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# The generalized Day norm
# Part I. Properties

Abstract. In this paper we introduce a modification of the Day norm in $c_0(\Gamma)$ and investigate properties of this norm.

**1. Introduction.** In 1955, M. M. Day introduced a new norm $|||\cdot|||$ in $c_0(\Gamma)$ to show that the Banach space $c_0(\Gamma)$ with the max-norm can be equivalently renormed to strictly convex space ([5]). In 1969, J. Rainwater showed that $(c_0(\Gamma), |||\cdot|||)$ is locally uniformly convex ([18]). Finally in 1978, M. A. Smith proved that this space is not uniformly convex in every direction ([19]). It is important to note that using this norm, one can construct Banach spaces with the claimed properties (see for example [15], [19] and [20]). In our paper we investigate properties of the modified Day norm $|||\cdot|||_{\beta,p}$ in $c_0$ and among others we extend the Day and Rainwater results.

**2. Basic notions and facts.** Throughout this paper all Banach spaces are infinite dimensional and real.

First we recall a few notions and facts from the geometry of Banach spaces. We begin this section with the following well-known definitions.

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**Definition 2.1** (see for example [9], [10], [12]). A Banach space $(X, \|\cdot\|)$ is strictly convex if $\left\|\frac{x+y}{2}\right\| < 1$, whenever $x, y \in X$, $\|x\| \leq 1$, $\|y\| \leq 1$ and $x \neq y$.

**Definition 2.2** ([8]). A Banach space $(X, \|\cdot\|)$ is said to be uniformly convex in every direction if for every nonzero element $z$ of $X$ and every $0 < \epsilon \leq 2$ there exists $\delta > 0$ such that $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$ whenever $\|x\| \leq 1, \|y\| \leq 1$, $x \neq y$, $x - y = \alpha z$ for some $\alpha \in \mathbb{R} \setminus \{0\}$ and $\|x - y\| \geq \epsilon$.

**Definition 2.3** ([14], see also [7]). We say that a Banach space $(X, \|\cdot\|)$ is locally uniformly convex (LUR) if for each $x \in X$ every sequence $\{x_n\}_n$ with $\lim_n \|x_n\| = \|x\|$ and $\lim_n \|x + x_n\| = 2\|x\|$ tends strongly to $x$.

**Remark 2.4.** Each locally uniformly convex Banach space and each uniformly convex in every direction Banach space is strictly convex (see for example [19]).

Let $\Gamma$ be an infinite set and let $c_0(\Gamma)$ denote the Banach space (with the max-norm) of all real-valued functions $u = \{u^i\}_{i\in\Gamma}$ on $\Gamma$ such that for each $\epsilon > 0$ the set $\{i \in \Gamma : |u^i| \geq \epsilon\}$ is finite. We denote the support of $u \in c_0(\Gamma)$ by $N(u)$. Recall that for $1 < p < \infty$ the Banach space $l^p(\Gamma)$ consists of all $u \in c_0(\Gamma)$ such that $\sum_{i\in N(u)} |u^i|^p < \infty$ (we set $\sum_{i\in N(u)} |u^i|^p = 0$ if $N(u) = \emptyset$) and then

$$\|u\|_p = \left( \sum_{i\in N(u)} |u^i|^p \right)^{\frac{1}{p}}$$

for $u \in l^p(\Gamma)$ (see for example [12]).

Now we recall a definition of the Day norm $|||\cdot|||$ in $c_0(\Gamma)$ (see [5]). If $u = \{u^i\}_{i\in\Gamma} \in c_0(\Gamma) \setminus \{0\}$, then we enumerate the support $N(u)$ of $u$ as $\{\tau(j,u)\}_{j\in J(u)}$ (for a detailed definition of $\tau(\cdot, u)$ see Remark 2.5) in such a way that $|u^{\tau(j,u)}| \geq |u^{\tau(j+1,u)}|$. Next we define $D(u) = \{D^i(u)\}_{i\in\Gamma} \in l^2(\Gamma)$ by

$$D^i(u) = \begin{cases} \frac{u^{\tau(j,u)}}{2^j}, & \text{if } i = \tau(j,u) \text{ for some } j \in J(u) \\ 0, & \text{otherwise} \end{cases}$$

and set $|||u||| = \|D(u)\|_2$. For $0 \in c_0(\Gamma)$ we set $D^i(0) = 0$ for each $i \in \Gamma$ and $D(0) = \{D^i(0)\}_i = 0 \in l^2(\Gamma)$. So $|||0||| = \|D(0)\|_2 = 0$. It is easy to observe that

$$\frac{1}{2}\|u\|_{c_0(\Gamma)} \leq |||u||| \leq \frac{1}{\sqrt{3}}\|u\|_{c_0(\Gamma)}$$

for each $u \in c_0(\Gamma)$, where $\|\cdot\|_{c_0(\Gamma)}$ is the standard max-norm in $c_0(\Gamma)$.

**Remark 2.5.** Throughout this paper we will use the following notation. Let $t = \{t^i\}_{i\in\Gamma} \in c_0(\Gamma)$, where the set $\Gamma$ is infinite. Then the $\{\tau(j,t)\}_j$ is defined as follows:

(1) if the support $N(t)$ of $t$ is infinite, then $N(t)$ is enumerated as $\{\tau(j,t)\}_j$ in such a way that $|t^{\tau(j,t)}| \geq |t^{\tau(j+1,t)}|$ for $j \in J(t) = \mathbb{N}$,
(2) if $N(t) = \{t^{\tilde{i}}\}$ is a singleton, then we set $J(t) = \{1\}$, $\tau(1,t) = \tilde{i}$ and extend $\tau(\cdot,t)$ onto $\mathbb{N}$ so that $\tau(\cdot,t) : \mathbb{N} \to \Gamma$ is an injection,
(3) if the support $N(t)$ of $t$ is finite and consists of $k(t) \geq 2$ elements, then $N(t)$ is enumerated as $\{\tau(j,t) : j \in J(t) = \{1,\dots,k(t)\}\}$ in such a way that $|t^{\tau(j,t)}| \geq |t^{\tau(j+1,t)}|$ for $1 \leq j \leq k(t)-1$ and we extend $\tau(\cdot,t)$ onto $\mathbb{N}$ so that $\tau(\cdot,t) : \mathbb{N} \to \Gamma$ is an injection,
(4) if $t = 0$, then $J(t) = \emptyset$ and $\tau(\cdot,t) : \mathbb{N} \to \Gamma$ is an arbitrarily chosen injection.

The following result is well known.

**Theorem 2.6** ([4], see also [1] and [11]). *For space $(l^p, \|\cdot\|_p)$ the following inequalities between the norms of two arbitrary $x$ and $y$ of the space are valid (here $q$ is the conjugate index $q = \frac{p}{p-1}$):*

(1) $\|x+y\|_p^p + \|x-y\|_p^p \leq 2^{p-1}\left(\|x\|_p^p + \|y\|_p^p\right)$ *for* $2 \leq p < \infty$,
(2) $\|x+y\|_p^q + \|x-y\|_p^q \leq 2\left(\|x\|_p^p + \|y\|_p^p\right)^{q-1}$ *for* $1 < p \leq 2$.

We will also use some elementary inequalities ([5] and see also [18]). We state them below. These inequalities will play a crucial role in the proofs of our theorems.

**Lemma 2.7** ([5] and [18]). *Assume that*

(1) $s = \{s^i\}_i$ *is a positive and non-increasing sequence,*
(2) $t = \{t^i\}_i \in c_0 \setminus \{0\}$,
(3) $t^i \geq 0$ *for each* $i \in \mathbb{N}$,
(4) $\emptyset \neq I \subset \mathbb{N}$,
(5) *functions* $f, g : I \to \mathbb{N}$ *are injective.*

*Then*

$$\sum_{i\in I} s^{f(i)} \cdot t^{g(i)} \leq \sum_{j=1}^{\infty} s^j \cdot t^{\tau(j,t)}.$$

**Corollary 2.8** ([5] and [18]). *Let $\Gamma$ be an infinite set. Assume that*

(1) $s = \{s^i\}_i$ *is a positive and non-increasing sequence,*
(2) $t = \{t^i\}_i \in c_0(\Gamma) \setminus \{0\}$,
(3) $t^i \geq 0$ *for each* $i \in \Gamma$,
(4) *a function* $f : \mathbb{N} \to \mathbb{N}$ *is injective,*
(5) *a function* $g : \mathbb{N} \to \Gamma$ *is injective.*

*Then*

$$\sum_{j=1}^{\infty} s^{f(j)} \cdot t^{g(j)} \leq \sum_{j=1}^{\infty} s^j \cdot t^{\tau(j,t)}.$$

**Lemma 2.9** ([5] and [18]). *If $\{s^j\}_j$ and $\{t^j\}_j$ are nonnegative and nonincreasing sequences and if a function $g : \mathbb{N} \to \mathbb{N}$ is injective, then*

(1) *for each $m \in \mathbb{N}$ either $g_{|\{1,\dots,m\}}$ permutes $\{1,\dots,m\}$ onto itself and*

$$\sum_{j=1}^{m} s^j t^j - \sum_{j=1}^{m} s^j t^{g(j)} \geq 0$$

*or*

$$\sum_{j=1}^{m} s^j t^j - \sum_{j=1}^{m} s^j t^{g(j)} \geq (s^m - s^{m+1})(t^m - t^{m+1}) \geq 0,$$

(2)
$$\sum_{j=1}^{\infty} s^j t^j \geq \sum_{j=1}^{\infty} s^j t^{g(j)}.$$

As a consequence of Corollary 2.8 and Lemma 2.9 we get

**Lemma 2.10** ([18]). *Assume that*

(1) *$s = \{s^i\}_i$ is a positive and strictly decreasing to $0$,*
(2) *$t = \{t^i\}_i \in c_0 \setminus \{0\}$,*
(3) *$t^i \geq 0$ for each $i \in \mathbb{N}$,*
(4) *$m \in \mathbb{N}$ is such that $t^{\tau(m,t)} > t^{\tau(m+1,t)}$,*
(5) *if $t^{\tau(1,t)} > t^{\tau(m,t)}$, then*
$\omega := \min\Big\{\sum_{j=1}^{m} s^j t^{\tau(j,t)} - \sum_{j=1}^{m} s^j t^{\sigma(j)} : \sigma$ maps $\{1,\dots,m\}$ onto $\{\tau(1,t),\dots,\tau(m,t)\}$ and $\sum_{j=1}^{m} s^j t^{\sigma(j)} < \sum_{j=1}^{m} s^j \cdot t^{\tau(j,t)}\Big\} > 0$
*and $\delta := \min\{(s^m - s^{m+1})(t^{\tau(m,t)} - t^{\tau(m+1,t)}), \omega\} > 0$,*
(6) *if $t^{\tau(1,t)} = t^{\tau(m,t)}$, then $\delta := (s^m - s^{m+1})(t^{\tau(m,t)} - t^{\tau(m+1,t)}) > 0$,*
(7) *$\varphi : \mathbb{N} \to \mathbb{N}$ is injective,*
(8) *$\sum_{j=1}^{m} s^j t^{\tau(j,t)} - \sum_{j=1}^{m} s^j t^{\varphi(j)} < \delta$.*

*Then*

$$\sum_{j=1}^{m} s^j t^{\tau(j,t)} = \sum_{j=1}^{m} s^j t^{\varphi(j)},$$

*$\varphi_{|\{1,\dots,m\}}$ maps $\{1,\dots,m\}$ onto $\{\tau(1,t),\dots,\tau(m,t)\}$ and $t^{\tau(j,t)} = t^{\varphi(j)}$ for $j = 1,\dots,m$.*

**3. A generalization of the Day norm.** In this section we introduce our modification of the Day norm $|||\cdot|||$ in $c_0(\Gamma)$. We replace $l^2(\Gamma)$ with $l^p(\Gamma)$. So fix $1 < p < \infty$ and choose a strictly decreasing positive sequence $\beta = \{\beta_j\}_j$ satisfying the following two conditions

- the series $\sum_{j=1}^{\infty} \beta_j^p$ is convergent,
- there exists a constant $L > 1$ such that for each $n \in \mathbb{N}$

$$\sum_{j=n+1}^{\infty} \beta_j^p \leq L\beta_{n+1}^p.$$

If $u = \{u^i\}_{i\in\Gamma} \in c_0(\Gamma) \setminus \{0\}$, then define $D_{\beta,p}(u) = \{D^i_{\beta,p}(u)\}_{i\in\Gamma} \in l^p(\Gamma)$ by

$$D^i_{\beta,p}(u) = \begin{cases} \beta_j u^{\tau(j,u)}, & \text{if } i = \tau(j,u) \text{ for some } j \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

and set $|||u|||_{\beta,p} = \|D_{\beta,p}(u)\|_p$. For $0 \in c_0$ we set $D^i_{\beta,p}(0) = 0$ for each $i \in \Gamma$ and $D_{\beta,p}(0) = \{D^i_{\beta,p}(0)\}_{i\in\Gamma} = 0 \in l^p(\Gamma)$ and therefore $|||0|||_{\beta,p} = \|D(0)\|_{\beta,p} = 0$.

**Theorem 3.1.** *For each $1 < p < \infty$, $|||\cdot|||_{\beta,p}$ is a norm in $c_0(\Gamma)$ and*

$$\beta_1\|u\|_{c_0(\Gamma)} \leq |||u|||_{\beta,p} \leq \left(\sum_{j=1}^{\infty}\beta_j^p\right)^{\frac{1}{p}}\|u\|_{c_0(\Gamma)}$$

*for each $u \in c_0(\Gamma)$, where $\|\cdot\|_{c_0(\Gamma)}$ is the standard norm in $c_0(\Gamma)$.*

**Proof.** It is obvious that

$$|||\alpha u|||_{\beta,p} = |\alpha|\,|||u|||_{\beta,p}$$

for each $\alpha \in \mathbb{R}$ and each $u \in c_0(\Gamma)$. Next by Corollary 2.8 we have

$$\begin{aligned} |||u+v|||_{\beta,p} = \|D_{\beta,p}(u+v)\|_p &= \left(\sum_{j=1}^{\infty}|\beta_j(u+v)^{\tau(j,u+v)}|^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{j=1}^{\infty}\left|\beta_j u^{\tau(j,u+v)}\right|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{\infty}\left|\beta_j v^{\tau(j,u+v)}\right|^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{j=1}^{\infty}\left|\beta_j u^{\tau(j,u)}\right|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{\infty}\left|\beta_j v^{\tau(j,v)}\right|^p\right)^{\frac{1}{p}} = |||u|||_{\beta,p} + |||v|||_{\beta,p} \end{aligned}$$

for $u = \{u^i\}_i$ and $v = \{v^i\}_i$ in $c_0(\Gamma)$.

Finally, it is easy to observe that

$$\beta_1\|u\|_{c_0(\Gamma)} \leq |||u|||_{\beta,p} \leq \left(\sum_{j=1}^{\infty}\beta_j^p\right)^{\frac{1}{p}}\|u\|_{c_0(\Gamma)}$$

for each $u \in c_0(\Gamma)$. □

**4. The modified Day norm is LUR.** Now we are ready to prove the main theorem of this paper. This theorem generalizes the Rainwater result ([18]).

**Theorem 4.1.** *The Banach space $(c_0(\Gamma), |||\cdot|||_{\beta,p})$ is LUR.*

**Proof.** The proof is based on the Rainwater concept ([18]).

We have to show that if $u \in c_0(\Gamma)$, $u_n \in c_0(\Gamma)$ for $n = 1, 2, \dots$, $\lim_n |||u_n|||_{\beta,p} = |||u|||_{\beta,p}$ and $\lim_n |||u + u_n|||_{\beta,p} = 2|||u|||_{\beta,p}$, then $\lim_n u_n = u$. Observe that without loss of generality we can assume that

(1) $\Gamma = \mathbb{N}$ and therefore $c_0(\Gamma) = c_0(\mathbb{N}) = c_0$,
(2) $|||u|||_{\beta,p} = \lim_n |||u_n|||_{\beta,p} = 1$,
(3) for each $n, i \in \mathbb{N}$ we have $u_n^i \neq 0$ and $u^i + u_n^i \neq 0$, i.e. the supports $N(u_n)$ and $N(u+u_n)$ are equal to $\mathbb{N}$ (in the other case we can replace the sequence $\{u_n\}_n$ by suitably chosen $\{\tilde{u}_n\}_n$ such that $\lim_n(u_n - \tilde{u}_n) = 0$).

Suppose that the sequence $\{u - u_n\}_n$ is not convergent to 0. Then, taking a subsequence if necessary, we see that there exists $\eta > 0$ such that

$$\|u\|_{c_0} \geq \eta \quad \text{and} \quad \|u - u_n\|_{c_0} \geq \eta \tag{i}$$

for each $n \in \mathbb{N}$. Let

$$0 < \lambda < \frac{1}{3(3L)^{\frac{1}{p}}} \tag{ii}$$

and $m$ be the largest integer which satisfies

$$\left| u^{\tau(m,u)} \right| \geq \lambda\eta.$$

Then we have

$$\lambda\eta < \frac{1}{3} \tag{iii}$$

$$\left| u^{\tau(j,u)} \right| < \lambda\eta \tag{iv}$$

for each $j > m$.

Now, by the Clarkson inequalities (see Theorem 2.6) for $2 \leq p < \infty$, we get

$$\begin{aligned}
\text{(v)}\quad & 2^{p-1}\left(|||u|||_{\beta,p}^p + |||u_n|||_{\beta,p}^p\right) - |||u + u_n|||_{\beta,p}^p \\
&= 2^{p-1}\left(\sum_{j=1}^{\infty}\left|\beta_j u^{\tau(j,u)}\right|^p + \sum_{j=1}^{\infty}\left|\beta_j u_n^{\tau(j,u_n)}\right|^p\right) - \sum_{j=1}^{\infty}\left|\beta_j (u+u_n)^{\tau(j,u+u_n)}\right|^p \\
&\geq 2^{p-1}\left(\sum_{j=1}^{\infty}\left|\beta_j u^{\tau(j,u+u_n)}\right|^p + \sum_{j=1}^{\infty}\left|\beta_j u_n^{\tau(j,u+u_n)}\right|^p\right) \\
&\quad - \sum_{j=1}^{\infty}\left|\beta_j (u+u_n)^{\tau(j,u+u_n)}\right|^p \\
&\geq \sum_{j=1}^{\infty}\left|\beta_j (u-u_n)^{\tau(j,u+u_n)}\right|^p = \sum_{j=1}^{\infty}\left|\beta_j\left(u^{\tau(j,u+u_n)} - u_n^{\tau(j,u+u_n)}\right)\right|^p \geq 0
\end{aligned}$$

and for $1 < p \leq 2$ we have

$$\text{(vi)} \quad 2\left(|||u|||_{\beta,p}^p + |||u_n|||_{\beta,p}^p\right)^{q-1} - |||u+u_n|||_{\beta,p}^q$$

$$= 2\left(\sum_{j=1}^{\infty}\left|\beta_j u^{\tau(j,u)}\right|^p + \sum_{j=1}^{\infty}\left|\beta_j u_n^{\tau(j,u_n)}\right|^p\right)^{q-1} - \left[\sum_{j=1}^{\infty}\left|\beta_j (u+u_n)^{\tau(j,u+u_n)}\right|^p\right]^{\frac{q}{p}}$$

$$\geq 2\left(\sum_{j=1}^{\infty}\left|\beta_j u^{\tau(j,u+u_n)}\right|^p + \sum_{j=1}^{\infty}\left|\beta_j u_n^{\tau(j,u+u_n)}\right|^p\right)^{q-1}$$

$$- \left[\sum_{j=1}^{\infty}\left|\beta_j (u+u_n)^{\tau(j,u+u_n)}\right|^p\right]^{\frac{q}{p}}$$

$$\geq \left[\sum_{j=1}^{\infty}\left|\beta_j (u-u_n)^{\tau(j,u+u_n)}\right|^p\right]^{\frac{q}{p}}$$

$$= \left[\sum_{j=1}^{\infty}\left|\beta_j \left(u^{\tau(j,u+u_n)} - u_n^{\tau(j,u+u_n)}\right)\right|^p\right]^{\frac{q}{p}} \geq 0$$

(here $q$ is the conjugate index $q = \frac{p}{p-1}$). Since

$$\lim_n \left[2^{p-1}\left(|||u|||_{\beta,p}^p + |||u_n|||_{\beta,p}^p\right) - |||u+u_n|||_{\beta,p}^p\right] = 0$$

for $p \geq 2$ and

$$\lim_n \left[2\left(|||u|||_{\beta,p}^p + |||u_n|||_{\beta,p}^p\right)^{q-1} - |||u+u_n|||_{\beta,p}^q\right] = 0$$

for $1 < p \leq 2$, we get

$$\text{(vii)} \qquad \lim_n \left[u^{\tau(j,u+u_n)} - u_n^{\tau(j,u+u_n)}\right] = 0$$

for each $j \in \mathbb{N}$ in both cases. Next we observe that (see (v) and (vi))

$$2^{p-1}\left(\sum_{j=1}^{\infty}\left|\beta_j u^{\tau(j,u)}\right|^p + \sum_{j=1}^{\infty}\left|\beta_j u_n^{\tau(j,u_n)}\right|^p\right) - \sum_{j=1}^{\infty}\left|\beta_j (u+u_n)^{\tau(j,u+u_n)}\right|^p$$

$$\geq 2^{p-1}\left(\sum_{j=1}^{\infty}\left|\beta_j u^{\tau(j,u+u_n)}\right|^p + \sum_{j=1}^{\infty}\left|\beta_j u_n^{\tau(j,u+u_n)}\right|^p\right)$$

$$- \sum_{j=1}^{\infty}\left|\beta_j (u+u_n)^{\tau(j,u+u_n)}\right|^p \geq 0$$

for $p \geq 2$ and

$$2\left(\sum_{j=1}^{\infty}\left|\beta_j u^{\tau(j,u)}\right|^p+\sum_{j=1}^{\infty}\left|\beta_j u_n^{\tau(j,u_n)}\right|^p\right)^{q-1}-\left[\sum_{j=1}^{\infty}\left|\beta_j (u+u_n)^{\tau(j,u+u_n)}\right|^p\right]^{\frac{q}{p}}$$

$$\geq 2\left(\sum_{j=1}^{\infty}\left|\beta_j u^{\tau(j,u+u_n)}\right|^p+\sum_{j=1}^{\infty}\left|\beta_j u_n^{\tau(j,u+u_n)}\right|^p\right)^{q-1}$$

$$-\left[\sum_{j=1}^{\infty}\left|\beta_j (u+u_n)^{\tau(j,u+u_n)}\right|^p\right]^{\frac{q}{p}}\geq 0$$

for $1<p\leq 2$. Consequently, since

$$\lim_n\left[2^{p-1}\left(\sum_{j=1}^{\infty}\left|\beta_j u^{\tau(j,u)}\right|^p+\sum_{j=1}^{\infty}\left|\beta_j u_n^{\tau(j,u_n)}\right|^p\right)\right.$$

$$\left.-\sum_{j=1}^{\infty}\left|\beta_j (u+u_n)^{\tau(j,u+u_n)}\right|^p\right]=0$$

and

$$\lim_n\left[2\left(\sum_{j=1}^{\infty}\left|\beta_j u^{\tau(j,u)}\right|^p+\sum_{j=1}^{\infty}\left|\beta_j u_n^{\tau(j,u_n)}\right|^p\right)^{q-1}\right.$$

$$\left.-\left[\sum_{j=1}^{\infty}\left|\beta_j (u+u_n)^{\tau(j,u+u_n)}\right|^p\right]^{\frac{q}{p}}\right]=0$$

for $p\geq 2$ and for $1<p\leq 2$, respectively, we obtain

$$\lim_n\left[\left(\sum_{j=1}^{\infty}\left|\beta_j u^{\tau(j,u)}\right|^p-\sum_{j=1}^{\infty}\left|\beta_j u^{\tau(j,u+u_n)}\right|^p\right)\right.$$

$$\left.+\sum_{j=1}^{\infty}\left(\left|\beta_j u_n^{\tau(j,u_n)}\right|^p-\left|\beta_j u_n^{\tau(j,u+u_n)}\right|^p\right)\right]=0$$

and

$$\lim_n\left[\left(\sum_{j=1}^{\infty}\left|\beta_j u^{\tau(j,u)}\right|^p+\sum_{j=1}^{\infty}\left|\beta_j u_n^{\tau(j,u_n)}\right|^p\right)^{q-1}\right.$$

$$\left.-\left(\sum_{j=1}^{\infty}\left|\beta_j u^{\tau(j,u+u_n)}\right|^p+\sum_{j=1}^{\infty}\left|\beta_j u_n^{\tau(j,u+u_n)}\right|^p\right)^{q-1}\right]=0,$$

respectively. Moreover, by Corollary 2.8

$$\sum_{j=1}^{\infty}\left|\beta_j u^{\tau(j,u)}\right|^p \geq \sum_{j=1}^{\infty}\left|\beta_j u^{\tau(j,u+u_n)}\right|^p$$

and

$$\sum_{j=1}^{\infty}\left|\beta_j u_n^{\tau(j,u_n)}\right|^p \geq \sum_{j=1}^{\infty}\left|\beta_j u_n^{\tau(j,u+u_n)}\right|^p$$

and therefore

$$\text{(viii)} \qquad \lim_n\left(\sum_{j=1}^{\infty}\left|\beta_j u^{\tau(j,u)}\right|^p - \sum_{j=1}^{\infty}\left|\beta_j u^{\tau(j,u+u_n)}\right|^p\right) = 0.$$

Here we can apply Lemma 2.10 with $m$ as above and with $t = \{|u^{\tau(j,u)}|^p\}_j$ and $s = \{\beta_j^p\}_j$ and get $\delta > 0$ such that if

$$\sum_{j=1}^{m} s^j t^j - \sum_{j=1}^{m} s^j t^{\varphi(j)} < \delta,$$

then

$$\sum_{j=1}^{m} s^j t^{\tau(j,t)} = \sum_{j=1}^{m} s^j t^{\varphi(j)},$$

where $\varphi_{|\{1,\dots,m\}}$ maps $\{1,\dots,m\}$ onto $\{\tau(1,t),\dots,\tau(m,t)\}$ and $t^{\tau(j,t)} = t^{\varphi(j)}$ for $j = 1,\dots,m$. By

$$\sum_{j=1}^{k}\left|\beta_j u_n^{\tau(j,u_n)}\right|^p \geq \sum_{j=1}^{k}\left|\beta_j u_n^{\tau(j,u+u_n)}\right|^p$$

for each $k \in \mathbb{N}$ (see Lemma 2.9) and by (viii) we have

$$\lim_n\left(\sum_{j=1}^{m}\left|\beta_j u^{\tau(j,u)}\right|^p - \sum_{j=1}^{m}\left|\beta_j u^{\tau(j,u+u_n)}\right|^p\right) = 0.$$

Hence there is $n_0 \in \mathbb{N}$ such that

$$\sum_{j=1}^{m}\left|\beta_j u^{\tau(j,u)}\right|^p - \sum_{j=1}^{m}\left|\beta_j u^{\tau(j,u+u_n)}\right|^p < \delta$$

for each $n \geq n_0$. This implies that

$$\sum_{j=1}^{m}\left|\beta_j u^{\tau(j,u)}\right|^p = \sum_{j=1}^{m}\left|\beta_j u^{\tau(j,u+u_n)}\right|^p,$$

$$\{\tau(1,u),\dots,\tau(m,u)\} = \{\tau(1,u+u_n),\dots,\tau(m,u+u_n)\}$$

and

$$\left|u^{\tau(j,u)}\right| = \left|u^{\tau(j,u+u_n)}\right|$$

for $j = 1, \dots, m$ and each $n \geq n_0$.

Taking once more a subsequence of $\{u_n\}$ if necessary, we can assume that $\tau(j, u + u_n) = \tilde{\tau}(j)$ for $j = 1, \dots, m$ and each $n \geq n_0$. Therefore, without loss of generality, we can also assume that

$$\tau(j, u) = \tau(j, u + u_n) = \tilde{\tau}(j)$$

for $j = 1, \dots, m$ and each $n \geq n_0$.

Now by (vii) and by $\lim_n |||u_n|||_{\beta,p} = |||u|||_{\beta,p} = 1$ there exists $n_1 \geq n_0$ such that

$$\left|u^{\tau(j,u)} - u_n^{\tau(j,u)}\right| < \eta \tag{ix}$$

for $j = 1, \dots, m$ and $n \geq n_1$,

$$\sum_{j=1}^{m} \beta_j^p \left(\left|u^{\tau(j,u)}\right|^p - \left|u_n^{\tau(j,u)}\right|^p\right) < \frac{\beta_{m+1}^p \eta^p}{3 \cdot 3^p} \tag{x}$$

for $n \geq n_1$ and

$$|||u_n|||_{\beta,p}^p - |||u|||_{\beta,p}^p < \frac{\beta_{m+1}^p \eta^p}{3 \cdot 3^p} \tag{xi}$$

for $n \geq n_1$. Next by (i) for each $n \geq n_1$ we choose $j_n \in \mathbb{N}$ such that

$$\left|u^{\tau(j_n,u-u_n)} - u_n^{\tau(j_n,u-u_n)}\right| = \left|(u - u_n)^{\tau(j_n,u-u_n)}\right| = \|u - u_n\|_{c_0} \geq \eta.$$

Hence by (ix) for each $n \geq n_1$ we have

$$\tau(j_n, u - u_n) \notin \{\tau(1, u), \dots, \tau(m, u)\} = \{\tau(1, u_n), \dots, \tau(m, u_n)\}$$

and therefore by Corollary 2.8 we have

$$|||u_n|||_{\beta,p}^p = \sum_{j=1}^{\infty} \beta_j^p \left|u_n^{\tau(j,u_n)}\right|^p \geq \sum_{j=1}^{m} \beta_j^p \left|u_n^{\tau(j,u)}\right|^p + \beta_{m+1}^p \left|u_n^{\tau(j_n,u-u_n)}\right|^p. \tag{xii}$$

By (ii) and (iv) we also have

$$\begin{aligned} |||u|||_{\beta,p}^p &= \sum_{j=1}^{\infty} \beta_j^p \left|u^{\tau(j,u)}\right|^p < \sum_{j=1}^{m} \beta_j^p \left|u^{\tau(j,u)}\right|^p + \lambda^p \eta^p \sum_{j=m+1}^{\infty} \beta_j^p \\ &< \sum_{j=1}^{m} \beta_j^p \left|u^{\tau(j,u)}\right|^p + \frac{\beta_{m+1}^p \eta^p}{3 \cdot 3^p}. \end{aligned} \tag{xiii}$$

The inequalities (iii), (iv) and (x)–(xiii) lead to the following contradiction

$$2\frac{\beta_{m+1}^p \eta^p}{3^p} < \frac{2^p \beta_{m+1}^p \eta^p}{3^p} = \beta_{m+1}^p \left|\eta - \frac{\eta}{3}\right|^p$$

$$
\begin{aligned}
&\le \beta_{m+1}^p \left| \left| u_n^{\tau(j_n,u-u_n)} - u^{\tau(j_n,u-u_n)} \right| - \left| u^{\tau(j_n,u-u_n)} \right| \right|^p \\
&\le \beta_{m+1}^p \left| u_n^{\tau(j_n,u-u_n)} \right|^p \le \|\|u_n\|\|_{\beta,p}^p - \sum_{j=1}^{m} \beta_j^p \left| u_n^{\tau(j,u)} \right|^p \\
&= \left( \|\|u_n\|\|_{\beta,p}^p - \|\|u\|\|_{\beta,p}^p \right) + \|\|u\|\|_{\beta,p}^p - \sum_{j=1}^{m} \beta_j^p \left| u_n^{\tau(j,u)} \right|^p \\
&< \frac{\beta_{m+1}^p \eta^p}{3 \cdot 3^p} + \left( \|\|u\|\|_{\beta,p}^p - \sum_{j=1}^{m} \beta_j^p \left| u^{\tau(j,u)} \right|^p \right) \\
&\quad + \left( \sum_{j=1}^{m} \beta_j^p \left| u^{\tau(j,u)} \right|^p - \sum_{j=1}^{m} \beta_j^p \left| u_n^{\tau(j,u)} \right|^p \right) \\
&< \frac{\beta_{m+1}^p \eta^p}{3 \cdot 3^p} + \frac{\beta_{m+1}^p \eta^p}{3 \cdot 3^p} + \frac{\beta_{m+1}^p \eta^p}{3 \cdot 3^p} = \frac{\beta_{m+1}^p \eta^p}{3^p}
\end{aligned}
$$

and the proof is complete. □

**Corollary 4.2.** *The Banach space $(c_0(\Gamma), \|\|\cdot\|\|_{\beta,p})$ is strictly convex.*

**Proof.** It is sufficient to use Theorem 4.1 and Remark 2.4. □

**Theorem 4.3.** *The Banach space $(c_0(\Gamma), \|\|\cdot\|\|_{\beta,p})$ is not uniformly convex in every direction.*

**Proof.** Without loss of generality we can assume that $\Gamma = \mathbb{N}$ and let $\{e_i\}_i$ be a standard basis in $c_0 = c_0(\mathbb{N})$. We set $z = e_1$, $u_n = \sum_{i=2}^{n+1} e_i$ and $v_n = u_n + z = \sum_{i=1}^{n+1} e_i$ for $n = 1, 2, \ldots$. Then we have

$$D^i(u_n) = \begin{cases} \beta_i, & \text{if } 2 \le i \le n+1 \\ 0, & \text{for } i > n+1, \end{cases}$$

$$D^i(v_n) = \begin{cases} \beta_i, & \text{if } 1 \le i \le n+1 \\ 0, & \text{for } i > n+1, \end{cases}$$

$$D^i\left(\frac{u_n + v_n}{2}\right) = \begin{cases} \frac{\beta_1}{2}, & \text{for } i = 1 \\ \beta_i, & \text{if } 2 \le i \le n+1 \\ 0, & \text{for } i > n+1 \end{cases}$$

and

$$D^i(z) = \begin{cases} \beta_1, & \text{if } i = 1 \\ 0, & \text{for } i > 1. \end{cases}$$

Hence we get

$$\|\|v_n - u_n\|\|_{\beta,p} = \|\|z\|\|_{\beta,p} = \beta_1 > 0,$$

$$|||u_n|||_{\beta,p} \leq \left(\sum_{j=1}^{\infty} \beta_j^p\right)^{\frac{1}{p}},$$

$$|||v_n|||_{\beta,p} \leq \left(\sum_{j=1}^{\infty} \beta_j^p\right)^{\frac{1}{p}}$$

for $n = 1, 2, \ldots$ and

$$\lim_n \left|\left|\left| \frac{u_n + v_n}{2} \right|\right|\right|_{\beta,p} = \left(\sum_{j=1}^{\infty} \beta_j^p\right)^{\frac{1}{p}}$$

and therefore the Banach space $(c_0, |||\cdot|||_{\beta,p})$ is not uniformly convex in every direction. $\square$

Finally, we recall that in [6] the following theorem is proved.

**Theorem 4.4.** *Let a set $\Gamma$ be uncountable. Then the Banach space $c_0(\Gamma)$ with the max-norm is not isomorphic to a space that is uniformly convex in every direction.*

**5. The modified Day norm and the non-strict Opial property.** Now we recall the Opial property of a Banach space.

**Definition 5.1** ([17]). A Banach space $(X, \|\cdot\|)$ has the Opial property if for each weakly null convergent sequence $\{x_n\}_n$ and each $x \neq 0$ in $X$

$$\limsup_n \|x_n\| < \limsup_n \|x_n - x\|.$$

A Banach space $(X, \|\cdot\|)$ has the non-strict Opial property if for each weakly null convergent sequence $\{x_n\}_n$ and each $x$ in $X$

$$\limsup_n \|x_n\| \leq \limsup_n \|x_n - x\|.$$

In this section we prove the following theorem.

**Theorem 5.2.** *The Banach space $(c_0(\Gamma), |||\cdot|||_{\beta,p})$ has the non-strict Opial property.*

**Proof.** Without loss of generality we can assume that $\Gamma = \mathbb{N}$ and $c_0 = c_0(\mathbb{N})$. Let $\{u_n\} \subset c_0$ tend weakly to $0 \in c_0$ and $u \in c_0 \setminus \{0\}$. Let us take $0 < \epsilon < 1$. Then there exists $\tilde{i} \in \mathbb{N}$ such that

$$|u^i(x)| < \epsilon$$

for each $\tilde{i} < i \in \mathbb{N}$. Therefore

$$|u_n^i| \leq |u_n^i - u^i| + |u^i| < |u_n^i - u^i| + \epsilon$$

for each $\tilde{i} < i \in \mathbb{N}$ and all $n \in \mathbb{N}$.

Now for each $1 \leq i \leq \tilde{i}$ we have either $u^i = 0$ or $u^i \neq 0$. In the second case setting $\eta_i = \min\{\epsilon, \frac{1}{2}|u^i|\}$ and taking into account the weak convergence of $\{u_n\}$ to 0, we find $\tilde{n}_i \in \mathbb{N}$ such that

$$|u_n^i| < \eta_i$$

for $\tilde{n}_i < n \in \mathbb{N}$ and hence we obtain

$$|u_n^i - u^i| \geq |u^i| - |u_n^i| > |u^i| - \eta_i > \frac{1}{2}|u^i| > |u_n^i|.$$

It is obvious that in the first case we have

$$|u_n^i| \leq |u_n^i - u^i|.$$

This implies that

$$|u_n^i| \leq |u_n^i - u^i|$$

for each $1 \leq i \leq \tilde{i}$ and all $\max\{\tilde{n}_1, \ldots, \tilde{n}_{\tilde{i}}\} < n \in \mathbb{N}$.

Putting together all above inequalities we get

$$\text{(xiv)} \qquad |u_n^i| \leq |u_n^i - u^i| + \epsilon$$

for each $i \in \mathbb{N}$ and for all $\max\{\tilde{n}_1, \ldots, \tilde{n}_{\tilde{i}}\} < n \in \mathbb{N}$.

Here observe that replacing $u$ and $u_n$ by suitably chosen $\tilde{v}_n$ and $\tilde{z}_n$ with $\lim_n \tilde{v}_n = u$, $\lim_n(\tilde{z}_n - u_n) = 0$ if necessary, we can assume that all numbers $u_n^i$ and $u_n^i - u^i$ are different from 0.

Now we fix $\max\{\tilde{n}_1, \ldots, \tilde{n}_{\tilde{i}}\} < n \in \mathbb{N}$. We have $D(u_n) = \left\{\beta_j u^{\tau(j,u_n)}\right\}_j$ and $D(u_n - u) = \left\{\beta_j\left(u_n^{\tau(j,u_n-u)} - u^{\tau(j,u_n-u)}\right)\right\}_j$, where $\{\tau(j,u_n)\}_j$ and $\{\tau(j,u_n-u)\}_j$ are suitable permutations of the set $\mathbb{N}$ of natural numbers. Using (xiv) and Corollary 2.8 with $\{s_j\}_j = \{\beta_j^p\}_j$, $\{t_j\}_j = \left\{|u_n^{\tau(j,u_n-u)} - u^{\tau(j,u_n-u)}|^p\right\}_j$ and $\{g(j)\}_j = \{\tau(j,u_n)\}_j$, we obtain

$$\begin{aligned}
\|\|u_n - u\|\|_{\beta,p} + \epsilon\left(\sum_{j=1}^{\infty}\beta_j^p\right)^{\frac{1}{p}} &= \left[\sum_{j=1}^{\infty}\left(\beta_j\left|(u_n-u)^{\tau(j,u_n-u)}\right|\right)^p\right]^{\frac{1}{p}} + \epsilon\left(\sum_{j=1}^{\infty}\beta_j^p\right)^{\frac{1}{p}} \\
&\geq \left[\sum_{j=1}^{\infty}\left(\beta_j\left|(u_n-u)^{\tau(j,u_n)}\right|\right)^p\right]^{\frac{1}{p}} + \epsilon\left(\sum_{j=1}^{\infty}\beta_j^p\right)^{\frac{1}{p}} \\
&\geq \left\{\sum_{j=1}^{\infty}\left[\beta_j\left(\left|u_n^{\tau(j,u_n)} - u^{\tau(j,u_n)}\right| + \epsilon\right)\right]^p\right\}^{\frac{1}{p}} \\
&\geq \left[\sum_{j=1}^{\infty}\left(\beta_j\left|u_n^{\tau(j,u_n)}\right|\right)^p\right]^{\frac{1}{p}} = \|\|u_n\|\|_{\beta,p}.
\end{aligned}$$

Since $0 < \epsilon < 1$ is arbitrarily chosen, by passing $n$ to $+\infty$, we get

$$\|\|u_n\|\|_{\beta,p} \leq \|\|u_n - u\|\|_{\beta,p}.$$

$\square$

Observe that the Banach space $(c_0(\Gamma), |||\cdot|||_{\beta,p})$ does not have the Opial property as the following example shows.

**Example 5.3.** Consider $(c_0, |||\cdot|||_{\beta,p})$ with the standard basis $\{e_i\}_i$. Let us take a sequence $\{u_n\}_n = \{e_{n+1} + \cdots + e_{n+n}\}_n$. This sequence is weakly convergent to $0 \in c_0$ and for $u = e_1$ we have

$$\lim_n |||u_n|||_{\beta,p} = \lim_n |||u_n - u|||_{\beta,p} = \left(\sum_{j=1}^{\infty} \beta_j^p\right)^{\frac{1}{p}}.$$

**6. The modified Day norm and smoothness.** We begin with the following definition.

**Definition 6.1** (see for example [12]). A Banach space $(X, \|\cdot\|_X)$ is smooth if for each $x \in X$ with $\|x\|_X = 1$ there exists a unique functional $x^* \in X^*$ with $\|x^*\|_{X^*} = 1$ such that $x^*(x) = 1$.

In this section we extend the Day result ([5]).

**Theorem 6.2.** *The Banach space* $(c_0(\Gamma), |||\cdot|||_{\beta,p})$ *is not smooth.*

**Proof.** Without loss of generality we can assume that $\Gamma = \mathbb{N}$, $c_0 = c_0(\mathbb{N})$ and $\beta_1 > \beta_2$, and let $\{e_i\}_i$ be a standard basis in $c_0$. Similarly as in [5] we take the plane $X_1 = \operatorname{span}\{e_1, e_2\}$. It is easy to observe that the point

$$\frac{1}{(\beta_1^p + \beta_2^p)^{\frac{1}{p}}} e_1 + \frac{1}{(\beta_1^p + \beta_2^p)^{\frac{1}{p}}} e_2$$

is a corner of the unit sphere $S_{|||\cdot|||_{\beta,p}}$ in $X_1$. So the Banach space $(c_0(\Gamma), |||\cdot|||_{\beta,p})$ is not smooth. $\square$

**7. The modified Day norm and normal structure.** Normal structure is strictly connected with the diameter of a set (see [9] and [10]).

**Definition 7.1.** Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space. For a nonempty, bounded and convex $C \subset X$ the number

$$r_{\|\cdot\|}(C, C) = \inf\{\sup\{\|y - y'\| : y' \in C\} : y \in C\}$$

is called the Chebyshev self-radius of $C$.

**Definition 7.2.** Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space and $C$ a nonempty, bounded and convex subset of $X$. We say that the set $C$ is diametral if $r_{\|\cdot\|}(C, C) = \operatorname{diam}_{\|\cdot\|}(C)$.

**Definition 7.3.** Let $(X, \|\cdot\|)$ be a Banach space. A convex set $C$ of $X$ has a normal structure if for every bounded and convex subset $C_1$ of $C$ with $\operatorname{diam}(C_1) > 0$ we have $r_{\|\cdot\|}(C_1, C_1) < \operatorname{diam}_{\|\cdot\|}(C_1)$.

In particular a Banach space $(X, \|\cdot\|)$ has a normal structure if it does not contain any diametral set, i.e. if $r_{\|\cdot\|}(C, C) < \operatorname{diam}_{\|\cdot\|}(C)$ for each nonempty, non-singleton, bounded and convex set $C \subset X$.

M. S. Brodski and D. P. Milman characterized the normal structure in terms of a diametral sequence.

**Definition 7.4** ([3]). Let $(X, \|\cdot\|)$ be a Banach space. A bounded and not eventually constant sequence $\{x_n\}$ in $(X, \|\cdot\|)$ is said to be diametral if

$$\lim_n \operatorname{dist}_{\|\cdot\|}(x_{n+1}, \operatorname{conv}\{x_1, \ldots, x_n\}) = \operatorname{diam}_{\|\cdot\|}\{x_1, x_2, \ldots\}.$$

**Theorem 7.5** ([3]). *A bounded and convex $C$ of a Banach space $(X, \|\cdot\|)$ has normal structure if and only if it does not contain a diametral sequence.*

**Theorem 7.6.** *The Banach space $(c_0(\Gamma), |||\cdot|||_{\beta,p})$ does not have normal structure.*

**Proof.** Without loss of generality we can assume that $\Gamma = \mathbb{N}$ and let $\{e_i\}_i$ be a standard basis in $c_0 = c_0(\mathbb{N})$. We set $x_1 = e_1$ and

$$x_n = \sum_{i=\frac{n(n+1)}{2}+1}^{\frac{(n+1)(n+2)}{2}} e_i$$

for $n = 2, \ldots$. Then we have

$$\lim_n \operatorname{dist}_{|||\cdot|||_{\beta,p}}(x_{n+1}, \operatorname{conv}\{x_1, \ldots, x_n\}) = \left(\sum_{j=1}^{\infty} \beta_j^p\right)^{\frac{1}{p}} = \operatorname{diam}_{|||\cdot|||_{\beta,p}}\{x_1, x_2, \ldots\}.$$

□

**8. The modified Day norm and asymptotic normal structure.** The notion of asymptotic normal structure was introduced in [2].

**Definition 8.1.** Let $(X, \|\cdot\|)$ be a Banach space. If for each nonempty, non-singleton, bounded and convex set $C \subset X$ and for each sequence $\{x_n\}_n$ in $C$ satisfying $x_n - x_{n+1} \to 0$ as $n \to \infty$, there exists a point $\tilde{x} \in C$ such that $\liminf_n \|x_n - \tilde{x}\| < \operatorname{diam}_{\|\cdot\|}(\mathrm{C})$, then we say that a Banach space $(X, \|\cdot\|)$ has asymptotic normal structure.

**Theorem 8.2.** *The Banach space $(c_0(\Gamma), |||\cdot|||_{\beta,p})$ does not have asymptotic normal structure.*

**Proof.** Without loss of generality we can assume that $\Gamma = \mathbb{N}$ and let $\{e_k\}_k$ be a standard basis in $c_0 = c_0(\mathbb{N})$. We set $u_1 = e_1$ and

$$u_i = \sum_{k=\frac{i(i+1)}{2}+1}^{\frac{(i+1)(i+2)}{2}} e_k$$

for $i = 2, 3, \dots,$

$$x_n = \begin{cases} (1 - \frac{j}{2^{2i}})u_i + u_{i+1}, & \text{if } n = 2^{2i} + j, \quad j = 1, 2, \dots, 2^{2i} \\ u_{i+1} + \frac{j}{2^{2i+1}} u_{i+2}, & \text{if } n = 2^{2i+1} + j, \quad j = 1, 2, \dots, 2^{2i+1}. \end{cases}$$

and

$$C = \overline{\text{conv}}\{x_n : n = 5, 6, \dots\}.$$

(see [16] and also [2]). Then we have

$$0 = \lim_n \|x_n - x_{n+1}\|_{c_0} = \lim_n |||x_n - x_{n+1}|||_{\beta,p}$$

and

$$\text{diam}_{|||\cdot|||_{\beta,p}}(C) = \left( \sum_{j=1}^{\infty} \beta_j^p \right)^{\frac{1}{p}} = \lim_n |||x_n - x|||_{\beta,p}$$

for each $x \in C$. □

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