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# The generalized Day norm
# Part II. Applications

Abstract. In this paper we prove that for each $1 < p, \tilde{p} < \infty$, the Banach space $(l^{\tilde{p}}, \|\cdot\|_{\tilde{p}})$ can be equivalently renormed in such a way that the Banach space $(l^{\tilde{p}}, \|\cdot\|_{L,\alpha,\beta,p,\tilde{p}})$ is LUR and has a diametrically complete set with empty interior. This result extends the Maluta theorem about existence of such a set in $l^2$ with the Day norm. We also show that the Banach space $(l^{\tilde{p}}, \|\cdot\|_{L,\alpha,\beta,p,\tilde{p}})$ has the weak fixed point property for nonexpansive mappings.

**1. Introduction.** Recently E. Maluta presented a reflexive LUR Banach space which contains a diametrically complete set with empty interior [11]. Namely she proved that the Banach space $l^2$ furnished with the Day norm $\|\cdot\|_L$ is such a Banach space. Next in [3] the authors introduced the generalized Day norm $|||\cdot|||_{\beta,p}$ in $c_0$ and showed its properties. In this paper applying the generalized Day norm, we prove that for each $1 < p, \tilde{p} < \infty$ in the Banach space $(l^{\tilde{p}}, \|\cdot\|_{\tilde{p}})$ there exists an equivalent norm $\|\cdot\|_{L,\alpha,\beta,p,\tilde{p}}$ such that $(l^{\tilde{p}}, \|\cdot\|_{L,\alpha,\beta,p,\tilde{p}})$ contains diametrically complete set with empty interior. We also show that the Banach spaces $(c_0, |||\cdot|||_{\beta,p})$ and $(l^{\tilde{p}}, \|\cdot\|_{L,\alpha,\beta,p,\tilde{p}})$ have the weak fixed point property for nonexpansive mappings.

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**2. Basic notions and facts.** Throughout this paper all Banach spaces are infinite dimensional and real. We will also use the notations, assumptions and facts from [3]. Additionally, let us recall the following definition.

**Definition 2.1** ([6] and [8], see also [1], [4] and [5]). Let $(X, \|\cdot\|)$ be a Banach space. We say that $(X, \|\cdot\|)$ has the *Kadec–Klee property* with respect to the weak topology (the Kadec–Klee property, for short) if each sequence $\{x_n\}$ with $\lim_n \|x_n\| = 1$, which converges weakly to a point $\xi$ with $\|\xi\| = 1$, tends strongly to $\xi$.

It is known that for $1 < p < +\infty$ the Banach space $l^p$ with the standard norm $\|\cdot\|_p$ has the Kadec–Klee property (see for example [4]).

We also need the definition of a diametrically complete set in a Banach space.

**Definition 2.2.** Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space and let $C$ be a nonempty and non-singleton subset of $X$. We say that $C$ is a diametrically complete set in $X$ if

$$\begin{aligned}\operatorname{diam}_{\|\cdot\|}(C \cup \{x\}) &= \sup\{\|y - y'\| : y, y' \in C \cup \{x\}\} \\ &> \operatorname{diam}_{\|\cdot\|}(C) = \sup\{\|y - y'\| : y, y' \in C\}\end{aligned}$$

for each $x \in X \setminus C$.

It is obvious that a diametrically complete set has to be bounded, closed and convex.

Next we give two results which establish relations between a diametral property of a set and the interior of a diametrically complete set. First in [14], J. P. Moreno, P. L. Papini and R. R. Phelps proved the following theorem.

**Theorem 2.3.** *Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space and $C \subset X$ be diametrically complete. If the interior of $C$ is empty, then $C$ is diametral.*

In [12], E. Maluta and P. L. Papini showed the following result.

**Theorem 2.4.** *Each infinite dimensional and reflexive Banach space $(X, \|\cdot\|)$, which satisfies the non-strict Opial property and lacks normal structure, contains diametrically complete sets whose interior is empty.*

**3. The generalized Day norm and renorming of Banach spaces.** In this section we apply the generalized Day norm $|||\cdot|||_{\beta,p}$ in $c_0$ to renorm a Banach space $(l^{\tilde{p}}, \|\cdot\|_{\tilde{p}})$, where $1 < p, \tilde{p} < +\infty$.

Fix $\alpha \in (0, 1)$ and fix $1 < p, \tilde{p} < \infty$. Next we choose a strictly decreasing positive sequence $\beta = \{\beta_j\}_j$ satisfying the following two conditions

- the series $\sum_{j=1}^{\infty} \beta_j^p$ is convergent,

- there exists a constant $L > 1$ such that for each $n \in \mathbb{N}$

$$\sum_{j=n+1}^{\infty} \beta_j^p \le L\beta_{n+1}^p.$$

Now we can observe that for each $x = \{x^k\}_k \in l^{\tilde{p}}$, the sequence given by

$$u(x) = \{u^i(x)\}_i = \{\alpha\|x\|_{\tilde{p}}, x^1, x^2, x^2, \ldots, x^k, \ldots, x^k, \ldots\}$$

is an element of $c_0$ (here the $k$-th coordinate of $x$ is repeated exactly $k$ times and $\|\cdot\|_{\tilde{p}}$ is the standard norm in $l^{\tilde{p}}$). So we can apply the Day norm $|||\cdot|||_{\beta,p}$ ([3]) to the element $u(x)$ and set

$$\|x\|_{L,\alpha,\beta,p,\tilde{p}} := |||u(x)|||_{\beta,p} = \|D(u(x))\|_p,$$

where $\|\cdot\|_p$ is the standard norm in $l^p$.

It is easy to note that

$$\|u(x)\|_{c_0} = \max\{\alpha\|x\|_{\tilde{p}}, |x^1|, |x^2|, \ldots\},$$

$$\beta_1\alpha\|x\|_{\tilde{p}} \le \|D(u(x))\|_p = \|x\|_{L,\alpha,\beta,p,\tilde{p}} \le \left(\sum_{j=1}^{\infty} \beta_j^p\right)^{\frac{1}{p}} \|x\|_{\tilde{p}},$$

and

$$\begin{aligned}\beta_1\|u(x)\|_{c_0} \le \|D(u(x))\|_p = |||u(x)|||_{\beta,p} = \|x\|_{L,\alpha,\beta,p,\tilde{p}} \\ \le \left(\sum_{j=1}^{\infty} \beta_j^p\right)^{\frac{1}{p}} \|u(x)\|_{c_0}\end{aligned}$$

for each $x \in l^{\tilde{p}}$. Therefore $\|\cdot\|_{L,\alpha,\beta,p,\tilde{p}}$ is a norm in $l^{\tilde{p}}$, which is equivalent to the original one.

**Remark 3.1.** The norm $\|\cdot\|_L$ connected with the Day norm $|||\cdot|||$ was introduced by M. Smith ([16]) and our norm $\|\cdot\|_{L,\alpha,\beta,p,\tilde{p}}$ is a generalization of the norm $\|\cdot\|_L$. In his paper M. Smith proves that $(l^2, \|\cdot\|_L)$ is a reflexive, locally uniformly rotund Banach space that is not uniformly convex in every direction. In [17], M. Smith and B. Turett show that $(l^2, \|\cdot\|_L)$ lacks normal structure.

**4. The norm $\|\cdot\|_{L,\alpha,\beta,p,\tilde{p}}$ and the non-strict Opial property.** To get the result about the non-strict Opial property of the Banach space $(l^{\tilde{p}}, \|\cdot\|_{L,\alpha,\beta,p,\tilde{p}})$ we modify the Maluta proof of Theorem 3.1 in [11] (see also the proof of Theorem 5.2 in [3]).

**Theorem 4.1.** *The Banach space $(l^{\tilde{p}}, \|\cdot\|_{L,\alpha,\beta,p,\tilde{p}})$ has the non-strict Opial property.*

**Proof.** Assume that $\{x_n\}_n = \{\{x_n^i\}_i\}_n \subset l^{\tilde{p}}$ tends weakly to $0 \in l^{\tilde{p}}$ and let $x = \{x^i\}_i \in l^{\tilde{p}} \setminus \{0\}$. Let us take $0 < \epsilon < 1$. By the Opial property of the Banach space $(l^{\tilde{p}}, \|\cdot\|_{\tilde{p}})$ there exists $\tilde{n}_0 \in \mathbb{N}$ such that for each $\tilde{n}_0 < n \in \mathbb{N}$ we have

$$\|x_n\|_{\tilde{p}} < \|x_n - x\|_{\tilde{p}} + \epsilon.$$

Next there exists $\tilde{i} \in \mathbb{N}$ such that

$$|x^i| < \epsilon$$

for each $\tilde{i} < i \in \mathbb{N}$. Therefore,

$$|x_n^i| \leq |x_n^i - x^i| + |x^i| < |x_n^i - x^i| + \epsilon$$

for each $\tilde{i} < i \in \mathbb{N}$ and all $n \in \mathbb{N}$.

Now for each $1 \leq i \leq \tilde{i}$ we have either $x^i = 0$ or $x^i \neq 0$. In the second case setting $\eta_i = \min\{\epsilon, \frac{1}{2}|x^i|\}$ and taking into account the weak convergence of $\{x_n\}_n$ to 0, we find $\tilde{n}_i \in \mathbb{N}$ such that

$$|x_n^i| < \eta_i$$

for $\tilde{n}_i < n \in \mathbb{N}$ and hence we obtain

$$\begin{aligned} |x_n^i - x^i| &\geq |x^i| - |x_n^i| \\ &> |x^i| - \eta_i \geq \frac{1}{2}|x^i| \geq \eta_i > |x_n^i|. \end{aligned}$$

In the first case, i.e., $x^i = 0$, we have

$$|x_n^i| = |x_n^i - x^i|$$

for each $n \in \mathbb{N}$.

So we have shown that

$$|x_n^i| \leq |x_n^i - x^i|$$

for each $1 \leq i \leq \tilde{i}$ and all $\max\{\tilde{n}_1, \dots, \tilde{n}_{\tilde{i}}\} < n \in \mathbb{N}$.

Putting together all above inequalities, we get

$$|x_n^i| < |x_n^i - x^i| + \epsilon$$

for each $i \in \mathbb{N}$ and for all $\max\{\tilde{n}_1, \dots, \tilde{n}_{\tilde{i}}\} < n \in \mathbb{N}$.

The above considerations yield the following inequalities

$$(*) \qquad |u^i(x_n)| < |u^i(x_n - x)| + \epsilon$$

for each $i \in \mathbb{N}$ and for all $\max\{\tilde{n}_0, \tilde{n}_1, \dots, \tilde{n}_{\tilde{i}}\} < n \in \mathbb{N}$.

Now let us take $\max\{\tilde{n}_0, \tilde{n}_1, \dots, \tilde{n}_{\tilde{i}}\} < n \in \mathbb{N}$. Then using Corollary 2.8 from [3] and applying $(*)$, we obtain

$$\begin{aligned} \|x_n\|_{L,\alpha,\beta,p,\tilde{p}} &= |||u(x_n)|||_{\beta,p} = \|D(u(x_n))\|_{\tilde{p}} \\ &= \left[\sum_{j=1}^{\infty} \left(\beta_j \left|u^{\tau(j,u(x_n))}(x_n)\right|\right)^p\right]^{\frac{1}{p}} \end{aligned}$$

$$
\begin{aligned}
&< \left\{ \sum_{j=1}^{\infty} \beta_j \left[ \left| u^{\tau(j,u(x_n))}(x_n - x) \right| + \epsilon \right]^p \right\}^{\frac{1}{p}} \\
&\leq \left\{ \sum_{j=1}^{\infty} \beta_j \left[ \left| u^{\tau(j,u(x_n - x))}(x_n - x) \right| + \epsilon \right]^p \right\}^{\frac{1}{p}} \\
&\leq \left[ \sum_{j=1}^{\infty} \left( \beta_j \left| u^{\tau(j,u(x_n - x))}(x_n - x) \right| \right)^p \right]^{\frac{1}{p}} + \epsilon \left[ \sum_{j=1}^{\infty} \beta_j^p \right]^{\frac{1}{p}} \\
&= \| D(u(x_n - x)) \|_p + \epsilon \left[ \sum_{j=1}^{\infty} \beta_j^p \right]^{\frac{1}{p}} \\
&= \| u(x_n - x) \|_{\beta,p} + \epsilon \left[ \sum_{j=1}^{\infty} \beta_j^p \right]^{\frac{1}{p}} \\
&= \| x_n - x \|_{L,\alpha,\beta,p,\tilde{p}} + \epsilon \left[ \sum_{j=1}^{\infty} \beta_j^p \right]^{\frac{1}{p}}.
\end{aligned}
$$

Finally, by passing $n$ to $+\infty$, we get

$$
\limsup_n \|x_n\|_{L,\alpha,\beta,p,\tilde{p}} \leq \limsup_n \|x_n - x\|_{L,\alpha,\beta,p,\tilde{p}} + \epsilon \left[ \sum_{j=1}^{\infty} \beta_j^p \right]^{\frac{1}{p}}
$$

and by arbitrariness of $0 < \epsilon < 1$, we obtain

$$
\limsup_n \|x_n\|_{L,\alpha,\beta,p,\tilde{p}} \leq \limsup_n \|x_n - x\|_{L,\alpha,\beta,p,\tilde{p}}.
$$

□

Observe that the Banach space $(l^{\tilde{p}}, |||\cdot|||_{L,\alpha,\beta,p,\tilde{p}})$ does not have the Opial property as the following example shows.

**Example 4.2.** Consider $(l^{\tilde{p}}, |||\cdot|||_{L,\alpha,\beta,p,\tilde{p}})$ with the standard basis $\{e_i\}_i$. Let us take a sequence $\{u_n\}_n = \{e_{n+1}\}_n$. This sequence is weakly convergent to $0 \in c_0$ and for

$$
u = \min \left\{ 1, \left( \frac{1}{\alpha^{\tilde{p}}} - 1 \right)^{\frac{1}{\tilde{p}}} \right\} e_1
$$

we have

$$
\lim_n |||u_n|||_{L,\alpha,\beta,p,\tilde{p}} = \lim_n |||u_n - u|||_{L,\alpha,\beta,p,\tilde{p}} = \left( \sum_{j=1}^{\infty} \beta_j^p \right)^{\frac{1}{p}}.
$$

**5. The norm $\|\cdot\|_{L,\alpha,\beta,p,\tilde{p}}$ and normal structure.** The following theorem is valid.

**Theorem 5.1.** *The Banach space* $(l^{\tilde{p}}, \|\cdot\|_{L,\alpha,\beta,p,\tilde{p}})$ *lacks normal structure for* $\alpha \leq 2^{-\frac{1}{\tilde{p}}}$.

**Proof.** As usual in $l^{\tilde{p}}$ we have the standard basis $\{e_i\}_i$. The proof is a small modification of the proof due to M. A. Smith and B. Turett ([17]). Observe that for $m_2 > m_1$ we have

$$e_{m_2} - e_{m_1} = \{0, \dots, 0, -1, 0, \dots, 0, 1, 0, \dots\}.$$

Clearly,

$$\alpha \|e_{m_2} - e_{m_1}\|_{\tilde{p}} = \alpha 2^{\frac{1}{\tilde{p}}} \leq 1$$

and therefore

$$\begin{aligned}\left[\sum_{j=1}^{m_1+m_2} \beta_j^p\right]^{\frac{1}{p}} &\leq \|e_{m_2} - e_{m_1}\|_{L,\alpha,\beta,p,\tilde{p}} = |||e_{m_2} - e_{m_1}|||_{\beta,p} \\ &\leq \left[\sum_{j=1}^{m_1+m_2+1} \beta_j^p\right]^{\frac{1}{p}} \leq \left[\sum_{j=1}^{\infty} \beta_j^p\right]^{\frac{1}{p}}.\end{aligned}$$

This means that

$$\operatorname{diam}_{\|\cdot\|_{L,\alpha,\beta,p,\tilde{p}}} \{e_i\}_i = \left[\sum_{j=1}^{\infty} \beta_j^p\right]^{\frac{1}{p}}.$$

Now we compute $\lim_m \operatorname{dist}_{\|\cdot\|_{L,\alpha,\beta,p,\tilde{p}}}(e_{m+1}, \operatorname{conv}\{e_1, \dots, e_m\})$. Let us take $a^1 + \dots + a^m = 1$, where $0 \leq a^k \leq 1$ for $1 \leq k \leq m$. Then we have

$$e_{m+1} - \sum_{k=1}^{m} a^k e_k = \left\{ e_j^* \left( e_{m+1} - \sum_{k=1}^{m} a^k e_k \right) \right\}_j = \{-a^1, \dots, -a^m, 1, 0, \dots\}$$

for all $m \in \mathbb{N}$. But we also have

$$\alpha \left\| e_{m+1} - \sum_{k=1}^{m} a^k e_k \right\|_{\tilde{p}} \leq 1.$$

Hence we get

$$\begin{aligned}\left[\sum_{j=1}^{m+1} \beta_j^p\right]^{\frac{1}{p}} &\leq \left\| e_{m+1} - \sum_{k=1}^{m} a^k e_k \right\|_{L,\alpha,\beta,p,\tilde{p}} \\ &\leq \operatorname{diam}_{\|\cdot\|_{L,\alpha,\beta,p,\tilde{p}}} \{e_i\}_i = \left[\sum_{j=1}^{\infty} \beta_j^p\right]^{\frac{1}{p}}.\end{aligned}$$

This means that

$$\lim_m \operatorname{dist}_{\|\cdot\|_{L,\alpha,\beta,p,\tilde{p}}}(e_{m+1}, \operatorname{conv}\{e_1,\dots,e_m\}) = \left[\sum_{j=1}^{\infty} \beta_j^p\right]^{\frac{1}{p}}.$$

□

**6. The norm $\|\cdot\|_{L,\alpha,\beta,p,\tilde{p}}$ and asymptotic normal structure.** We have the following result.

**Theorem 6.1.** *The Banach space $(l^{\tilde{p}}, \|\cdot\|_{L,\alpha,\beta,p,\tilde{p}})$ fails asymptotic normal structure for $\alpha \le 4^{-\frac{1}{\tilde{p}}}$.*

**Proof.** Once again $\{e_i\}_i$ is the standard basis in $l^{\tilde{p}}$. Analogously as in [13] (see also [2]) we put

$$(**) \qquad x_n = \begin{cases} (1-\frac{j}{2^{2i}})e_i + e_{i+1}, & \text{if } n = 2^{2i}+j, \quad j=1,2,\dots,2^{2i} \\ e_{i+1} + \frac{j}{2^{2i+1}})e_{i+2}, & \text{if } n = 2^{2i+1}+j, \quad j=1,2,\dots,2^{2i+1}. \end{cases}$$

and

$$C = \overline{\operatorname{conv}}\{x_n : n = 5,6,\dots\}.$$

Directly from $(**)$ we get

$$0 = \lim_n \|x_n - x_{n+1}\|_{\tilde{p}} = \lim_n \|x_n - x_{n+1}\|_{L,\alpha,\beta,p,\tilde{p}}$$

and

$$\alpha \operatorname{diam}_{\|\cdot\|_{\tilde{p}}}(C) \le \alpha 4^{\frac{1}{\tilde{p}}} \le 1.$$

Next it is easy to see that $|x_n^k - x_m^k| \le 1$ for every $k,m,n \in \mathbb{N}$ and therefore

$$\|x_n - x_m\|_{L,\alpha,\beta,p,\tilde{p}} \le \left[\sum_{j=1}^{\infty} \beta_j^p\right]^{\frac{1}{p}}.$$

On the other hand, for example, if $i_1+2 < i_2$, $n_1 = 2^{2i_1}+1$, $n_2 = 2^{2i_2}+1 > 2^{2i_1+4}+1$, then we have

$$\begin{aligned} x_{n_2} - x_{n_1} &= \{x_{n_2}^k - x_{n_1}^k\}_k \\ &= \{0,\dots,0, x_{n_2}^{i_1} - x_{n_1}^{i_1}, x_{n_2}^{i_1+1} - x_{n_1}^{i_1+1}, 0,\dots,0, x_{n_2}^{i_2} - x_{n_1}^{i_2}, x_{n_2}^{i_2+1} - x_{n_1}^{i_2+1}, 0,\dots\} \\ &= \left\{0,\dots,0, \frac{1}{2^{2i_1}} - 1, -1, 0,\dots,0, 1 - \frac{1}{2^{2i_2}}, 1, 0,\dots\right\}. \end{aligned}$$

and

$$\alpha \|x_{n_2} - x_{n_1}\|_{\tilde{p}} \le \alpha \operatorname{diam}_{\|\cdot\|_{\tilde{p}}}(C) \le 1.$$

Thus directly from the definition of the norm $\|\cdot\|_{L,\alpha,\beta,p,\tilde{p}}$ we get

$$\left[\sum_{j=1}^{i_1+i_2+2} \beta_j^p\right]^{\frac{1}{p}} \le \|x_{n_2} - x_{n_1}\|_{L,\alpha,\beta,p,\tilde{p}} \le \left[\sum_{j=1}^{\infty} \beta_j^p\right]^{\frac{1}{p}}.$$

This means that

$$\operatorname{diam}_{L,\alpha,\beta,p,\tilde{p}}(C) = \operatorname{diam}_{\|\cdot\|_{L,\alpha,\beta,p,\tilde{p}}}\{x_n\} = \left[\sum_{j=1}^{\infty} \beta_j^p\right]^{\frac{1}{p}}.$$

Similarly, we obtain

$$\left\|\sum_{l=1}^{m} a_l x_l - x_n\right\|_{L,\alpha,\beta,p,\tilde{p}} \geq \left[\sum_{j=1}^{i+1} \beta_j^p\right]^{\frac{1}{p}}$$

for each convex combination $\sum_{l=1}^{m} a_l x_l$ and for every sufficiently large $n$, where $i$ is connected with $n$ by $(**)$. This implies

$$\lim_n \left\|\sum_{l=1}^{m} a_l x_l - x_n\right\|_{L,\alpha,\beta,p,\tilde{p}} = \left[\sum_{j=1}^{\infty} \beta_j^p\right]^{\frac{1}{p}} = \operatorname{diam}_{\|\cdot\|_{L,\alpha,\beta,p,\tilde{p}}}(C).$$

Hence we have

$$\lim_n \|x - x_n\|_{L,\alpha,\beta,p,\tilde{p}} = \operatorname{diam}_{\|\cdot\|_{L,\alpha,\beta,p,\tilde{p}}}(C)$$

for each $x \in C$ and therefore the set $C$ lacks asymptotic normal structure. □

**7. The norm $\|\cdot\|_{L,\alpha,\beta,p,\tilde{p}}$ and LUR.** Here applying the Smith method (Example 1 in [16]), we arrive at the following theorem.

**Theorem 7.1.** *The Banach space $(l^{\tilde{p}}, \|\cdot\|_{L,\alpha,\beta,p,\tilde{p}})$ is LUR.*

**Proof.** Let $x \in l^{\tilde{p}}$ and $\{x_n\}_n \subset l^{\tilde{p}}$ be such that $\|x\|_{L,\alpha,\beta,p,\tilde{p}} = 1$, $\lim_n \|x_n\|_{L,\alpha,\beta,p,\tilde{p}} = 1$ and $\lim_n \|x + x_n\|_{L,\alpha,\beta,p,\tilde{p}} = 2$. Then we also have $|||u(x)|||_{\beta,p} = 1$, $\lim_n |||u(x_n)|||_{\beta,p} = 1$ and $\lim_n |||u(x) + u(x_n)|||_{\beta,p} = 2$. Applying local uniform convexity (LUR) of $(c_0, |||\cdot|||_{\beta,p})$, we immediately obtain the strong convergence of the sequence $\{u(x_n)\}$ to $u(x)$ in the norm $|||\cdot|||_{\beta,p}$. But we have

$$\begin{aligned}\beta_1\|u(x) - u(x_n)\|_{c_0} &\leq |||u(x) - u(x_n)|||_{\beta,p} \\ &\leq \left[\sum_{j=1}^{\infty} \beta_j^p\right]^{\frac{1}{p}} \|u(x) - u(x_n)\|_{c_0}\end{aligned}$$

and

$$\|u(x) - u(x_n)\|_{c_0} = \max\{\alpha\,|\|x\|_{\tilde{p}} - \|x_n\|_{\tilde{p}}|\,, |x^1 - x_n^1|, |x^2 - x_n^2|, \dots\}.$$

This implies that $\lim_n \|x_n\|_{\tilde{p}} = \|x\|_{\tilde{p}}$ and $\lim_n x_n^i = x^i$ for $i = 1, 2, \dots$. Hence the sequence $\{x_n\}$ tends weakly to $x$. Finally, by the Kadec–Klee property of $(l^{\tilde{p}}, \|\cdot\|_{\tilde{p}})$ we have $\lim_n x_n = x$ in $(l^{\tilde{p}}, \|\cdot\|_{\tilde{p}})$ and therefore in $(l^{\tilde{p}}, \|\cdot\|_{L,\alpha,\beta,p,\tilde{p}})$, too. □

**8. Diametrically complete sets with empty interior in the space $(l^{\tilde{p}}, \|\cdot\|_{L,\alpha,\beta,p,\tilde{p}})$.** Using the properties of the norm $\|\cdot\|_{L,\alpha,\beta,p,\tilde{p}}$ and the Maluta–Papini Theorem, we obtain

**Theorem 8.1.** *For $\alpha \leq 2^{-\frac{1}{p}}$ the Banach space $(l^{\tilde{p}}, \|\cdot\|_{L,\alpha,\beta,p,\tilde{p}})$ is LUR and contains a diametrically complete set whose interior is empty.*

**Proof.** It is a simple consequence of Theorems 2.4, 4.1, 5.1 and 7.1. □

**9. The weak fixed point property.** We begin with the following definition.

**Definition 9.1.** Let $(X, \|\cdot\|)$ be a Banach space with a Schauder basis $\{e_i\}_i$. We say that a Schauder basis $\{e_i\}_i$ is unconditional if whenever the series $\sum_{i=1}^{\infty} x^i e_i$ converges, it converges unconditionally, i.e., $\sum_{i=1}^{\infty} x^{\tilde{\sigma}(i)} e_{\tilde{\sigma}(i)}$ converges for every permutation $\tilde{\sigma}$ of $\mathbb{N}$.

The following theorem shows a few equivalent definitions of unconditionality of a Schauder basis (see for example [10] and [15]).

**Theorem 9.2.** *Let $(X, \|\cdot\|)$ be a Banach space with a Schauder basis $\{e_i\}_i$. The following statements are equivalent:*

(1) *the basis $\{e_i\}_i$ is an unconditional basis;*
(2) *for every choice of signs $\{\varepsilon_i\}_i$ (i.e. $\varepsilon_i = \pm 1$) $\sum_{i=1}^{\infty} \varepsilon_i x^i e_i$ converges whenever $\sum_{i=1}^{\infty} x^i e_i$ converges;*
(3) *for every convergent series $\sum_{i=1}^{\infty} x^i e_i$ and for every sequence of scalars $\{b^i\}_i$ such that $|b^i| \leq |x^i|$ for all $i$ the series $\sum_{i=1}^{\infty} b^i e_i$ converges.*

It is known that if $\{e_i\}_i$ is an unconditional Schauder basis in a Banach space $(X, \|\cdot\|)$, then

$$\sup\left\{\left\|\sum_{i=1}^{\infty} \varepsilon_i x^i e_i\right\| : \left\|\sum_{i=1}^{\infty} x^i e_i\right\| = 1 \quad \text{and} \quad \varepsilon_i = \pm 1\right\}$$

is finite (see for example [10] and [15]). Hence we state

**Definition 9.3.** If a Schauder basis $\{e_i\}_i$ is an unconditional basis in a Banach space $(X, \|\cdot\|)$, then the number

$$K := \sup\left\{\left\|\sum_{i=1}^{\infty} \varepsilon_i x^i e_i\right\| : \left\|\sum_{i=1}^{\infty} x^i e_i\right\| = 1 \quad \text{and} \quad \varepsilon_i = \pm 1\right\}$$

is called the unconditional constant of $\{e_i\}_i$ . If this constant $K$ is equal to 1, then we say that $\{e_i\}_i$ is a 1-unconditional basis.

Next we recall the definition of the weak fixed point property.

**Definition 9.4.** Let $(X, \|\cdot\|)$ be a Banach space and let $C$ be a weakly compact and convex subset of $X$. We say that $C$ has the fixed point property if each nonexpansive mappings $T : C \to C$ (i.e. $\|Tx_1 - Tx_2\| \leq \|x_1 - x_2\|$ for every $x_1, x_2 \in C$) has a fixed point.

If each weakly compact and convex subset $C$ of $X$ has the fixed point property, then we say that a Banach space $(X, \|\cdot\|)$ has the weak fixed point property.

The following theorem is generally known.

**Theorem 9.5** ([4]). *Let $(X, \|\cdot\|)$ be a Banach space, $C \subset X$ be weakly compact and convex and let $T : C \to C$ be a nonexpansive mapping. Then there exists a separable subspace $X_1 \subset X$ such that the set $C_1 = C \cap X_1$ is a weakly compact, convex and $T$-invariant, i.e. $T(C_1) \subset C_1$.*

In 1965, W. A. Kirk [7] published his famous fixed point theorem.

**Theorem 9.6.** *Let $C$ be a nonempty, weakly compact, convex subset of a Banach space $(X, \|\cdot\|)$, and suppose $C$ has normal structure. Then each nonexpansive mapping $T : C \to C$ has a fixed point.*

Next in 1981, J. B. Baillon and R. Schöneberg [2] extended Kirk's Theorem, using asymptotic normal structure.

**Theorem 9.7.** *Every reflexive Banach space with asymptotic normal structure has the weak fixed point property for nonexpansive mappings.*

Observe that by Theorems 7.6 and 8.2 from [3] we are not able to apply the above fixed point theorems to the Banach space $(c_0, |||\cdot|||_{\beta,p})$.

However, the following theorem, which is due to P.-K. Lin ([9]), is valid.

**Theorem 9.8.** *Each Banach space $(X, \|\cdot\|)$ with a 1-unconditional Schauder basis $\{e_i\}_i$ has the weak fixed point property.*

Directly from the definitions of a 1-unconditional Schauder basis and the norm $|||\cdot|||_{\beta,p}$ we get

**Theorem 9.9.** *Let $\{e_i\}_i$ be a standard basis in $c_0 = c_0(\mathbb{N})$. Then $\{e_i\}_i$ is a 1-unconditional Schauder basis in $(c_0, |||\cdot|||_{\beta,p})$.*

Thus we are ready to prove

**Theorem 9.10.** *The Banach space $(c_0(\Gamma), |||\cdot|||_{\beta,p})$ has the weak fixed point property.*

**Proof.** By Theorem 9.5 we can assume that $\Gamma = \mathbb{N}$ and $c_0(\Gamma) = c_0$. Since by Theorem 9.9 the standard basis $\{e_i\}$ in $c_0$ is a 1-unconditional Schauder basis in $(c_0, |||\cdot|||_{\beta,p})$, we can apply Theorem 9.8 to get our result. □

Next, if $\alpha \le 4^{-\frac{1}{\tilde{p}}}$, then by Theorem 5.1 and 6.1 we are not able to apply Theorems 9.6 and 9.7 to the Banach space $(l^{\tilde{p}}, \|\cdot\|_{L,\alpha,\beta,p,\tilde{p}})$. But directly from the definitions of a 1-unconditional Schauder basis and the norm $|||\cdot|||_{\beta,p}$, we obtain

**Theorem 9.11.** *Let $\{e_i\}_i$ be a standard basis in $(l^{\tilde{p}}, \|\cdot\|_{\tilde{p}})$. Then $\{e_i\}_i$ is a 1-unconditional Schauder basis in $(l^{\tilde{p}}, \|\cdot\|_{L,\alpha,\beta,p,\tilde{p}})$.*

So we are ready to prove

**Theorem 9.12.** *The Banach space $(l^{\tilde{p}}, \|\cdot\|_{L,\alpha,\beta,p,\tilde{p}})$ has the weak fixed point property.*

**Proof.** Since by Theorem 9.11 the standard basis $\{e_i\}$ in $(l^{\tilde{p}}, \|\cdot\|_{\tilde{p}})$ is a 1-unconditional Schauder basis in $(l^{\tilde{p}}, \|\cdot\|_{L,\alpha,\beta,p,\tilde{p}})$, the Lin Theorem implies our result. □

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