ANNALES
UNIVERSITATIS MARIAE CURIE – SKŁODOWSKA
LUBLIN – POLONIA

VOL. LVI, 6 SECTIO A 2002

WŁODZIMIERZ M. MIKULSKI

The natural affinors on dual r-jet prolongations of bundles of 2-forms

Abstract. Let $J^r(\Lambda^2 T^*)M$ be the r-jet prolongation of $\Lambda^2 T^* M$ of an n-dimensional manifold M. For natural numbers r and $n \geq 3$ all natural affinors on $(J^r(\Lambda^2 T^*)M)^*$ are the constant multiples of the identity affinor only.

0. Let us recall the following definitions (see e.g. [4]).

Let $F : \mathcal{M}f_n \to \mathcal{FM}$ be a functor from the category $\mathcal{M}f_n$ of all n-dimensional manifolds and their local diffeomorphisms into the category $\mathcal{FM}$ of fibered manifolds. Let B be the base functor from the category of fibered manifolds to the category of manifolds.

A *natural bundle* over n-manifolds is a functor F satisfying $B \circ F = \mathrm{id}$ and the localization condition: for every inclusion of an open subset $i_U : U \to M$, FU is the restriction $p_M^{-1}(U)$ of $p_M : FM \to M$ over U and Fi_U is the inclusion $p_M^{-1}(U) \to FM$.

An *affinor* Q on a manifold M is a tensor type $(1,1)$, i.e. a linear morphism $Q : TM \to TM$ over id_M.

1991 *Mathematics Subject Classification.* 58A20, 53A55.
Key words and phrases. natural bundles, natural transformations, natural affinors.

A *natural affinor* on a natural bundle F is a system of affinors $Q : TFM \to TFM$ on FM for every n-manifold M satisfying $TFf \circ Q = Q \circ TFf$ for every local diffeomorphism $f : M \to N$.

A *connection* on a fibre bundle Y is an affinor $\Gamma : TY \to TY$ on Y such that $\Gamma \circ \Gamma = \Gamma$ and $im(\Gamma) = VY$, the vertical bundle of Y.

A *natural connection* on a natural bundle F is a system of connections $\Gamma : TFM \to TFM$ on FM for every n-manifold M which is (additionally) a natural affinor on F.

In [5] it was shown how natural affinors Q on some natural bundles FM can be used to study the torsion $\tau = [\Gamma, Q]$ of connections Γ on the same bundles FM. That is why natural affinors have been classified in many papers, [1]-[3], [6]-[11].

In this paper one considers the natural bundle $F = (J^r(\Lambda^2T^*))^*$ which associates to every n-manifold M the vector bundle $(J^r(\Lambda^2T^*))^*M = (J^r(\Lambda^2T^*)M)^*$, where $J^r(\Lambda^2T^*)M = \{j^r_x\omega \mid \omega \text{ is a 2-form on } M\ ,\ x \in M\}$, and to every embedding $\varphi : M \to N$ of n-manifolds the induced vector bundle mapping $(J^r(\Lambda^2T^*))^*\varphi = (J^r(\Lambda^2T^*)\varphi^{-1})^* : (J^r(\Lambda^2T^*)M)^* \to (J^r(\Lambda^2T^*)N)^*$, where the map $J^r(\Lambda^2T^*)\varphi : J^r(\Lambda^2T^*)M \to J^r(\Lambda^2T^*)N$ is given by $j^r_x\omega \to j^r_{\varphi(x)}(\varphi_*\omega)$.

For integers $r \geq 1$ and $n \geq 3$ we classify all natural affinors on $(J^r(\Lambda^2T^*))^*M$. We prove that every natural affinor Q on $(J^r(\Lambda^2T^*))^*M$ is proportional to the identity affinor.

We note that the classification of natural affinors on $(J^rT^*M)^*$ is different. In [9] we proved that for $n \geq 2$ the vector space of all natural affinors on $(J^rT^*M)^*$ is 2-dimensional.

The above result shows that "torsion" of a connection Γ on $(J^r(\Lambda^2T^*))^*M$ makes no sense because of $[\Gamma, \mathrm{id}] = 0$.

The above result also shows that for integers $r \geq 1$ and $n \geq 3$ there are no natural connections on $(J^r(\Lambda^2T^*))^*$ over n-manifolds.

The usual coordinates on $\mathbf{R}^n$ are denoted by x^i and $\partial_i = \frac{\partial}{\partial x^i}$, $i = 1, ..., n$.

All manifolds and maps are assumed to be of class C^∞.

1. We start with the classification of all linear natural transformations $A : T(J^r(\Lambda^2T^*))^*M \to (J^r(\Lambda^2T^*))^*M$ in the sense of [4] over n-manifolds M.

A *natural transformation* $T(J^r(\Lambda^2T^*))^* \to (J^r(\Lambda^2T^*))^*$ over n-manifolds is a system of fibered maps $A : T(J^r(\Lambda^2T^*))^*M \to (J^r(\Lambda^2T^*))^*M$ over id_M for every n-manifold M satisfying $(J^r(\Lambda^2T^*))^*f \circ A = A \circ T(J^r(\Lambda^2T^*))^*f$ for every local diffeo. $f : M \to N$. The linearity means that A gives a linear map $T_y(J^r(\Lambda^2T^*))^*M \to (J^r(\Lambda^2T^*))^*_xM$ for any $y \in (J^r(\Lambda^2T^*))^*_xM$, $x \in M$.

Proposition 1. *If $n \geq 3$ and r are natural numbers then every linear natural transformation $A : T(J^r(\Lambda^2T^*))^* \to (J^r(\Lambda^2T^*))^*$ over n-manifolds is* 0.

Proof. Every element from the fibre $(J^r(\Lambda^2T^*))^*_0\mathbf{R}^n$ is a linear combination of the $(j^r_0(x^\alpha dx^i \wedge dx^j))^*$ for all $\alpha \in (\mathbf{N} \cup \{0\})^n$ with $|\alpha| \leq r$ and $i, j = 1, ..., n$, $i < j$, where the $(j^r_0(x^\alpha dx^i \wedge dx^j))^*$ form the basis dual to the $j^r_0(x^\alpha dx^i \wedge dx^j) \in (J^r(\Lambda^2T^*))_0\mathbf{R}^n$ for α and i, j as beside.

Consider a linear natural transformation $A : T(J^r(\Lambda^2T^*))^* \to (J^r(\Lambda^2T^*))^*$ over n-manifolds.

Clearly, A is uniquely determined by the values $\langle A(u), j^r_0(x^\alpha dx^i \wedge dx^j)\rangle \in \mathbf{R}$ for $u \in (T(J^r(\Lambda^2T^*))^*\mathbf{R}^n)_0 \tilde{=} \mathbf{R}^n \times (V(J^r(\Lambda^2T^*))^*\mathbf{R}^n)_0 \tilde{=} \mathbf{R}^n \times (J^r(\Lambda^2T^*))^*_0 \mathbf{R}^n \times (J^r(\Lambda^2T^*))^*_0\mathbf{R}^n$, $\alpha \in (\mathbf{N} \cup \{0\})^n$ with $|\alpha| \leq r$ and $i, j = 1, ..., n$, $i < j$, where $\tilde{=}$ is the standard trivialization and the canonical identification.

Since A is invariant with respect to the coordinate permutations, A is uniquely determined by the values $\langle A(u), j^r_0(x^\alpha dx^1 \wedge dx^2)\rangle$, where u and α are as above.

If $|\alpha| \geq 1$, then the local diffeomorphisms $\varphi_\alpha = (x^1, x^2, x^3 + x^\alpha, x^4, ..., x^n)^{-1}$ sends $j^r_0(x^3dx^1 \wedge dx^2)$ into $j^r_0(x^3dx^1 \wedge dx^2) + j^r_0(x^\alpha dx^1 \wedge dx^2)$. Then by the invariance of A with respect to the φ's, A is uniquely determined by the values $\langle A(u), j^r_0(x^3dx^1 \wedge dx^2)\rangle \in \mathbf{R}$ and $\langle A(u), j^r_0(dx^1 \wedge dx^2)\rangle \in \mathbf{R}$, where $u \in (T(J^r(\Lambda^2T^*))^*\mathbf{R}^n)_0 \tilde{=} \mathbf{R}^n \times (J^r(\Lambda^2T^*))^*_0\mathbf{R}^n \times (J^r(\Lambda^2T^*))^*_0\mathbf{R}^n$.

The proof of Proposition 1 will be complete after proving that $\langle A(u), j^r_0(dx^1 \wedge dx^2)\rangle = 0$ and $\langle A(u), j^r_0(x^3dx^1 \wedge dx^2)\rangle = 0$ for any $u \in (T(J^r(\Lambda^2T^*))^*\mathbf{R}^n)_0 \tilde{=} \mathbf{R}^n \times (J^r(\Lambda^2T^*))^*_0\mathbf{R}^n \times (J^r(\Lambda^2T^*))^*_0\mathbf{R}^n$. We will prove these conditions in Lemmas 1 — 6.

At first we study the values $\langle A(u), j^r_0(dx^1 \wedge dx^2)\rangle$.

Lemma 1. *There exist the numbers $\lambda, \mu, \nu \in \mathbf{R}$ such that*

$$(1) \qquad \langle A(u), j^r_0(dx^1 \wedge dx^2)\rangle = \lambda u^1_1 u^2_1 + \mu u_{2,(0),1,2} + \nu u_{3,(0),1,2}$$

for every $u = (u_1, u_2, u_3) \in \mathbf{R}^n \times (J^r(\Lambda^2T^*))^*_0\mathbf{R}^n \times (J^r(\Lambda^2T^*))^*_0\mathbf{R}^n$, where $u_1 = (u^1_1, ..., u^n_1) \in \mathbf{R}^n$, $u_{\tau,\alpha,i,j}$ is the coefficient of $u_\tau \in (J^r(\Lambda^2T^*))^*_0\mathbf{R}^n$ on $(j^r_0(x^\alpha dx^i \wedge dx^j))^*$, $\tau = 2, 3$, $(0) = (0, ..., 0) \in (\mathbf{N} \cup \{0\})^n$.

Proof of Lemma 1. By the naturality of A with respect to the homotheties $a_t = (t^1x^1, ..., t^nx^n)$ for $t = (t^1, ..., t^n) \in \mathbf{R}^n_+$,

$$\langle A(T(J^r(\Lambda^2T^*))^*(a_t)(u)), j^r_0(dx^1 \wedge dx^2)\rangle = t^1t^2\langle A(u), j^r_0(dx^1 \wedge dx^2)\rangle$$

for any $t = (t^1, ..., t^n) \in \mathbf{R}^n_+$. For $t \in \mathbf{R}^n$, $i, j = 1, ..., n$, $i < j$ and $\alpha \in (\mathbf{N} \cup \{0\})^n$ we have $T(J^r(\Lambda^2T^*))^*(a_t)((j^r_0(x^\alpha dx^i \wedge dx^j))^*) = t^{\alpha+e_i+e_j}(j^r_0(x^\alpha dx^i \wedge dx^j))^*$. Then the lemma follows from the homogeneous function theorem, [4]. □

Lemma 2. *We have* $\lambda=\mu=\nu=0$.

Proof of Lemma 2. Since $\langle A(u_1,u_2,u_3), j^r_0(dx^1\wedge dx^2)\rangle$ is linear in (u_1,u_3) for u_2, we have $\lambda=\mu=0$. Then (in particular) we have

$$\langle A(\partial^C_{1\ |w}), j^r_0(dx^1\wedge dx^2)\rangle = \langle A(e_1,w,0), j^r_0(dx^1\wedge dx^2)\rangle = 0 \tag{2}$$

for $w\in (J^r(\Lambda^2T^*))^*_0\mathbf{R}^n$, where $(\ )^C$ is the complete lift.

To prove $\nu=0$ it is sufficient to show that

$$\langle A(0,0,(j^r_0(dx^1\wedge dx^2))^*), j^r_0(dx^1\wedge dx^2)\rangle = 0.$$

But we have

$$\begin{aligned} 0 &= \langle A(((x^1)^{r+1}\partial_1)^C_{|w}), j^r_0(dx^1\wedge dx^2)\rangle \\ &= (r+1)\langle A(0,w,(j^r_0(dx^1\wedge dx^2))^* +\ \dots\ ), j^r_0(dx^1\wedge dx^2)\rangle \\ &= (r+1)\langle A(0,0,(j^r_0(dx^1\wedge dx^2))^*), j^r_0(dx^1\wedge dx^2)\rangle\ , \end{aligned} \tag{3}$$

where $w=(j^r_0((x^1)^r dx^1\wedge dx^2))^*$ and the dots mean the linear combination of the $(j^r_0(x^\alpha dx^i\wedge dx^j))^*$ with $(j^r_0(x^\alpha dx^i\wedge dx^j))^*\neq (j^r_0(dx^1\wedge dx^2))^*$.

Let us explain (3).

Let φ_t be the flow of $(x^1)^{r+1}\partial_1$. We have

$$\begin{aligned} &\langle ((x^1)^{r+1}\partial_1)^C_{|w}, j^r_0(dx^1\wedge dx^2)\rangle \\ &= \langle \frac{d}{dt}_{|t=0}(J^r(\Lambda^2T^*))^*_0(\varphi_t)(w), j^r_0(dx^1\wedge dx^2)\rangle \\ &= \frac{d}{dt}_{|t=0}\langle (J^r(\Lambda^2T^*))^*_0(\varphi_t)(w), j^r_0(dx^1\wedge dx^2)\rangle \\ &= \frac{d}{dt}_{|t=0}\langle w, j^r_0((\varphi_{-t})_*(dx^1\wedge dx^2))\rangle \\ &= \langle w, j^r_0(\frac{d}{dt}_{|t=0}(\varphi_{-t})_*(dx^1\wedge dx^2))\rangle \\ &= \langle w, j^r_0(L_{(x^1)^{r+1}\partial_1}(dx^1\wedge dx^2))\rangle \\ &= (r+1)\langle w, j^r_0((x^1)^r dx^1\wedge dx^2)\rangle = r+1\ . \end{aligned}$$

Then $((x^1)^{r+1}\partial_1)^C_{|w} = (r+1)(j^r_0(dx^1\wedge dx^2))^* +\ \dots$ under the canonical isomorphism $V_w((J^r(\Lambda^2T^*))^*\mathbf{R}^n)\tilde{=}(J^r(\Lambda^2T^*))^*_0\mathbf{R}^n$, i.e. $\langle A(((x^1)^{r+1}\partial_1)^C_{|w}), j^r_0(dx^1\wedge dx^2)\rangle = (r+1)\langle A(0,w,(j^r_0(dx^1\wedge dx^2))^* +\ \dots\ ), j^r_0(dx^1\wedge dx^2)\rangle$.

The equality $(r+1)\langle A(0,w,(j^r_0(dx^1\wedge dx^2))^* +\ \dots\ ), j^r_0(dx^1\wedge dx^2)\rangle = (r+1)\langle A(0,0,(j^r_0(dx^1\wedge dx^2))^*), j^r_0(dx^1\wedge dx^2)\rangle$ is clear because of (1) and $\mu=0$.

We can prove the equality $0 = \langle A(((x^1)^{r+1}\partial_1)^C_{|w}), j^r_0(dx^1 \wedge dx^2)\rangle$ as follows. Vector fields $\partial_1 + (x^1)^{r+1}\partial_1$ and ∂_1 have the same r-jets at 0. Then by [11], there exists a diffeomorphism $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ such that $j^{r+1}_0\varphi = \mathrm{id}$ and $\varphi_*\partial_1 = \partial_1 + (x^1)^{r+1}\partial_1$ near 0. Clearly, φ preserves $j^r_0(dx^1 \wedge dx^2)$ because of the jet argument. Then, by the naturality of A with respect to φ, it follows from (2) that

$$\langle A((\partial_1 + (x^1)^{r+1}\partial_1)^C{}_{|w}), j^r_0(dx^1 \wedge dx^2)\rangle = 0$$

for any $w \in (J^r(\Lambda^2T^*))^*_0\mathbf{R}^n$. Now, applying the linearity of A, we end the proof of the equality. □

Now, we study the values $\langle A(u), j^r_0(x^3dx^1 \wedge dx^2)\rangle = 0$.

Lemma 3. *There exist the numbers $a, b, c, e, f, g \in \mathbf{R}$ such that*

$$\begin{aligned}\langle A(u), j^r_0(x^3dx^1 \wedge dx^2)\rangle &= au^1_1u_{2,(0),2,3} + bu^2_1u_{2,(0),1,3} \\ &\quad + cu^3_1u_{2,(0),1,2} + eu_{3,e_1,2,3} + fu_{3,e_2,1,3} + gu_{3,e_3,1,2}\end{aligned} \tag{4}$$

for any $u = (u_1, u_2, u_3)$, where $u_1 = (u^1_1, ..., u^n_1) \in \mathbf{R}^n$, $u_2, u_3 \in (J^r(T^* \wedge T^*))^*_0\mathbf{R}^n$, $u_{\tau,\alpha,i,j}$ is as in Lemma 1 and $e_i = (0, ..., 1, 0, ..., 0) \in (\mathbf{N} \cup \{0\})^n$, 1 in i-position.

Proof of Lemma 3. The proof is similar to the proof of Lemma 1. We apply the naturality of A with respect to the homotheties $a_t = (t^1x^1, ..., t^nx^n)$ for $t = (t^1, ..., t^n) \in \mathbf{R}^n_+$, the homogeneous function theorem and the linearity of A. □

To prove $g = f = e = a = b = c = 0$ we shall use the following

Lemma 4. *For every $u \in (T(J^r(\Lambda^2T^*))^*\mathbf{R}^n)_0$ we have*

$$\langle A(u), j^r_0(x^3dx^1 \wedge dx^2)\rangle = \langle A(u'), j^r_0(x^3dx^1 \wedge dx^2)\rangle\ , \tag{5}$$

where u' is the image of u by $(x^2, x^3, x^1) \times \mathrm{id}_{\mathbf{R}^{n-3}}$.

Proof of Lemma 4. We consider $u \in (T(J^r(\Lambda^2T^*))^*\mathbf{R}^n)_0$. Let $\tilde{u}$ be the image of u by $(x^1 + x^1x^3, x^2, ..., x^n)$. By Lemma 2 we have $\lambda = \mu = \nu = 0$, i.e. $\langle A(\tilde{u}), j^r_0(dx^1 \wedge dx^2)\rangle = \langle A(u), j^r_0(dx^1 \wedge dx^2)\rangle = 0$. Then by the invariance of A with respect to $(x^1 + x^1x^3, x^2, ..., x^n)^{-1}$ we get

$$0 = \langle A(u), j^r_0(dx^1 \wedge dx^2)\rangle + \langle A(u), j^r_0(x^3dx^1 \wedge dx^2)\rangle - \langle A(u), j^r_0(x^1dx^2 \wedge dx^3)\rangle$$

as $(x^1 + x^1x^3, x^2, ..., x^n)^{-1}$ sends $j^r_0(dx^1 \wedge dx^2)$ into $j^r_0(dx^1 \wedge dx^2) + j^r_0(x^3dx^1 \wedge dx^2) - j^r_0(x^1dx^2 \wedge dx^3)$. Hence $\langle A(u), j^r_0(x^3dx^1 \wedge dx^2)\rangle = \langle A(u), j^r_0(x^1dx^2 \wedge dx^3)\rangle$. Therefore we have (5) because $(x^2, x^3, x^1) \times \mathrm{id}_{\mathbf{R}^{n-3}}$ sends $j^r_0(x^1dx^2 \wedge dx^3)$ into $j^r_0(x^3dx^1 \wedge dx^2)$. □

Lemma 5. *We have* $g = f = e = 0$.

Proof of Lemma 5. We have to show

$$\begin{aligned}&\langle A(0,0,(j^r_0(x^3dx^1\wedge dx^2))^*), j^r_0(x^3dx^1\wedge dx^2)\rangle\\&\quad=\langle A(0,0,-(j^r_0(x^2dx^1\wedge dx^3))^*), j^r_0(x^3dx^1\wedge dx^2)\rangle\\&\quad=\langle A(0,0,(j^r_0(x^1dx^2\wedge dx^3))^*), j^r_0(x^3dx^1\wedge dx^2)\rangle=0.\end{aligned}$$

We see that $(x^2,x^3,x^1)\times\mathrm{id}_{\mathbf{R}^{n-3}}$ sends $(j^r_0(x^3dx^1\wedge dx^2))^*$ into $-(j^r_0(x^2dx^1\wedge dx^3))^*$ and $-(j^r_0(x^2dx^1\wedge dx^3))^*$ into $(j^r_0(x^1dx^2\wedge dx^3))^*$. Then due to (5) it suffices to verify that $\langle A(0,0,(j^r_0(x^3dx^1\wedge dx^2))^*), j^r_0(x^3dx^1\wedge dx^2)\rangle=0$. But we have

$$\begin{aligned}0&=\langle A(((x^1)^r\partial_1)^C_{|w}), j^r_0(x^3dx^1\wedge dx^2)\rangle\\&=r\langle A(0,w,(j^r_0(x^3dx^1\wedge dx^2))^*), j^r_0(x^3dx^1\wedge dx^2)\rangle\\&=r\langle A(0,0,(j^r_0(x^3dx^1\wedge dx^2))^*), j^r_0(x^3dx^1\wedge dx^2)\rangle\ ,\end{aligned}\tag{6}$$

where $w=(j^r_0(x^3(x^1)^{r-1}dx^1\wedge dx^2))^*\in(J^r(\Lambda^2T^*))^*_0\mathbf{R}^n$.

Let us explain (6).

That $\langle A(0,w,(j^r_0(x^3dx^1\wedge dx^2))^*), j^r_0(x^3dx^1\wedge dx^2)\rangle=\langle A(0,0,(j^r_0(x^3dx^1\wedge dx^2))^*), j^r_0(x^3dx^1\wedge dx^2)\rangle$ is clear, see (4).

We can prove $0=\langle A(((x^1)^r\partial_1)^C_{|w}), j^r_0(x^3dx^1\wedge dx^2)\rangle$ as follows. Vector fields $\partial_1+(x^1)^r\partial_1$ and ∂_1 have the same $r-1$-jets at 0. Then by [11] there exists a diffeomorphism $\varphi=\varphi_1\times\mathrm{id}_{\mathbf{R}^{n-1}}:\mathbf{R}^n=\mathbf{R}\times\mathbf{R}^{n-1}\to\mathbf{R}^n=\mathbf{R}\times\mathbf{R}^{n-1}$ such that $\varphi_1:\mathbf{R}\to\mathbf{R}$, $j^r_0\varphi=\mathrm{id}$ and $\varphi_*\partial_1=\partial_1+(x^1)^r\partial_1$ near 0. Let φ^{-1} send w into $\tilde w$. Then $\tilde w$ is the linear combination of the $(j^r_0(x^\alpha dx^i\wedge dx^j))^*\in(J^r(\Lambda^2T^*))^*_0\mathbf{R}^n$ for $|\alpha|\ge 1$ and $i,j=1,...,n$ with $i<j$. (For, $\langle\tilde w, j^r_0(dx^i\wedge dx^j)\rangle=\langle w, j^r_0(d(x^i\circ\varphi^{-1})\wedge d(x^j\circ\varphi^{-1}))\rangle=0$.) Then, by (4), $\langle A(\partial^C_{1\,|\tilde w}), j^r_0(x^3dx^1\wedge dx^2)\rangle=\langle A(e_1,\tilde w,0), j^r_0(x^3dx^1\wedge dx^2)\rangle=0$. Clearly, φ preserves $j^r_0(x^3dx^1\wedge dx^2)$. Then, using the naturality of A with respect to φ we get $\langle A((\partial_1+(x^1)^r\partial_1)^C{}_{|w}), j^r_0(x^3dx^1\wedge dx^2)\rangle=0$. Now, applying the linearity of A, we end the proof of equality.

Using the flow argument one can prove $\langle A(((x^1)^r\partial_1)^C_{|w}), j^r_0(x^3dx^1\wedge dx^2)\rangle = r\langle A(0,w,(j^r_0(x^3dx^1\wedge dx^2))^*), j^r_0(x^3dx^1\wedge dx^2)\rangle$ as follows. For any $\alpha\in(\mathbf{N}\cup\{0\})^n$ with $|\alpha|\le r$ and any $i,j=1,...,n$ with $i<j$ we have

$$\begin{aligned}\langle((x^1)^r\partial_1)^C{}_{|w}, j^r_0(x^\alpha dx^i\wedge dx^j)\rangle&=\langle w, j^r_0(L_{(x^1)^r\partial_1}x^\alpha dx^i\wedge dx^j)\rangle\\&=\langle w,\alpha_1 j^r_0((x^1)^{r-1}x^\alpha dx^i\wedge dx^j)\rangle\\&\quad+\langle w, j^r_0(x^\alpha\delta^i_1 r(x^1)^{r-1}dx^1\wedge dx^j)\rangle\ .\end{aligned}$$

Since $w=(j^r_0(x^3(x^1)^{r-1}dx^1\wedge dx^2))^*$, the sum is equal to r if $\alpha=e_3$ and $(i,j)=(1,2)$ and equal to 0 in the other cases. Hence $((x^1)^r\partial_1)^C{}_{|w}=r(j^r_0(x^3dx^1\wedge dx^2))^*\in V_w(J^r(\Lambda^2T^*))^*\mathbf{R}^n$. This ends the proof of $\langle A(((x^1)^r\partial_1)^C_{|w}), j^r_0(x^3dx^1\wedge dx^2)\rangle=r\langle A(0,w,(j^r_0(x^3dx^1\wedge dx^2))^*), j^r_0(x^3dx^1\wedge dx^2)\rangle$. □

Lemma 6. *We have $a = b = c = 0$.*

Proof of Lemma 6. By (5), similarly as for $e = f = g = 0$, it is sufficient to prove that $c = 0$, i.e. $\langle A(\partial^C_3{}_{|(j^r_0(dx^1\wedge dx^2))^*}), j^r_0(x^3dx^1\wedge dx^2)\rangle = 0$. But we have

$$\begin{aligned}0 &= \langle A(\partial^C_3{}_{|(j^r_0((x^1)^r dx^1\wedge dx^2))^*}), j^r_0(x^3dx^1\wedge dx^2)\rangle\\ &= \langle A(\partial^C_3{}_{|(j^r_0(dx^1\wedge dx^2))^*+\dots}), j^r_0(x^3dx^1\wedge dx^2)\rangle\\ &= \langle A(\partial^C_3{}_{|(j^r_0(dx^1\wedge dx^2))^*}), j^r_0(x^3dx^1\wedge dx^2)\rangle\ ,\end{aligned}\tag{7}$$

where the dots denote the linear combination of the $(j^r_0(x^\alpha dx^i\wedge dx^j))^* \neq (j^r_0(dx^1\wedge dx^2))^*$ for $|\alpha|\leq r$ and $i, j = 1, ..., n$, $i < j$.

Let us explain (7).

The equality $0 = \langle A(\partial^C_3{}_{|(j^r_0((x^1)^r dx^1\wedge dx^2))^*}), j^r_0(x^3dx^1\wedge dx^2)\rangle$ follows from (4). Similarly, from (4) we obtain $\langle A(\partial^C_3{}_{|(j^r_0(dx^1\wedge dx^2))^*+\dots}), j^r_0(x^3dx^1\wedge dx^2)\rangle = \langle A(\partial^C_3{}_{|(j^r_0(dx^1\wedge dx^2))^*}), j^r_0(x^3dx^1\wedge dx^2)\rangle$.

We consider the local diffeomorphism $\varphi = (x^1+\frac{1}{r+1}(x^1)^{r+1}, x^2, ..., x^n)^{-1}$. We see that φ^{-1} preserves $j^r_0(x^3dx^1\wedge dx^2)$ and ∂_3. Moreover, we see that φ^{-1} sends $(j^r_0((x^1)^r dx^1\wedge dx^2))^*$ into $(j^r_0(dx^1\wedge dx^2))^*+...$, where the dots are as above, because of $\langle (j^r_0((x^1)^r dx^1\wedge dx^2))^*, j^r_0(\varphi_*(dx^1\wedge dx^2))\rangle = 1$. Now, by the invariance of A with respect to φ^{-1} we get $\langle A(\partial^C_3{}_{|(j^r_0((x^1)^r dx^1\wedge dx^2))^*}), j^r_0(x^3dx^1\wedge dx^2)\rangle = \langle A(\partial^C_3{}_{|(j^r_0(dx^1\wedge dx^2))^*+\dots}), j^r_0(x^3dx^1\wedge dx^2)\rangle$. □

The proof of Proposition 1 is complete. □

2. The tangent map $T\pi : T(J^r(\Lambda^2T^*))^*M \to TM$ of the bundle projection $\pi : (J^r(\Lambda^2T^*))^*M \to M$ defines a linear natural transformation $T\pi : T(J^r(\Lambda^2T^*))^* \to T$ over n-manifolds. (The definition of linear natural transformations $T(J^r(\Lambda^2T^*))^* \to T$ over n-manifolds is similar to the one of Section 1.)

Proposition 2. *If r and $n \geq 2$ are natural numbers, then every linear natural transformation $B : T(J^r(\Lambda^2T^*))^* \to T$ over n-manifolds is proportional to $T\pi$.*

Proof. Due to similar arguments as in the proof of Proposition 1, B is uniquely determined by the values $\langle B(u), d_0x^1\rangle$ for $u\in(T(J^r(\Lambda^2T^*))^*\mathbf{R}^n)_0 \tilde{=} \mathbf{R}^n\times(J^r(\Lambda^2T^*))^*_0\mathbf{R}^n\times(J^r(\Lambda^2T^*))^*_0\mathbf{R}^n$.

By the naturality of B with respect to the homotheties $(t^1x^1, ..., t^nx^n)$ for $t\in\mathbf{R}^n_+$ and the homogeneous function theorem we deduce that $\langle B(.), dx^1\rangle = x^1\circ p_1$, where $p_1 : \mathbf{R}^n\times(J^r(\Lambda^2T^*))^*_0\mathbf{R}^n\times(J^r(\Lambda^2T^*))^*_0\mathbf{R}^n \to \mathbf{R}^n$ is the canonical projection.

Then the vector space of all B as above is 1-dimensional. □

3. The main result of this paper is the following theorem.

Theorem 1. *If $n \geq 3$ and r are natural numbers, then every natural affinor Q on $(J^r(\Lambda^2T^*))^*$ over n-manifolds is a constant multiple of* id.

Proof. Let $Q : T(J^r(\Lambda^2T^*))^*M \to T(J^r(\Lambda^2T^*))^*M$ be a natural affinor on $(J^r(\Lambda^2T^*))^*$ over n-manifolds. Then $B = T\pi \circ Q : T(J^r(\Lambda^2T^*))^* \to T$ is a linear natural transformation. By Proposition 2, $B = T\pi \circ Q = \lambda T\pi$ for some λ. Clearly, $T\pi \circ \mathrm{id} = T\pi$. Then $Q - \lambda\mathrm{id}$ is an affinor of vertical type. Now, applying Proposition 1 we deduce that $Q - \lambda\mathrm{id}$ is the zero affinor. □

From Theorem 1 we obtain immediately

Corollary 1. *If $n \geq 3$ and r are natural numbers, then there is no natural connection on $(J^r(\Lambda^2T^*))^*$ over n-manifolds.*

References

[1] Doupovec, M., *Natural transformations between TTT^*M and TT^*TM*, Czech Math. J. **43(118)** (1993), 599–613.

[2] Doupovec, M., I. Kolář, *Natural affinors on time-dependent Weil bundles*, Arch. Math. (Brno) **27** (1991), 205–209.

[3] Gancarzewicz, J., I. Kolář, *Natural affinors on the extended r-th order tangent bundles*, Suppl. Rendiconti Circolo Mat. Palermo **30** (1993), 95–100.

[4] Kolář I., P.W. Michor and J. Slovák, *Natural operations in differential geometry*, Springer-Verlag, Berlin, 1993.

[5] Kolář, I., M. Modugno, *Torsions of connections on some natural bundles*, Diff. Geom. and Appl. **2** (1992), 1–16.

[6] Kurek, J., *Natural affinors on higher order cotangent bundles*, Arch. Math. (Brno) **28** (1992), 175–180.

[7] Mikulski, W.M., *Natural affinors on r-jet prolongation of the tangent bundle*, Arch. Math. (Brno) **34(2)** (1998), 321–328.

[8] Mikulski, W.M., *The natural affinors on $\otimes^k T^{(r)}$*, Note di Matematica **19(2)** (1999), 269–274.

[9] Mikulski, W.M., *The natural affinors on $(J^rT^*)^*$*, Arch. Math. (Brno) **36(4)** (2000), 261–267.

[10] Mikulski, W.M., *The natural affinors on generalized higher order tangent bundles*, Rend. Mat. Roma **21(VII)** (2001), 339–349.

[11] Zajtz, A., *On the order of natural operators and liftings*, Ann. Polon. Math. **49** (1988), 169–178.

Institute of Mathematics
Jagiellonian University
Kraków, Reymonta 4, Poland
e-mail: mikulski@im.uj.edu.pl

received February 7, 2001