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On strong proximality in normed linear spaces

ABSTRACT. The paper deals with strong proximality in normed linear spaces. It is proved that in a compactly locally uniformly rotund Banach space, proximality, strong proximality, weak approximative compactness and approximative compactness are all equivalent for closed convex sets. How strong proximality can be transmitted to and from quotient spaces has also been discussed.

1. Introduction. Let W be a closed subset of a normed linear space $(X, \|\cdot\|)$. The *metric projection* of X onto W is the set-valued map P_W defined by $P_W(x) = \{y \in W : \|x - y\| \leq \|x - w\| \text{ for all } w \in W\}$. The set W is said to be *proximal* (*Chebyshev*) if for every $x \in X$, $P_W(x)$ is non-empty (a singleton).

A stronger form of proximality, called strong proximality by Godefroy and Indumathi [6] has been discussed by several researchers (see e.g. [1], [3], [5]–[8] and references cited therein). Vlasov [11] has also studied this concept under the name H-sets.

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A proximal subset W of a normed linear space $(X, \|\cdot\|)$ is said to be *strongly proximal* at $x \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $y \in P_W(x, \delta)$ there is some $y' \in P_W(x)$ satisfying $\|y - y'\| < \varepsilon$ or equivalently, $P_W(x, \delta) \subseteq P_W(x) + \varepsilon B_X$, where $P_W(x, \delta) = \{y \in W : \|x - y\| < d(x, W) + \delta\}$ and B_X is the unit ball in X . The set W is said to be strongly proximal in X if it is strongly proximal at all points of X .

A proximal set need not be strongly proximal (see [3]), even a Chebyshev set need not be strongly proximal (see [8]).

A subset W of a normed linear space X is said to be *approximately compact* [4] (*weakly approximately compact*) for $x \in X$, if every minimizing sequence $\{y_n\} \subseteq W$ for x , i.e. $\|x - y_n\| \rightarrow d(x, W)$, has a convergent subsequence in W (a weakly convergent subsequence in W).

A subset W of a normed linear space X is said to be *strongly Chebyshev* [1] for $x \in X$, if every minimizing sequence $\{y_n\} \subseteq W$ for x is convergent in W .

The set W is said to be approximately compact or weakly approximately compact or strongly Chebyshev in X if it is so at every point $x \in X$.

It is known (see [1]) that approximately compact sets are strongly proximal and strongly Chebyshev sets are precisely the sets which are strongly proximal and Chebyshev.

A normed linear space X is said to be *locally uniformly rotund (LUR)* if for every $x \in S_X = \{x \in X : \|x\| = 1\}$ and every sequence $\{x_n\}$ in S_X satisfying $\|x + x_n\| \rightarrow 2$, we have $x_n \rightarrow x$.

A normed linear space X is said to be *compactly locally uniformly rotund (CLUR)* if for every $x \in S_X$ and every sequence $\{x_n\}$ in S_X satisfying $\|x + x_n\| \rightarrow 2$, the sequence $\{x_n\}$ has a convergent subsequence.

A normed linear space X is said to be *compactly weakly locally uniformly rotund (CWLUR)* if for every $x \in S_X$ and every sequence $\{x_n\}$ in S_X satisfying $\|x + x_n\| \rightarrow 2$, the sequence $\{x_n\}$ has a weakly convergent subsequence.

A normed linear space X is said to have *property (H)* if for any $\{x_n\} \subseteq S_X$ and $x \in S_X$ satisfying $x_n \rightarrow x$ weakly, we have $x_n \rightarrow x$.

Clearly, every finite-dimensional normed linear space is CLUR, and LUR normed linear spaces are CLUR. It is easy to prove (see [9]) that a normed linear space is LUR if and only if it is rotund and CLUR. Moreover, CLUR spaces have property (H).

In this paper, we prove some results concerning strong proximality in normed linear spaces. We see how strong proximality can be transmitted to and from quotient spaces, and prove that for a proximal linear subspace M of a normed linear space X , if $W \supseteq M$ is strongly Chebyshev in X then W/M is also strongly Chebyshev in X/M . We also prove that in a CLUR Banach space, proximality, strong proximality, weak approximative compactness and approximative compactness are equivalent for closed convex sets.

2. Main results. We start with proving some basic results concerning strong proximality.

Proposition 2.1. *Let W be a linear subspace of a normed linear space $(X, \|\cdot\|)$. If W is strongly proximal at $x \in X$, then $W + y$ is strongly proximal at $x + y$ for all $y \in X$.*

Proof. Let $\varepsilon > 0$ be given. Since W is strongly proximal at x , there exists a $\delta > 0$ such that $P_W(x, \delta) \subseteq P_W(x) + \varepsilon B_X$. This implies that $P_W(x, \delta) + y \subseteq [P_W(x) + \varepsilon B_X] + y$ for all $y \in X$, i.e., $P_{W+y}(x + y, \delta) \subseteq P_{W+y}(x + y) + \varepsilon B_X$ for all $y \in X$. Hence $W + y$ is strongly proximal at $x + y$. \square

Proposition 2.2. *Let W be a linear subspace of a normed linear space $(X, \|\cdot\|)$. If W is strongly proximal at x , then W is strongly proximal at αx for every scalar α .*

Proof. Suppose $\alpha = 0$, then $\alpha x = 0$. As $0 \in W$, $P_W(0) = \{0\}$. For any $\varepsilon > 0$, take $\delta = \varepsilon$. Then the inclusion $P_W(0, \varepsilon) \subseteq P_W(0) + \varepsilon B_X$ implies that W is strongly proximal at 0. Now, suppose $\alpha \neq 0$. Let $\varepsilon > 0$ be arbitrary and $x \in X$. Since W is strongly proximal at x , for $\frac{\varepsilon}{|\alpha|} > 0$, there exists some $\delta_1 > 0$ such that for every $y \in P_W(x, \delta_1)$ there is some $y' \in P_W(x)$ satisfying $\|y - y'\| < \frac{\varepsilon}{|\alpha|}$.

Let $\delta = |\alpha|\delta_1$ and $z \in P_W(\alpha x, \delta)$, then $\|\alpha x - z\| < \|\alpha x - w\| + \delta$ for all $w \in W$. This implies that $\|x - \frac{z}{|\alpha|}\| < \|x - w'\| + \frac{\delta}{|\alpha|}$ for all $w' \in W$, i.e., $\frac{z}{|\alpha|} \in P_W(x, \delta_1)$. Since W is strongly proximal at x , there exists $z' \in P_W(x)$ satisfying $\|\frac{z}{|\alpha|} - z'\| < \frac{\varepsilon}{|\alpha|}$. Then for any $z \in P_W(\alpha x, \delta)$ there exists $z' \in P_W(x)$, i.e., $\alpha z' \in P_W(\alpha x)$ satisfying $\|z - \alpha z'\| = |\alpha| \|\frac{z}{|\alpha|} - z'\| < \varepsilon$. Therefore, W is strongly proximal at αx for $\alpha \neq 0$ and hence for every scalar α . \square

It is known (see [10]) that if W is a Chebyshev subset of a normed linear space $(X, \|\cdot\|)$, then $P_W(x) = P_W(\alpha x + (1 - \alpha)P_W(x))$ for every scalar $\alpha \in [0, 1]$. Using this property, we show that a similar result is true for strong proximality.

Theorem 2.3. *Let W be a Chebyshev subset of a normed linear space $(X, \|\cdot\|)$. If W is strongly proximal at x , then W is strongly proximal at $\alpha x + (1 - \alpha)P_W(x)$ for every scalar $\alpha \in [0, 1]$.*

Proof. Let $\varepsilon > 0$ be arbitrary and $x \in X$. Since W is strongly proximal at x , there exists a $\delta_1 > 0$ such that for every $y \in P_W(x, \delta_1)$ there is some $y' \in P_W(x)$ satisfying $\|y - y'\| < \varepsilon$.

Let $z = \alpha x + (1 - \alpha)P_W(x)$, $0 \leq \alpha \leq 1$ and $P_W(x) = \{y'\}$. Then $P_W(z) = \{y'\}$ and

$$(2.1) \quad \|x - z\| + \|z - y'\| = \|x - y'\|.$$

Suppose $z' \in P_W(z, \delta)$, $\delta = \delta_1$. Then

$$(2.2) \quad \|z - z'\| < \|z - w\| + \delta$$

for all $w \in W$.

We claim that $z' \in P_W(x, \delta)$. Using (2.2), we obtain

$$\|x - z'\| \leq \|x - z\| + \|z - z'\| < \|x - z\| + \|z - w\| + \delta$$

for all $w \in W$, i.e. $\|x - z'\| < \|x - z\| + \|z - y'\| + \delta$, as $y' \in W$. By (2.1), this gives

$$\|x - z'\| < \|x - y'\| + \delta,$$

i.e., $\|x - z'\| < \|x - w\| + \delta$ for all $w \in W$, as $y' \in P_W(x)$. Therefore, $z' \in P_W(x, \delta)$. Since W is strongly proximal at x , for $\{y'\} = P_W(x)$, we have $\|z' - y'\| < \varepsilon$. Thus for any $z' \in P_W(z, \delta)$ there exists $\{y'\} = P_W(z)$ satisfying $\|z' - y'\| < \varepsilon$. Hence W is strongly proximal at $z = \alpha x + (1 - \alpha)P_W(x)$ for every scalar α , $0 \leq \alpha \leq 1$. \square

Concerning the strong proximality in quotient spaces, we have the following result.

Theorem 2.4. *Let M be a closed linear subspace of a normed linear space $(X, \|\cdot\|)$ and W a linear subspace of X such that $W \supseteq M$. If W is strongly proximal at x , then W/M is strongly proximal at $x + M$.*

Proof. Let $\varepsilon > 0$ be arbitrary and $x \in X$. Since W is strongly proximal at x , there exists a $\delta_1 > 0$ such that for every $y \in P_W(x, \delta_1)$ there is some $y' \in P_W(x)$ satisfying $\|y - y'\| < \varepsilon$.

Let $z + M \in P_{W/M}(x + M, \delta)$, $\delta = \delta_1$. Then

$$\|(x + M) - (z + M)\| < \|(x + M) - (w + M)\| + \delta$$

for all $w + M \in W/M$. This implies

$$\inf_{m \in M} \|(x - z) - m\| < \|x - w\| + \delta$$

for all $w \in W$. Then there exists $m' \in M$ such that

$$\|(x - z) - m'\| < \|x - w\| + \delta$$

for all $w \in W$.

This gives $z + m' \in P_W(x, \delta)$. Since W is strongly proximal at x , there exists $z' \in P_W(x)$ satisfying $\|(z + m') - z'\| < \varepsilon$. Also $z' \in P_W(x)$ gives $z' + M \in P_{W/M}(x + M)$ (see [2]). Therefore,

$$\|(z + M) - (z' + M)\| = \inf_{m \in M} \|(z - z') - m\| \leq \|(z - z') + m'\| < \varepsilon.$$

Hence W/M is strongly proximal at $x + M$. \square

Remarks. (i) If M is a closed linear subspace of a normed linear space X and $W \supseteq M$ is a strongly proximal subspace in X , then W/M is strongly proximal in X/M .

(ii) The authors do not know whether the converse of Theorem 2.4 hold? However, it was proved in [8] that if M is an infinite dimensional proximal Banach space, then M can be embedded isometrically as a nonstrongly proximal hyperplane in another Banach space W . Thus, $\dim W/M = 1$ and so it is strongly proximal in all its super spaces (see [8]). Then W/M is proximal in all its super spaces and so W is proximal in all its super spaces (see [2]). Using the same technique, W can be embedded as a nonstrongly proximal hyperplane in another Banach space.

We require the following lemma given in [2] for our next result.

Lemma 2.5. *Let M be a proximal subspace of a normed linear space $(X, \|\cdot\|)$ and W a linear subspace of X such that $W \supseteq M$. If W is Chebyshev in X , then W/M is Chebyshev in X/M .*

Using the above lemma and Theorem 2.4, we obtain the following theorem.

Theorem 2.6. *Let M be a proximal linear subspace of a normed linear space $(X, \|\cdot\|)$ and W a linear subspace of X such that $W \supseteq M$. If W is strongly Chebyshev in X , then W/M is strongly Chebyshev in X/M .*

It is well known (see [1]) that a Banach X is reflexive if and only if every closed convex subset of X is proximal or if and only if every closed convex subset of X is weakly approximatively compact. Analogously, the following result shows that in a CLUR Banach space X , a closed convex set is proximal if and only if it is weakly approximatively compact or if and only if it is strongly proximal.

Theorem 2.7. *Let W be a closed convex subset of a CLUR Banach space $(X, \|\cdot\|)$ then the following are equivalent:*

- (i) W is proximal.
- (ii) W is weakly approximatively compact.
- (iii) W is approximatively compact.
- (iv) W is strongly proximal.

Proof. (i) \Rightarrow (ii) Let $x \in X$ be arbitrary. If $x \in W$, then the result is obvious, so suppose $x \in X \setminus W$. Without loss of generality, we may assume that $x = 0$. Let $y \in P_W(0)$ and $\delta = d(0, W)$. Suppose that $\{x_n\}$ is a minimizing sequence in W for 0, i.e.,

$$\lim_{n \rightarrow \infty} \|x_n\| = d(0, W) = \delta.$$

Notice that

$$(2.3) \quad \delta \leq \left\| \frac{x_n + y}{2} \right\| \leq \frac{\|x_n\| + \|y\|}{2} \rightarrow \delta.$$

For every $n \in \mathbb{N}$, put $p_n = \delta \frac{x_n}{\|x_n\|}$. Then

$$\left\| \frac{p_n + y}{2} \right\| = \left\| \frac{\delta x_n + y}{2\|x_n\|} \right\|.$$

Using (2.3), we have

$$\lim_{n \rightarrow \infty} \left\| \frac{p_n + y}{2} \right\| = \lim_{n \rightarrow \infty} \left\| \frac{\delta x_n + y}{2\|x_n\|} \right\| = \delta.$$

Since X being CLUR is CWLUR, $\{p_n\}$ has a weakly convergent subsequence $p_{n_i} \rightarrow p$ weakly. This gives $x_{n_i} \rightarrow p$ weakly and hence W is weakly approximatively compact.

(ii) \Rightarrow (iii) Let $\{y_n\} \subseteq W$ be any minimizing sequence for $x \in X \setminus W$, i.e.,

$$\lim_{n \rightarrow \infty} \|x - y_n\| = d(x, W).$$

Since W is weakly approximatively compact, $\{y_n\}$ has a subsequence $y_{n_i} \rightarrow y$ weakly. Since W is closed and convex $y \in W$. Based on the weak lower semi-continuity of the norm, we get

$$\|x - y\| \leq \liminf_{i \rightarrow \infty} \|x - y_{n_i}\| = d(x, W),$$

i.e., $y \in P_W(x)$. Therefore,

$$(2.4) \quad \|x - y\| = d(x, W) = \lim_{i \rightarrow \infty} \|x - y_{n_i}\|.$$

Also as $y_{n_i} \rightarrow y$ weakly, we have $x - y_{n_i} \rightarrow (x - y)$ weakly. Since X is CLUR, it has property (H). Therefore using (2.4), we obtain

$$\|y - y_{n_i}\| = \|(x - y_{n_i}) - (x - y)\| \rightarrow 0.$$

Hence W is approximatively compact.

(iii) \Rightarrow (iv) is proved in [1].

(iv) \Rightarrow (i) is obvious. □

Remark. If W is a closed convex subset of a LUR Banach space X , then the proximality of W implies that every minimizing sequence in W is convergent.

Since for a closed convex subset of a LUR Banach space, best approximation if it exist, is always unique, we obtain

Corollary 2.8. *Let W be a closed convex subset of a LUR Banach space X then the following statements are equivalent:*

- (i) W is weakly approximatively compact.
- (ii) W is approximatively compact.
- (iii) W is strongly proximinal.
- (iv) W is strongly Chebyshev.
- (v) W is Chebyshev.

In general, strong proximality need not imply approximative compactness.

Example 2.9. Let $X = l_\infty$, $W = c_0$. Then W being an M -ideal is strongly proximal (see [5]) in X . But, for $x = (1, 1, 1, \dots) \in l_\infty$, the sequence $y_n = (1, 1, \dots, 1, 0, 0, \dots) \in W$ is minimizing sequence for x but $\{y_n\}$ has no convergent subsequence.

Analogous to Theorem 2.7, we have the following result.

Theorem 2.10. *Let W be a closed convex subset of a CWLUR Banach space $(X, \|\cdot\|)$ then the following are equivalent:*

- (i) W is proximal.
- (ii) W is weakly approximatively compact.

Proof. (i) \Rightarrow (ii) The proof runs on similar lines as that of Theorem 2.7. (ii) \Rightarrow (i) is proved in [12]. \square

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