f(z) - S(f, z)$ is a bounded harmonic function in $\mathbb{C}$ and hence a constant which concludes the proof of the theorem. □

The following example shows that Theorem 2.1 can be strenghtened: There are p-valent and light harmonic mappings on the plane $\mathbb{C}$ which are not polynomials and satisfy $\lim_{z \to \infty} f(z) = \infty$.

Example 2.3. Let $f(z)=z+\mathrm{Re}(e^z)$. Then f is a 2-valent harmonic mapping in $\mathbb{C}$ that is not a harmonic polynomial. Indeed, for each fixed y,f is a convex or a concave non-constant function of x. On the other hand, we have $\lim_{z\to\infty} f(z)=\infty$. Finally, note that $f(\mathbb{C})$ omits open subsets of $\mathbb{C}$.

We are now aiming to get explicit bounds for the number of zeros of a harmonic polynomial f satisfying $|a(\infty)|\neq 1$. We shall start with a generalized argument principle for harmonic functions.

Lemma 2.4 (Generalized argument principle for harmonic mappings). *Let $f(z)$ be a harmonic mapping defined on the closure $\overline{D}$ of a Jordan domain D. Consider the sets*

$$K_1=\{z:|a(z)|>1\}\cap\overline{D} \text{ and } K_2=\{z:|a(z)|<1\}\cap\overline{D}.$$

Fix $w\in C$ such that $Z(f-w,C)\cap(\partial K_1\cup\partial K_2)$ is empty. Then we have

$$NZ(f-w,K_2)-NZ(f-w,K_1)=\frac{1}{2\pi}\oint_{\partial D} d\arg(f-w). \tag{2.1}$$

Proof. Since $\{z:|a(z)|=1\}$ is a finite union of piecewise analytic curves, K_1 and K_2 are finite unions of Jordan domains. Let ∂K_i denote the boundary of $K_i, i=1,2$, endowed with the positive orientation on each of its components. Then we have

$$\frac{1}{2\pi}\oint_{\partial K_1} d\arg(f-w)+\frac{1}{2\pi}\oint_{\partial K_2} d\arg(f-w)=\frac{1}{2\pi}\oint_{\partial D} d\arg(f-w)$$

and (2.1) follows. □

Let now f be a polynomial of degree n satisfying $|a(\infty)|\neq 1$. Then without loss of generality we shall assume that $|a(\infty)|<1$. Fix $w\in\mathbb{C}$ such that $Z(f-w,\mathbb{C})\cap(\partial K_1\cup\partial K_2)$ is empty. As already noted $\lim_{z\to\infty} f(z)=\infty$ and therefore there exists a sufficiently large $R(w)>0$ such that $|a|<1$ on $\Delta_R=\{z:|z|>R\}$ and $N(f-w,\Delta_R)=0$. For $D=\{z:|z|\leq R\}$, (2.1) yields

Corollary 2.5. *Let f be a harmonic polynomial of degree n. Then for fixed $w\in C$ such that $Z(f-w,C)\cap(\partial K_1\cup\partial K_2)$ is empty, we have*

$$NZ(f-w,C)\geq\frac{1}{2\pi}\oint_{\partial D} d\arg(f-w)=n. \tag{2.2}$$

We are now ready to state a result about the valency of a harmonic polynomial.

Theorem 2.6. *Let f be a harmonic polynomial of degree n and assume that $|a(\infty)| \neq 1$. Then f is p-valent on $\mathbb{C}$ where $n \leq p \leq n^2$. Both bounds are best possible. Furthermore, we have $p = n$ if and only if $\overline{s}$ is analytic on $\mathbb{C}$ or f is of the form*

$$s(z) = t(z) + b\overline{t(z)}, |b| \neq 1 \tag{2.3}$$

where $t(z)$ is an analytic polynomial and b is a constant.

Proof. If $\overline{f}$ is analytic on $\mathbb{C}$ then $p = n$. Suppose hence that $a(z)$ is not identically infinity. Without loss of generality, we may assume that $|a(\infty)| < 1$. The upper bound n^2 and its sharpness were shown in [6]. Denote by K the set $K = \{z : |a(z)| \geq 1\}$. If K is empty, then by Liouville's theorem, a is a constant and hence f is an affine transformation of an analytic polynomial, i.e. of the form (2.3) so that $p = n$. Suppose now that K is not empty. By the maximum principle for analytic functions, we conclude that there is a $z_0 \in K^o$, the interior of K, such that $a(z_0) = \infty$. Put $w = s(z_0)$. If $f^{-1}(w) \cap \partial K$ is empty, then Corollary 2.5 shows that $p \geq NZ(f - w, \mathbb{C}) > n$. If not, there is a z_1 in the same component of K^o such that $Z(f - w_1, \mathbb{C}) \cap \partial K$ is not empty and the theorem is proved. □

3. p-valent harmonic mappings which preserve the orientation

3.1. Introduction

Recall, that a nonconstant function f of the class $C"(D)$ is a harmonic transformation on D which preserves the orientation if and only if it is the solution of the elliptic partial differential equation (1.4) where $a(z)$ is an analytic function on D satisfying $|a| < 1$. Such a mapping is locally quasiregular and admits the representation

$$f(z) = A \circ \psi(z) \tag{3.1}$$

where ψ is a sense-preserving homeomorphism defined on D and A is analytic on $\psi(D)$. Hence, such functions behave topologically like analytic functions. In particular, the zeros of f are isolated and the order of a zero of f coincidies with the order of the zero of A. Furthermore, the classical argument principle holds and f is locally p-valent at z_0 if and only if $f_z = h'$ has a zero of order $p - 1$ at z.

Observe that p-valent mappings need not to be sense-preserving. For example, $f(z) = z + \overline{z}^2$ is 4-valent on $D = \{z : |z| < 2\}$ and we have $|a(0)| = 0$ and $|a(1.5)| = 3$.

Definition 3.1. Let Ω be a simply connected Jordan domain of $\mathbb{C}$.

(1) A function $f^*(e^{it})$ is called a sense-preserving quasihomeomorphism from the unit circle ∂U into the boundary $\partial\Omega$ if it is a pointwise limit of sense-preserving homeomorphisms from ∂U onto $\partial\Omega$ and if its image contains at least three non-collinear points. If in addition, $f^*(\partial U) = \partial\Omega$, then we say that f^* is a sense-preserving quasihomeomorphism from ∂U onto $\partial\Omega$.

(2) A function $f^*(e^{it})$ is called a sense-preserving weak homeomorphism from the unit circle ∂U onto the boundary $\partial\Omega$ if it is a sense-preserving quasihomeomorphism which maps ∂U continuously onto $\partial\Omega$.

Since the image of a sense-preserving quasihomeomorphism from ∂U into $\partial\Omega$ contains at least three non collinear points, $f^*(\partial U)$ is the boundary of a simply connected domain Ω_1 which is contained in Ω.

In the rest of this article we shall study p-valent harmonic transformations from the unit disk U onto a simply connected Jordan domain Ω.

Definition 3.2. Let Ω be a simply connected Jordan domain of $\mathbb{C}$ and let ϕ be a conformal mapping from the unit disk U onto D.

(1) A function $f^*(e^{it})$ is a p-valent sense-preserving local homeomorphism from the unit circle ∂U onto the boundary $\partial\Omega$ if

$$\theta(t) = \phi^{-1} \circ f^*(e^{it})$$

is a continuous strictly increasing function from $[0, 2\pi]$ onto $[0, 2p\pi]$.

(2) A function $f^*(e^{it})$ is called a p-valent sense-preserving local quasihomeomorphism from the unit circle ∂U into $\partial\Omega$ if it is a pointwise limit of p-valent sense-preserving local homeomorphism from ∂U onto $\partial\Omega$ and if its image is the boundary of a subdomain Ω_1 of Ω. If, in addition, $f^*(e^{it})$ is a continuous map from ∂U onto $\partial\Omega$ then we call $f^*(e^{it})$ a p-valent sense-preserving local weak homeomorphism from ∂U onto $\partial\Omega$

The solution of the Dirichlet problem for a p-valent sense-preserving local homeomorphism from ∂U onto $\partial\Omega$ need not be p-valent. One further condition must be enforced.

Definition 3.3. Let Ω be a simply connected Jordan domain of $\mathbb{C}$ and let $f^*(e^{it})$ be a p-valent sense-preserving local quasihomeomorphism from ∂U onto $\partial\Omega$. If the second dilatation function $a(z)$ of the solution f of the Dirichlet problem satisfies $|a| < 1$ on U, then we call f a p-valent harmonic and sense-preserving mapping from U onto Ω.

Let us first show that this definition makes sense.

Theorem 3.4. *Let Ω be a simply connected Jordan domain of $\mathbf{C}$ and let $f^*(e^{it})$ be a p-valent sense-preserving local quasihomeomorphism from ∂U onto $\partial\Omega$. Suppose that the second dilatation function $a(z)$ of the solution f of the Dirichlet problem satisfies $|a| < 1$ on U, then*

(1) *f is p-valent on U.*
(2) *$NZ(\phi'_\alpha, U) = p - 1$, where*

$$\phi_\alpha(z) = e^{i\alpha}h(z) - e^{-i\alpha}g(z), \quad \alpha \in \mathbb{R}, \tag{3.2}$$

(3) *If, in addition, Ω is a convex Jordan domain, then $\phi_\alpha(z)$, $\alpha \in \mathbb{R}$, are m-valent for some m, $2 \le m \le p$.*

Proof. Since $|a(z)| < 1$ on U, we can apply the classical argument principle and statement (1) follows immediately. Furthermore, for $w \in \Omega$ and $0 < r < 1$, r close to one,

$$\begin{aligned} 2\pi p &= \oint_{|z|=r} d\arg(f-w) = \oint_{|z|=r} d\arg(df) = \oint_{|z|=r} d\arg(h'dz + \overline{g'dz}) \\ &= \oint_{|z|=r} d\arg(h'dz) + \oint_{|z|=r} d\arg\left(1 + \frac{\overline{ah'dz}}{h'dz}\right) \\ &= \oint_{|z|=r} d\arg(h'dz) = 2\pi + \oint_{|z|=r} d\arg(h') \\ &= 2\pi + \oint_{|z|=r} d\arg(h') + \oint_{|z|=r} d\arg(e^{i\alpha} - ae^{-i\alpha}) \\ &= 2\pi + \oint_{|z|=r} d\arg(e^{i\alpha}h' - e^{-i\alpha}g') = 2\pi(1 + NZ(\phi'_\alpha)). \end{aligned} \tag{3.3}$$

This proves statement (2).

Suppose now that Ω is a convex Jordan domain. Then we have

$$\operatorname{Im} e^{i\alpha} f(z) = \operatorname{Im} \phi_\alpha(z).$$

Each local maximum (local minimum, respectively) of $\operatorname{Im} e^{i\alpha} f(e^{it})$ is a local maximum (local minimum respectively) of $\operatorname{Im} \phi_\alpha(e^{it})$. These maxima (minima respectively) are attained p times. Hence, $\frac{d}{dt} \operatorname{Im} \phi_\alpha(e^{it})$ changes sign $2p$ times . Since $\phi_\alpha(U)$ has to be a domain, the argument principle applied to $\phi_\alpha - w$ and $iz\phi'_\alpha$ implies that for each $\alpha \in \mathbb{R}$, ϕ_α is m-valent, $2 \le m \le p$.

□

3.2. Boundary behaviour of harmonic maps on an interval where $|a| = 1$

Suppose that $a(z)$ admits an analytic extension across an interval I of the unit circle and that its absolute value there is one. Then the boundary values of f depend strongly on the values of a.

Theorem 3.5. *Let $a(z)$ be an analytic function with $|a| < 1$ on U and suppose that $a(z)$ has an analytic extension across an open subinterval $J = \{e^{it} : \alpha < t < \beta\}$ of the unit circle ∂U such that $|a(z)| \equiv 1$ on J.*

(1) *Let $f(z)$ be a bounded solution of (1.4) and suppose that the radial limits $f^*(e^{it})$ exist and are of bounded variation on J. Then we have*

$$f^*(e^{it}) - \overline{a(e^{it})f^*(e^{it}) + \int^t f^*(e^{it})da(e^{it})} = const. \tag{3.4}$$

almost everywhere on J.

(2) *If in addition, f is a p-valent harmonic mapping from U onto a Jordan domain Ω then the radial limit can be replaced in (3.4) by any accumulation point of f at e^{it}. In other words, (3.4) holds for each limit value of f when z tends to $e^{it} \in J$.*

(3) *If f^* jumps at e^{it}, (which must and can happen only when $f^*(J)$ contains a linear segment) then we have*

$$\arg[f^*(e^{i(t+0)}) - f^*(e^{i(t-0)})] = -\frac{1}{2}\arg(a(e^{it})) \mod \pi. \tag{3.5}$$

(4) *If f^* is continuous at e^{it}, then we have*

$$\lim_{h\to 0} \operatorname{Im}\left\{\sqrt{a(e^{it})}\left[\frac{f^*(e^{i(t+h)}) - f^*(e^{i(t-h)})}{h}\right]\right\} = 0. \tag{3.6}$$

(5) *If f^* is not constant on a subinterval of J, then the right limit*

$$\lim_{h\downarrow 0} \arg[f^*(e^{i(t+h)}) - f^*(e^{i(t-0)})] = -\frac{1}{2}\arg a(e^{it}) \mod \pi. \tag{3.7}$$

Proof. The proof is essentially the same as the one given in Corollary 2.5 in [1] which is stated for univalent harmonic mappings □

The relation (3.4) can be expressed in the differential form

$$df^*(e^{it}) - \overline{a(e^{it})df^*(e^{it})} = 0, \tag{3.8}$$

or equivalently by

$$\operatorname{Im}(\sqrt{a(e^{it})}df^*(e^{it})) = 0. \tag{3.9}$$

on J. Hence

$$\arg df^*(e^{it}) = -\frac{1}{2}\arg a(e^{it}) \mod \pi, \tag{3.10}$$

whenever $df^*(e^{it}) \neq 0$ on J.

3.3. The inverse image of a boundary point

We first introduce the notion of a regulated domain.

Definition 3.6.

(1) We call $\beta(\tau)$ a regulated function on the interval $[a,b]$ if the one-sided limits $(\beta(\tau+0)$ and $(\beta(\tau-0)$ exist for all $t \in [a,b]$.

(2) Let Ω be a simply connected domain of $\mathbb{C}$ and suppose that the boundary $\partial\Omega$ is locally connected (every prime end is a singleton). Let ϕ be a conformal mapping from U onto Ω. We call Ω a regulated domain if for each prime end $q = w(\tau) = \phi(e^{i\tau})$ of $\partial\Omega$ the direction angle of the forward (half-)tangent at $w(\tau)$,

$$\beta(q) = \lim_{s\downarrow\tau} \arg[w(s) - w(\tau)] = \lim_{s\downarrow\tau} \arg[w(s) - q], \tag{3.11}$$

exists and defines a regulated function. For more details see [5].

Let Ω be a simply connected regulated domain of $\mathbb{C}$ and let f be a p-valent harmonic sense-preserving mapping from U onto Ω. Let q be a prime end of $\partial\Omega$. Then we have $(f^*)^{-1}(q) = \cup_{k=1}^{p} J_k(q)$. In other words, the preimage of q is the union of p mutually disjoint closed intervals $J_k(q) = \{e^{it}, \gamma_k(q) \le t \le \delta_k(q)\}, 1 \le k \le p$. of ∂U.

Remarks 3.7.

(1) If f^* has a jump at e^{it_k} then we have for all interior points q of this jump, $\gamma_k(q) = \delta_k(q) = e^{it_k}$.

(2) Observe that the cluster sets $C(f^*, e^{i\gamma_k(q)})$ and $C(f^*, e^{i\delta_k(q)})$ contain q but they may contain other points if a jump occurs.

(3) If $J_k(q)$ is a continuum then $|a| \equiv 1$ on $J_k(q)$.

(4) If a has an analytic continuation across an interval J of ∂U such that $|a| \equiv 1$ on J and $J_k(q) \subset J$, then we conclude from (3.7) that

$$\begin{aligned}\beta(q) &= \lim_{h\downarrow 0} \arg\left[f^*(e^{i(\delta_k(q)+h)}) - f^*(e^{i(\delta_k(q)-0)})\right] \\ &= -\frac{1}{2}\arg\, a(e^{i\delta_k(q)}) \mod \pi.\end{aligned} \tag{3.12}$$

The next theorem states that under the conditions of item 4 of the previous remark, the total change of $-\frac{1}{2}\arg\, a(e^{it})$ over the interval $J_k(q)$ is either equal to the opening angle $\alpha(q)$ as seen from the inside of the domain or, if $\pi \le \alpha(q) \le 2\pi$, it can also be $\alpha(q) - \pi$.

Theorem 3.8. *Let Ω be a simply connected regulated domain of C and let f be a p-valent harmonic orientation-preserving mapping from U onto Ω. Suppose that $a(z)$, as defined by (1.4), admits an analytic continuation across an open interval $J \subset \partial U$ such that $|a(z)| \equiv 1$ there. Let q be a prime end of $\partial\Omega$ and suppose that $J_k(q) \subset J$ for at least one $k, 1 \leq k \leq p$. Denote by $\alpha(q)$ the opening angle at q as seen from the inside of Ω. Set $A(t) = \arg\ a(e^{it}), e^{it} \in J$ as a continuous function and define $\triangle A_k(q) = \frac{1}{2}[A(\delta_k(q)) - A(\gamma_k(q))]$. Then we have the following relation between $\alpha(q)$ and $\triangle A_k(q)$.*

(1) *If $0 \leq \alpha(q) < \pi$. then $\alpha(q) = \triangle A_k(q)$.*
(2) *If $\pi \leq \alpha(q) \leq 2\pi$. then either $\alpha(q) = \triangle A_k(q)$ or $\alpha(q) = \triangle A_k(q) - \pi$.*

Proof. The proof goes along the same lines as in Theorem 2.13 in [1]. We have

$$\frac{\partial f}{\partial r}(e^{it}) = e^{it}h'(e^{it}) + \overline{e^{it}g'(e^{it})} = 2e^{it}h'(e^{it})$$

and

$$a(e^{it}) = \frac{e^{it}g'(e^{it})}{e^{it}h'(e^{it})} = \frac{\overline{e^{it}h'(e^{it})}}{e^{it}h'(e^{it})} = \frac{\overline{\frac{\partial f}{\partial r}(e^{it})}}{\frac{\partial f}{\partial r}(e^{it})}$$

on $J_k(q) \setminus Z(h', \partial U)$. In other words, we get

$$\arg\ \frac{\partial f}{\partial r}(e^{it}) = -\frac{1}{2}\arg\ a(e^{it}) \mod \pi$$

on $J_k(q) \setminus Z(h', \partial U)$. Hence, $\arg\ \frac{\partial f}{\partial r}(e^{it})$ is a monotone decreasing function on $J_k(q) \setminus Z(h', \partial U)$. If $e^{it} \in Z(h', \partial U)$, then a jump of the magnitude π appears. From the geometry, we also get $\beta(q) \leq \arg\ \frac{\partial f}{\partial r}(e^{it}) \leq \beta_L(q) + 2\pi$ on $J_k(q) \setminus Z(h', \partial U)$ where $\beta_L(q) - \pi$ is the direction angle of the backward half-tangent of $\partial\Omega$. Fix t and s, $\gamma_k(q) < t < s < \delta_k(q)$ and suppose that $h'(e^{is})h'(e^{it}) \neq 0$. Then by the continuity of $\arg\ a(e^{it})$, we conclude that

$$0 \leq \frac{1}{2}[\arg\ a(e^{is}) - \arg\ (e^{it})] \leq \beta_L(q) + \pi - \beta(q) = \alpha(q).$$

Passing to the limit, we obtain

$$\triangle A_k(q) = \frac{1}{2}[A(\delta_k(q)) - A(\gamma_k(q))] \leq \alpha(q),$$

for all $0 \leq \alpha(q) \leq 2\pi$. Since we have $\beta(q) = -\frac{1}{2}\arg\ a(e^{i\delta_k(q)})$ and $\beta_L(q) = -\frac{1}{2}\arg\ a(e^{i\gamma_k(q)})\mod \pi$ for all prime ends $q \in \partial\Omega$, we conclude that equality holds in (3.12) and we get

$$\alpha(q) = \pi - \beta(q) + \beta_L(q) = \frac{1}{2}\arg\ a(e^{i\delta_k(q)} - \frac{1}{2}\arg\ a(e^{i\gamma_k(q)}) \mod \pi.$$

Finally, the same argument as given in the proof of Theorem 4.5 in [1] shows that the case $\alpha(q) = 2\pi$ and $\delta_k(q) = \gamma_k(q)$ cannot occur, since $\partial\Omega$ is a regulated domain. □

4. p-valent harmonic mappings with finite Blaschke dilatations

In the first paragraph of this chapter we describe the image domains Ω of p-valent harmonic and sense-preserving mappings from the unit disk U onto Ω whose dilatation function $a(z)$ is a finite Blaschke product.

4.1 A GEOMETRIC CHARACTERIZATION OF THE IMAGE DOMAIN

Let

$$a(z) = e^{i\gamma} \prod_{k=1}^{N} \frac{z - p_k}{1 - \overline{p_k} z} = \sum_{k=0}^{\infty} \alpha_k z^k, \quad |p_k| < 1, \quad 1 \leq k \leq N, \tag{4.1}$$

be a finite Blaschke product of degree N and let f be a p-valent solution of the partial differential equation (1.4), $\overline{f_{\bar{z}}} = a(z) f_z(z)$, which maps U onto a Jordan domain Ω. Since a is analytic on the closed unit disk $\overline{U}$ satisfying $|a| \equiv 1$ on ∂U, (3.9) holds for all $t \in [0, 2\pi]$. Furthermore, the relation (3.12),

$$\begin{aligned}\beta(q) &= \lim_{h \downarrow 0} \arg\left[f^*(e^{i(\delta_k(q)+h)}) - f^*(e^{i(\delta_k(q)-0)})\right] \\ &= -\frac{1}{2} \arg\, a(e^{i\delta_k(q)}) \mod \pi,\end{aligned}$$

holds for all $k, 1 \leq k \leq p$. Hence, Ω is a regulated Jordan domain. On the other hand, Theorem 3.8 says that

(1) If $0 \leq \alpha(q) < \pi$. then $\alpha(q) = \triangle A_k(q)$.

(2) If $\pi \leq \alpha(q) \leq 2\pi$. then either $\alpha(q) = \triangle A_k(q)$ or $\alpha(q) = \triangle A_k(q) - \pi$.

independently of $k, 1 \leq k \leq p$, where $A(t) = \arg\ a(e^{it})$ is defined as a continuous function and $\triangle A_k(q) = \frac{1}{2}[A(\delta_k(q)) - A(\gamma_k(q))]$.

We shall use the following notation.

Definition 4.1. Let Ω be a simply connected regulated domain of $\mathbb{C}$. We say that a prime end $q \in \partial\Omega$ is a point of convexity (with respect to Ω) if there is a line segment L containing q as an interior point such that $L \setminus \{q\}$ lies in the exterior of Ω.

Remarks 4.2.

(1) A Jordan domain of $\mathbb{C}$ has at least three points of convexity.

(2) A boundary point q of a bounded convex domain Ω is a point of convexity if and only if it is an extreme point of $\overline{\Omega}$.

Definition 4.3. A prime end $q \in \partial\Omega$ is said to be a complete resting point of order $n(q)$ of f^* if $\triangle A_k(q) = \alpha(q)$ holds for $n(q)$ intervals $J_k(q)$.

Remarks 4.4.

(1) If the prime end q is an interior point of a linear segment of $\partial\Omega$, then either q is an interior point of a jump of f^* at e^{it_k} in which case $\triangle A_k(q) = 0$ or the $J_k(q)$ is not a singleton and we have $\triangle A_k(q) = \pi$.

(2) Each prime end with an opening angle $\alpha(q)$ strictly less than π is a complete resting point of order p of f^*. In particular, if $\alpha(q) = 0$, then $J_k(q)$ is a singleton for all $k, 1 \leq k \leq p$, yet q is still a complete resting point of order p of f^*. On the other hand, if $\alpha(q) > \pi$, $J_k(q)$ is a continuum for all k, yet it may happen that $n(q) = 0$.

The main result of this section is:

Theorem 4.5. *Let Ω be a simply connected regulated domain of $\mathbf{C}$ and let $a(z)$ be a finite Blaschke product. Let f be a p-valent solution of (1.4) which maps U onto Ω. Then we have*

(1) $\sum_{q\in\partial\Omega} n(q) = N + 2p$.

(2) *$\partial\Omega$ contains at most $2 + N/p$ points of convexity.*

Proof. Let CRP be the set of complete resting points $q \in \partial\Omega$ of order $n(q)$ and let E be the set of prime ends $q \in \partial\Omega$ such that $\alpha(q) = 2\pi$ and for which $\gamma_k(q) = \delta_k(q)$ for as many as $m(q)$ $k's$. As atore-said, define $A(t) = \arg\, a(e^{it})$ as a continuous function of $t, 0 \leq t \leq 2\pi$. Then we have $A(2\pi) - A(0) = 2Np$. Choose a prime end $q_0 \in \partial\Omega$ and put

$$B(t) = \pi \sum_{q\in CRP} \sum_{k=1}^{n(q)} H_{qk}(t) - \pi \sum_{q\in E} \sum_{k=1}^{m(q)} H_{qk}(t) - \frac{1}{2}A(t), \tag{4.2}$$
$$B(\delta_1(q_0)) = \beta(q_0),$$

where $H_{qk}(t)$ is the Heaviside function $H_{qk}(t) = 0$ if $\delta_1(q_0) \leq t < \delta_k(q)$ and $H_{qk}(t) = 1$ if $\delta_k(q) \leq t < \delta_1(q_0) + 2\pi$. Observe that for each $q \in \partial\Omega$ and each $k, 1 \leq k \leq p$, we have $B(\delta_k(q)) = \beta(q)$ and $B(\gamma_k(q)) = \beta_L(q)$ where $\beta_L(q) - \pi$ is the direction angle of the backward half-tangent of $\partial\Omega$.

a) We begin by showing that the second sum contains only finitely many terms. Indeed, if not, there is a sequence of pime ends $q_j \in E$ converging from one side to a prime end $q^* \in \partial\Omega$ for which $\beta(q_j) - \beta_L(q_j) = -\pi$. Therefore, the sequence $\beta(q_j)$ does not converge which contadicts the hypothesis that Ω is a regulated domain. Hence E is a finite point set.

b) The relation

$$B(\delta_1(q_0)+2\pi)-B(\delta_1(q_0))=2p\pi$$

$$(4.3)\qquad =\pi\sum_{q\in CRP} n(q)-\pi\sum_{q\in E} m(q)-N\pi,$$

shows that there are only finitely many complete resting points of positive order.

c) Next, we show that there is no prime end in $\partial\Omega$ with $\alpha(q)=2\pi$. Indeed, if there is one, then there must be infinitely many points of convexity with opening angles $\alpha(q)<\pi$. Since each of them is a complete resting point, we are led to a contradicition.

d) Finally, the first statement of the theorem follows from

$$(4.4)\qquad B(\delta_1(q_0)+2\pi)-B(\delta_1(q_0))=2p\pi=\pi\sum_{q\in CRP} n(q)-N\pi.$$

The second part follows from the facts that each point of convexity is a complete resting point of order p and that f^* is a p-valent local quasi-homeomorphism from ∂U onto $\partial\Omega$. In other words, we have $2p\pi+N\pi=\pi\sum_{q\in CRP} n(q)\geq pn_c\pi$, where n_c is the number of points of convexity of $\partial\Omega$. □

Example 4.6. The function $f(z)=z^p+\overline{z}^{p((n+1)}/(n+1)$ is a p-valent harmonic mapping defined on U. Its dilatation function is $a(z)=z^{np}=z^N$ and the boundary of its image domain Ω contains the $n_c=n+2=2+N/p$ points of convexity $q_j=\frac{n+2}{n+1}e^{2\pi ik/(n+1)}, 0\leq k\leq n+1$.

Example 4.7. The function

$$f(z)=\frac{1}{2\sqrt{2}}\arg\left[\frac{1+\sqrt{2}iz-z^2}{1-\sqrt{2}iz-z^2}\right]+\frac{i}{2}\arg\left[\frac{1+z}{1-z}\right], z\in U,$$

is a univalent ($p=1$) harmonic mapping from U onto a rectangle. Its dilatation function is $a(z)=-z^4$ and its image domain has $n_c=4<2+N/p=6$ points of convexity. The points $w=i/4$ and $w=-i/4$ are complete resting points yet they are not points of convexity.

4.2 The inverse problem

Let $a(z)$ be a Blaschke product of degree N and let f be a p-valent solution of (1.4) which maps U onto a regulated Jordan domain Ω. We have seen that the relation (3.9),

$$\operatorname{Im}\left(\sqrt{a(e^{it})}df^*(e^{it})\right)\equiv 0$$

on ∂U is a necessary condition for f which implies that $\partial\Omega$ contains at most $2+N/p$ points of convexity and that f^* contains $2p+N$ complete resting points. It is natural to ask if the condtion (3.9) is also sufficient. Or else, for a given regulated Jordan domain whose boundary has at most $2+N/p$ points of convexity must there be a p-valent solution of (1.4) which maps U onto Ω. The answer for both questions is negative. For instance, it is shown in [1] that there is no univalent solution of $\overline{f_{\overline{z}}}(z) = z^2 f_z(z)$ which maps U onto the rectangle $R = (0,2)$ x $(0,1)$. The next result gives a complete answer to these questions.

Theorem 4.8. *Let $a(z)$ be a Blaschke product of degree N and let Ω be a simply connected regulated Jordan domain whose boundary contains at most $2+N/p$ points of convexity. Let $f^*(e^{it})$ be a p-valent sense-preserving local quasihomeomorphism from ∂U onto $\partial\Omega$ satisfying the relation (3.9). Then the solution of the Dirichlet problem is a p-valent harmonic sense-preserving mapping from U onto Ω if and only if the relation*

$$\int_0^{2\pi} \frac{a(e^{it}) - a(z)}{e^{it} - z} df^*(e^{it}) \equiv 0 \tag{4.5}$$

holds true on U.

Proof. The proof is exactly the same as the one given for Theorem 3.6 in [1]. Essentially one shows that $\overline{f_{\overline{z}}}(z) - a(z) f_z(z)$ is analytic on $\overline{U}$ and is identically zero there. The argument principle yealds the statement of the theorem. □

Since $a(z)$ is rational function of degree N, we will show that the condition (4.5) can be replaced by a system of $[N/2]$ (the integer part of $N/2$) equations. Let

$$a(z) = e^{i\gamma} \prod_{k=1}^{m} [\frac{z - p_k}{1 - \overline{p_k} z}]^{n_k}, \sum_{k=0}^{m} n_k = N, \; |p_k| < 1, \quad 1 \le k \le m,$$

$$p(z) = \prod_{k=1}^{m} (z - p_k)^{n_k},$$

$$q(z) = \prod_{k=1}^{m} (1 - \overline{p_k} z)^{n_k}, \quad \text{and}$$

$$\begin{aligned} t(z) &= e^{-i\gamma/2} z q(z) \int_0^{2\pi} \frac{a(e^{it}) - a(z)}{e^{it} - z} df^*(e^{it}) \\ &= z \int_0^{2\pi} e^{-i\gamma/2} \frac{q(z) a(e^{it})}{e^{it} - z} df^*(e^{it}) - z \int_0^{2\pi} e^{i\gamma/2} \frac{p(z)}{e^{it} - z} df^*(e^{it}) \\ &= z \int_0^{2\pi} e^{-i\gamma/2} \frac{q(z)}{e^{it} - z} \overline{df^*(e^{it})} - z \int_0^{2\pi} e^{i\gamma/2} \frac{p(z)}{e^{it} - z} df^*(e^{it}). \end{aligned}$$

Observe that $t(z)$ is a polynomial of degre N and that the condition (4.5) is equivalent to the condition

$$t(z) \equiv 0, \; z \in U. \tag{4.6}$$

Hence, we may replace (4.5) by $N+1$ equations of the form $L_k(t) = 0, 1 \leq k \leq N$, where the L_k's are $N+1$ linearly independent continuous linear functionals defined on the linear space $H(U)$ of analytic functions on U. Next, we have

$$q(z) = z^N \overline{p(\frac{1}{\bar{z}})},$$

$$p(z) = z^N \overline{q(\frac{1}{\bar{z}})},$$

$$\int_0^{2\pi} \frac{e^{it}}{e^{it} - z} df^*(e^{it}) = \int_0^{2\pi} \frac{z}{e^{it} - z} df^*(e^{it}) \quad \text{and}$$

$$\int_0^{2\pi} \frac{e^{it}}{e^{it} - z} \overline{df^*(e^{it})} = \int_0^{2\pi} \frac{z}{e^{it} - z} \overline{df^*(e^{it})}$$

which implies that $z^N \overline{t(1/\bar{z})} \equiv t(z)$. Finally, the condition $t(0) = 0$ is automatically satisfied and we are left with only $[N/2]$ equations. Summarizing we have shown

Theorem 4.9. *The necessary and sufficient condition* (4.5) *in* Theorem 4.8 *can be replaceed by any linearly independent set of* $[N/2]$ *linear functionals. In particular, we may choose them from the set of point evaluations of* $t(z)$ *and their derivatives, i.e., from the set*

$$L_{kj}(t) = \int_0^{2\pi} \frac{e^{ijt} df^*(e^{it})}{(1 - \overline{p_k} e^{it})^j}, \; 1 \leq j \leq n_k, 1 \leq k \leq m, \tag{4.7}$$

where

$$a(z) = e^{i\gamma} \prod_{k=1}^{m} \left[\frac{z - p_k}{1 - \overline{p_k} z}\right]^{n_k}, \sum_{k=0}^{m} n_k = N, \; |p_k| < 1, \quad 1 \leq k \leq m.$$

Application 4.10. Let $a(z) = z^2$ and $p = 2$. Then the sum of the orders of the complete resting points is $2p + N = 6$. Since the number of convexity points of $\partial\Omega$ is at least three and at most $2 + N/p = 3$ we conclude that $\partial\Omega$ has exactly 3 points of convexity, say w_1, w_2 and w_3. Put $w_4 = w_1$ and choose a $t = t_k$ from $e^{it} = f^{-1}(w_k)$. The necessary and sufficient conditions for the existence of a 2-valent solution of (1.4) are the relations (3.9) and (4.5). The first one determines the boundary correspondence while the second one can be expressed by the single equation

$$\int_0^{2\pi} e^{it} df^*(e^{it}) = 0. \tag{4.8}$$

By (3.9), we have $\mathrm{Im}[e^{it}df^*(e^{it})] = \mathrm{Im}[\sqrt{a(e^{it})}df^*(e^{it})] = 0$ and we get

$$\int_0^{2\pi} e^{it}df^*(e^{it}) = \sum_{k=1}^{3}\int_{t_k}^{t_{k+1}} e^{it}df^*(e^{it}) = \pm\sum_{k=1}^{3}\int_{t_k}^{t_{k+1}} (-1)^k |df^*(e^{it})|$$

$$= \pm[l_1 - l_2 + l_3 - l_1 + l_2 - l_3] = 0 \tag{4.9}$$

where l_k denotes the euclidean length of the positively oriented arc of $\partial\Omega$ joining w_k to w_{k+1}. Hence, given any Jordan domain $\partial\Omega$ with three points of convexity, there is a unique boundary correspondence $f^*(e^{it})$ which satisfies (3.9). On the other hand (4.9) shows that the necessary and sufficient condition (4.8) is always satisfied. Therefore, the solution of the Dirichlet problem is automatically a 2-valent solution of (1.4) which maps U onto Ω. Finally let us remark that f can be obtained by the relation $f(z) = f_1(z^2)$ where f_1 is a univalent solution of $\overline{f_{\bar z}}(z) = zf_z(z)$ which maps U onto Ω.

Application 4.11. Let $a(z) = z^3$ and $p = 2$. Then the sum of the orders of the complete resting points is exactly $2p + N = 7$. Since the number of convexity points of $\partial\Omega$ is at least three and at most $[2+N/p] = 3$ we conclude that $\partial\Omega$ has exactly 3 points of convexity, say w_1, w_2 and w_3. As in the previous application, put $w_4 = w_1$ and choose a $t = t_k$ from $e^{it} = f^{-1}(w_k)$. In addition, we need to find a complete resting point of order one such that together with (3.9) the condition (4.5) also holds. The first one determines the boundary correspondence while the second one can be expressed again by the single equation

$$\int_0^{2\pi} e^{it}df^*(e^{it}) = 0.$$

Note that the nice geometric property of (4.8) in the last example does no more hold. Moreover, it is not the case that for every Ω there exists a two-valent solution of $\overline{f_{\bar z}}(z) = z^3 f_z(z)$ which maps U onto Ω.

References

[1] Bshouty, D. and W. Hengartner, *Boundary values versus dilatations of harmonic mappings*, J. d'Analyse **72** (1997), 141-164.

[2] Gauthier, P. M. and W.Hengartner, *Approximation uniforme qualitative sur des ensembles non bornés*, Séminaire de mathématiques supérieures. Les Presses de l'Université de Montréal, 1982.

[3] Hengartner, W. and G. Schober, *Harmonic mappings with given dilatation*, J. London Math. Soc. **33** (1986), 473-483.

[4] Lyzzaik, A., *Local properties of light harmonic mappings*, Canad. J. Math. **44** (1992), 135-153.

[5] Pommerenke, C., *Boundary Behaviour of Conformal Maps*, Grundlehren der mathematischen Wissenschaften, Springer Verlag, 1991.

[6] Wilmshurst, A. S., *The valence of harmonic polynomials*, Proc. Amer. Math. Soc. **126** (1998), 2077-2081.

Department of Mathematics
Technion, Haifa 32000, Israel
e-mail:daoud@techunix.technion.ac.il

received October 22, 1998

Département des Mathématiques
Université Laval, Québec, Canada
e-mail:walheng@mat.ulaval.ca