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Multivalent harmonic starlike functions[††]

Dedicated to Professor Walter Kurt Hayman (FRS)
on his 75-th birthday

Abstract. We give sufficient coefficient conditions for complex-valued harmonic functions that are multivalent, sense-preserving, and starlike in the unit disk. These coefficient conditions are also shown to be necessary if the coefficients of the analytic part of the harmonic functions are negative and the coefficients of the co-analytic part of the harmonic functions are positive. We also determine the extreme points, distortion and covering theorems, convolution and convex combination conditions for these functions.

1. Introduction. A continuous complex valued function $f = u+iv$ defined in a simply connected domain $\mathcal{D}$ is said to be harmonic in $\mathcal{D}$ if u and v are

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real harmonic in $\mathcal{D}$. There is a close interrelation between analytic functions and harmonic functions. For example, for real harmonic functions u and v there exist analytic functions H and G so that $u = \mathrm{Re}(H)$ and $v = \mathrm{Im}(G)$. Consequently, we can write

$$\begin{aligned} f = u + iv = \mathrm{Re}(H) + i\ \mathrm{Im}(G) &= \frac{1}{2}(H + \bar{H}) + i\frac{1}{2i}(G - \bar{G}) \\ &= \frac{H+G}{2} + \frac{\bar{H} - \bar{G}}{2} = h + \bar{g}. \end{aligned}$$

An elementary calculation gives the Jacobian of f as $J_f = |h'|^2 - |g'|^2$. A harmonic function f is sense-preserving at a point z_o if $h'(z) \not\equiv 0$ and $w = g'/h'$ is analytic at z_o (possibly with a removable singularity), and $|w(z_o)| < 1$. Examples of sense-preserving harmonic functions are all non-constant analytic functions, and the functions $\alpha z^n + \beta \bar{z}^m$ for $|z| < 1$, with $n \leq m$ and $m|\beta| < n|\alpha|$. Notice that we do not require f to be univalent in $\mathcal{D}$.

For a sense-preserving harmonic function f, the order of a zero can be defined in terms of the local decomposition $f = h + \bar{g}$. Suppose $f(z_o) = 0$ at some z_o where f is sense-preserving, and write the power series expansion of h and g as

$$h(z) = a_o + \sum_{k=1}^{\infty} a_k (z - z_o)^k \ , \quad g(z) = b_o + \sum_{k=1}^{\infty} b_k (z - z_o)^k.$$

Actually, $b_o = -\overline{a_o}$ because $f(z_o) = 0$. Some a_k $(k \geq 0)$ must be non-zero since $h'(z) \not\equiv 0$.

Let a_m be the first such non-zero coefficient. Then $b_k = 0$ for $0 \leq k < m$, since $w = g'/h'$ is analytic at z_o and $|b_m| < |a_m|$ because $|w(z_o)| < 1$. In this case, we will say that f has a zero of order m at z_o. Now, for $0 < |z - z_o| < \delta$ it is possible to write

$$f(z) = h(z) + \overline{g(z)} = a_m (z - z_o)^m \{1 + \psi(z)\}$$

where

$$\psi(z) = \frac{\bar{b}_m}{a_m} (\bar{z} - \bar{z}_o)^m (z - z_o)^{-n} + O(z - z_o).$$

But it is clear that $|\psi(z)| < 1$ for z sufficiently close to z_o, since $|b_m/a_m| < 1$. Hence $f(z) \neq 0$ near z_o.

Recently, Duren, Hengartner, and Laugesen [3] proved the following argument principle for harmonic functions which is formulated as a direct generalization of the classical result for analytic functions.

Theorem A. *Let f be a harmonic function in a Jordan domain $\mathcal{D}$ with boundary $\mathcal{C}$. Suppose f is continuous in $\overline{\mathcal{D}}$ and $f(z) \neq 0$ on $\mathcal{C}$. Suppose f has no singular zeros in $\mathcal{D}$, and let m be the sum of the orders of the zeros of f in $\mathcal{D}$. Then $\Delta_{\mathcal{C}} \arg(f(z)) = 2\pi m$, where $\Delta_{\mathcal{C}} \arg(f(z))$ denotes the change in argument of $f(z)$ as z traverses $\mathcal{C}$.*

In [3] it is also shown that if f is a sense-preserving harmonic function near a point where $f(z_o) = w_o$, and if $f(z) - w_o$ has a zero of order m $(m \geq 1)$ at z_o, then to each sufficiently small $\epsilon > 0$ there corresponds a $\delta > 0$ with the following property. For each $\alpha \in \mathcal{N}_\delta(w_o) = \{w : |w - w_o| < \delta\}$, the function $f(z) - \alpha$ has exactly m zeros, counted according to multiplicity, in $\mathcal{N}_\epsilon(z_o)$. In particular, f has the open mapping property, that is, it carries open sets to open sets.

Without loss of generality, we let $\mathcal{D}$ be the open unit disk $\mathcal{D} = \{z : |z| < 1\}$. We also let $a_k = b_k = 0$ for $0 \leq k < m$ and $a_m = 1$. Denote by $\mathcal{H}(m)$ the set of all multivalent harmonic functions $f = h + \bar{g}$ that are sense-preserving in $\mathcal{D}$ and h and g are of the form

(1)
$$h(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1}\ , \quad g(z) = \sum_{n=1}^{\infty} b_{n+m-1} z^{n+m-1}, \ |b_m| < 1.$$

According to Theorem A and the above argument, functions in $\mathcal{H}(m)$ are harmonic and sense-preserving in $\mathcal{D}$ if $J_f > 0$ in $\mathcal{D}$. The class $\mathcal{H}(1)$ of harmonic univalent functions was studied in details by Clunie and Sheil-Small [2]. In this note, we look at two subclasses of $\mathcal{H}(m)$, $m \geq 1$, and provide coefficient conditions, extreme points, and distortion bounds for functions in these classes. We also examine their convolution and convex combination properties.

For $m \geq 1$, let $\mathcal{SH}(m)$ denote the subclass of $\mathcal{H}(m)$ consisting of harmonic starlike functions that map each circle $|z| = r < 1$ onto a closed curve that is starlike with respect to the origin. A function with such a property must satisfy the condition (e.g. see [6], p. 244)

$$\frac{\partial}{\partial\theta}\left(\arg(f(re^{i\theta}))\right) \geq 0, \tag{2}$$

for each $z = re^{i\theta}$, $0 \leq \theta < 2\pi$, and $0 \leq r < 1$.

Also let $\mathcal{TH}(m)$, $m \geq 1$, denote the class of functions $f = h + \bar{g}$ in $\mathcal{SH}(m)$ so that h and g are of the form

$$h(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}| z^{n+m-1}\ , \quad g(z) = \sum_{n=1}^{\infty} |b_{n+m-1}| z^{n+m-1}\ . \tag{3}$$

2. Coefficient bounds. It was shown by Sheil-Small [6, Theorem 7] that $|a_n| \le (n+1)(2n+1)/6$ and $|b_n| \le (n-1)(2n-1)/6$ if $f \in \mathcal{SH}^o(1)$. The subclass of $\mathcal{SH}(m)$ where $m = 1$ and $b_1 = 0$ is denoted by $\mathcal{SH}^o(1)$. These bounds are sharp and thus give necessary coefficient conditions for the class $\mathcal{SH}^o(1)$. Avci and Zlotkiewicz [1] proved that the coefficient condition $\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \le 1$ is sufficient for functions in $\mathcal{SH}^o(1)$. Silverman [7] showed that this coefficient condition is also necessary for $m = 1$ and harmonic functions of the form (3). Jahangiri [5] and Silverman and Silvia [8] extended this result to the case b_1 not necessarily zero. The arguments presented in this section provide sufficient coefficient bounds for functions $f = h + \bar{g}$ of the form (1) to be in $\mathcal{SH}(m),\ m \ge 1$. It is also shown that these bounds are necessary if $f \in \mathcal{TH}(m)$.

Theorem 1. *Let $f = h + \bar{g}$ be given by (1). If*

$$\sum_{n=1}^{\infty}(n+m-1)\left(|a_{n+m-1}| + |b_{n+m-1}|\right) \le 2m \tag{4}$$

where $a_m = 1$ and $m \ge 1$, then the harmonic function f is sense-preserving in $\mathcal{D}$ and $f \in \mathcal{SH}(m)$.

Proof. Write $z = re^{i\theta}$ where $0 \le r < 1$ and θ is real. For h and g given by (1) we have

$$\begin{aligned}
&|h'(z)| - |g'(z)| \\
&\quad = \left| mz^{m-1} + \sum_{n=2}^{\infty}(n+m-1)a_{n+m-1}z^{n+m-2} \right| - \left| \sum_{n=1}^{\infty}(n+m-1)b_{n+m-1}z^{n+m-2} \right| \\
&\quad \ge \left| mz^{m-1} \right| - \left| \sum_{n=2}^{\infty}(n+m-1)a_{n+m-1}z^{n+m-2} \right| - \left| \sum_{n=1}^{\infty}(n+m-1)b_{n+m-1}z^{n+m-2} \right| \\
&\quad \ge mr^{m-1} - \sum_{n=2}^{\infty}(n+m-1)|a_{n+m-1}|r^{n+m-2} - \sum_{n=1}^{\infty}(n+m-1)|b_{n+m-1}|r^{n+m-2} \\
&\quad = r^{m-1}\left[m - \sum_{n=2}^{\infty}(n+m-1)|a_{n+m-1}|r^{n-1} - \sum_{n=1}^{\infty}(n+m-1)|b_{n+m-1}|r^{n-1} \right] \\
&\quad > r^{m-1}\left[m - \sum_{n=2}^{\infty}(n+m-1)|a_{n+m-1}| - \sum_{n=1}^{\infty}(n+m-1)|b_{n+m-1}| \right] \\
&\quad = r^{m-1}\left\{ 2m - \left[\sum_{n=1}^{\infty}(n+m-1)\left(|a_{n+m-1}| + |b_{n+m-1}|\right) \right] \right\} \ge 0,
\end{aligned}$$

by (4).

Therefore, by Theorem A, the harmonic function $f = h + \bar{g}$ is sense-preserving in $\mathcal{D}$.

Now we show that $f \in \mathcal{SH}(m)$. According to the required condition (2), we only need to show that

$$\frac{\partial}{\partial\theta}\left(\arg(f(re^{i\theta}))\right) = \operatorname{Im}\left(\frac{\partial}{\partial\theta}\log(f(re^{i\theta}))\right) = \operatorname{Re}\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right) \geq 0.$$

The case $r = 0$ is obvious. For $0 < r < 1$, it follows that

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(\arg(f(re^{i\theta}))\right) &= \operatorname{Re}\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right)\\ &= \frac{mz^m + \sum_{n=2}^{\infty}(n+m-1)a_{n+m-1}z^{n+m-1} - \sum_{n=1}^{\infty}(n+m-1)\bar{b}_{n+m-1}\bar{z}^{n+m-1}}{z^m + \sum_{n=2}^{\infty}a_{n+m-1}z^{n+m-1} + \sum_{n=1}^{\infty}\bar{b}_{n+m-1}\bar{z}^{n+m-1}}\\ &= \operatorname{Re}\left[\frac{m + A(z)}{1 + B(z)}\right].\end{aligned}$$

For $z = re^{i\theta}$ we have

$$A(re^{i\theta}) = \sum_{n=2}^{\infty}(n+m-1)a_{n+m-1}r^{n-1}e^{(n-1)\theta i} - \sum_{n=1}^{\infty}(n+m-1)\bar{b}_{n+m-1}r^{n-1}e^{-(n+2m-1)\theta i}$$

and

$$B(re^{i\theta}) = \sum_{n=2}^{\infty}a_{n+m-1}r^{n-1}e^{(n-1)\theta i} + \sum_{n=1}^{\infty}\bar{b}_{n+m-1}r^{n-1}e^{-(n+2m-1)\theta i}.$$

Setting

$$\frac{m + A(z)}{1 + B(z)} = m\frac{1 + w(z)}{1 - w(z)}$$

the proof will be complete if we can show that $|w(z)| \leq r < 1$. This is the case since, by the condition (4), we can write

$$\begin{aligned}|w(z)| &= \left|\frac{A(z) - mB(z)}{A(z) + mB(z) + 2m}\right|\\ &= \left|\frac{\sum_{n=2}^{\infty}(n-1)a_{n+m-1}r^{n-1}e^{(n-1)\theta i} - \sum_{n=1}^{\infty}(n+2m-1)\bar{b}_{n+m-1}r^{n-1}e^{-(n+2m-1)\theta i}}{2m + \sum_{n=2}^{\infty}(n+2m-1)a_{n+m-1}r^{n-1}e^{(n-1)\theta i} - \sum_{n=1}^{\infty}(n-1)\bar{b}_{n+m-1}r^{n-1}e^{-(n+2m-1)\theta i}}\right|\\ &\leq \frac{\sum_{n=2}^{\infty}(n-1)|a_{n+m-1}|r^{n-1} + \sum_{n=1}^{\infty}(n+2m-1)|b_{n+m-1}|r^{n-1}}{2m - \sum_{n=2}^{\infty}(n+2m-1)|a_{n+m-1}|r^{n-1} - \sum_{n=1}^{\infty}(n-1)|b_{n+m-1}|r^{n-1}}\end{aligned}$$

$$= \frac{\sum_{n=1}^{\infty} [(n-1)|a_{n+m-1}| + (n+2m-1)|b_{n+m-1}|]\, r^{n-1}}{4m - \sum_{n=1}^{\infty} [(n+2m-1)|a_{n+m-1}| + (n-1)|b_{n+m-1}|]\, r^{n-1}}$$

$$< \frac{\sum_{n=1}^{\infty} [(n-1)|a_{n+m-1}| + (n+2m-1)|b_{n+m-1}|]}{4m - \sum_{n=1}^{\infty} [(n+2m-1)|a_{n+m-1}| + (n-1)|b_{n+m-1}|]} \leq 1 \,. \quad \square$$

For $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = m$, we use the starlike harmonic mappings

$$f(z) = z^m + \sum_{n=2}^{\infty} \frac{x_n}{n+m-1} z^{n+m-1} + \sum_{n=1}^{\infty} \frac{\bar{y}_n}{n+m-1} \bar{z}^{n+m-1}, \tag{5}$$

to show that the coefficient bound given by (4) is sharp.

The functions of the form (5) are in $\mathcal{SH}(m)$ because

$$\sum_{n=1}^{\infty} (n+m-1)\left(|a_{n+m-1}| + |b_{n+m-1}|\right) = m + \sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 2m.$$

The restriction in Theorem 1 placed on the moduli of the coefficients of $f = h + \bar{g}$ enables us to conclude for arbitrary rotation of the coefficients of f that the resulting functions would still be a member of $\mathcal{SH}(m)$. Our next theorem establishes that such coefficient bounds cannot be improved.

Theorem 2. *Let $f = h + \bar{g}$ be given by (3). Then $f \in \mathcal{TH}(m)$ if and only if*

$$\sum_{n=1}^{\infty} (n+m-1)\left(|a_{n+m-1}| + |b_{n+m-1}|\right) \leq 2m \tag{6}$$

where $a_m = 1$ and $m \geq 1$.

Proof. The "*if*" part follows from Theorem 1 upon noting that $\mathcal{TH}(m) \subset \mathcal{SH}(m),\ m \geq 1$.

For the "*only if*" part, we show that $f \notin \mathcal{TH}(m)$ if the condition (6) does not hold. We examine the required condition (2) for $f = h + \bar{g} \in \mathcal{TH}(m)$. This is equivalent to

$$\mathrm{Re}\, \frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}$$

$$= \mathrm{Re}\, \frac{mz^m - \sum_{n=2}^{\infty} (n+m-1)|a_{n+m-1}| z^{n+m-1} - \sum_{n=1}^{\infty} (n+m-1)|b_{n+m-1}| \bar{z}^{n+m-1}}{z^m - \sum_{n=2}^{\infty} |a_{n+m-1}| z^{n+m-1} + \sum_{n=1}^{\infty} |b_{n+m-1}| \bar{z}^{n+m-1}} \geq 0.$$

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$ we must have

$$\frac{m-\sum\limits_{n=2}^{\infty}(n+m-1)|a_{n+m-1}|r^{n-1} - \sum\limits_{n=1}^{\infty}(n+m-1)|b_{n+m-1}|r^{n-1}}{1-\sum\limits_{n=2}^{\infty}|a_{n+m-1}|r^{n-1}+\sum\limits_{n=1}^{\infty}|b_{n+m-1}|r^{n-1}} \geq 0\,. \tag{7}$$

If the condition (6) does not hold then the numerator in (7) is negative for r sufficiently close to 1. Thus there exists a $z_o = r_o$ in $(0,1)$ for which the quotient in (7) is negative. This contradicts the required condition for $f \in \mathcal{TH}(m)$ and so the proof is complete. $\square$

3. Extreme points and distortion bounds. Next we determine the extreme points of the closed convex hull of $\mathcal{TH}(m)$, denoted by clco$\mathcal{TH}(m)$.

Theorem 3. *$f = h + \bar{g} \in$ clco$\mathcal{TH}(m)$ if and only if f can be expressed in the form*

$$f = \sum_{n=1}^{\infty}\left(X_{n+m-1}h_{n+m-1} + Y_{n+m-1}\ g_{n+m-1}\right) \tag{8}$$

where $h_m(z)=z^m$, $h_{n+m-1}(z) = z^m - \frac{m}{n+m-1}z^{n+m-1}$ $(n=2,3,...)$, $g_{n+m-1}(z) = z^m+\frac{m}{n+m-1}\bar{z}^{n+m-1}$ $(n=1,2,3,...)$, $\sum_{n=1}^{\infty}(X_{n+m-1}+Y_{n+m-1})=1$, $X_{n+m-1} \geq 0$, and $Y_{n+m-1} \geq 0$. In particular, the extreme points of $\mathcal{TH}(m)$ are $\{h_{n+m-1}\}$ and $\{g_{n+m-1}\}$.

Proof. For functions f of the form (8) we have

$$\begin{aligned} f(z) &= z^m - \sum_{n=2}^{\infty}\frac{m}{n+m-1}X_{n+m-1}z^{n+m-1}+\sum_{n=1}^{\infty}\frac{m}{n+m-1}Y_{n+m-1}\bar{z}^{n+m-1} \\ &= z^m-\sum_{n=2}^{\infty}a_{n+m-1}z^{n+m-1} + \sum_{n=1}^{\infty}b_{n+m-1}\bar{z}^{n+m-1}\ . \end{aligned}$$

Then

$$\begin{aligned} \sum_{n=1}^{\infty}(n+m-1)\left(|a_{n+m-1}|+|b_{n+m-1}|\right) &= m+\sum_{n=2}^{\infty}m|X_{n+m-1}| + \sum_{n=1}^{\infty}m|Y_{n+m-1}| \\ &= m+m(1-X_m)\leq 2m, \end{aligned}$$

and so $f \in$ clco$\mathcal{TH}(m)$.

Conversely, suppose that $f \in \mathrm{clco}\mathcal{TH}(m)$. Set

$$X_{n+m-1} = \frac{n+m-1}{m}|a_{n+m-1}| \quad (n = 2, 3, ...),$$

$$Y_{n+m-1} = \frac{n+m-1}{m}|b_{n+m-1}| \quad (n = 1, 2, 3, ...),$$

and

$$X_m = 1 - \sum_{n=2}^{\infty} X_{n+m-1} - \sum_{n=2}^{\infty} Y_{n+m-1}.$$

Then, as required, we obtain

$$f(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}| + \sum_{n=1}^{\infty} |b_{n+m-1}|\, \bar{z}^{n+m-1}. \quad \square$$

Our next theorem is on the distortion bounds for functions in $\mathcal{TH}(m)$, which yields a covering result for the family $\mathcal{TH}(m)$.

Theorem 4. *If $f \in \mathcal{TH}(m)$, then*

$$|f(z)| \le (1 + |b_m|)\, r^m + \frac{m(1 - |b_m|)}{m+1} r^{m+1}, \quad |z| = r < 1,$$

and

$$|f(z)| \ge (1 - |b_m|)\, r^m - \frac{m(1 - |b_m|)}{m+1} r^{m+1}, \quad |z| = r < 1.$$

Proof. In view of Theorem 2, we have

$$\begin{aligned} m(1+|b_m|)+(m+1) \sum_{n=2}^{\infty} (|a_{n+m-1}|+|b_{n+m-1}|) &\le \sum_{n=1}^{\infty} (n+m-1)\,(|a_{n+m-1}|+|b_{n+m-1}|) \\ &\le 2m, \end{aligned}$$

so that

$$\sum_{n=2}^{\infty} (|a_{n+m-1}|+|b_{n+m-1}|) \le \frac{m(1-|b_m|)}{m+1}. \tag{9}$$

Now, taking the absolute values of f in $\mathcal{TH}(m)$ we obtain

$$\begin{aligned}
|f(z)| &= \left| z^m - \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1} + \sum_{n=1}^{\infty} b_{n+m-1} \bar{z}^{n+m-1} \right| \\
&\leq r^m + \sum_{n=2}^{\infty} |a_{n+m-1}| r^{n+m-1} + \sum_{n=1}^{\infty} |b_{n+m-1}| r^{n+m-1} \\
&= (1+|b_m|) r^m + \sum_{n=2}^{\infty} \left(|a_{n+m-1}| + |b_{n+m-1}|\right) r^{n+m-1} \\
&\leq (1+|b_m|) r^m + \sum_{n=2}^{\infty} \left(|a_{n+m-1}| + |b_{n+m-1}|\right) r^{m+1} \\
&\leq (1+|b_m|) r^m + \frac{m(1-|b_m|)}{m+1} r^{m+1}, \quad \text{by (9)}
\end{aligned}$$

and

$$\begin{aligned}
|f(z)| &= \left| z^m - \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1} + \sum_{n=1}^{\infty} b_{n+m-1} \bar{z}^{n+m-1} \right| \\
&\geq r^m - \sum_{n=2}^{\infty} |a_{n+m-1}| r^{n+m-1} - \sum_{n=1}^{\infty} |b_{n+m-1}| r^{n+m-1} \\
&= (1-|b_m|) r^m - \sum_{n=2}^{\infty} \left(|a_{n+m-1}| + |b_{n+m-1}|\right) r^{n+m-1} \\
&\geq (1-|b_m|) r^m - \sum_{n=2}^{\infty} \left(|a_{n+m-1}| + |b_{n+m-1}|\right) r^{m+1} \\
&\geq (1-|b_m|) r^m - \frac{m(1-|b_m|)}{m+1} r^{m+1}, \quad \text{by (9).}
\end{aligned}$$

The bounds given in Theorem 4 for the functions $f = h + \bar{g}$ in $\mathcal{TH}(m)$ also hold for functions in $\mathcal{SH}(m)$ if the coefficient condition (4) is satisfied. The function

$$f(z) = z^m + |b_m| \bar{z}^m + \frac{m(1-|b_m|)}{m+1} \bar{z}^{m+1}$$

and its rotations show that the bounds given in Theorem 4 are sharp. $\square$

The following covering result follows from the left hand inequality in Theorem 4.

Corollary. *If $f \in \mathcal{TH}(m)$ then*

$$\left\{ w : |w| < \frac{1-|b_m|}{m+1} \right\} \subset f(\mathcal{D}).$$

Remark 1. For $m = 1$ and $b_1 = 0$ the covering result in the above corollary coincides with that given in ([6] Theorem 5.9) for harmonic convex functions.

A harmonic function f is convex in $\mathcal{D}$ (see [4] or [6]) if for each $z,\ |z| = r < 1$,

$$\frac{\partial}{\partial\theta}\arg\left(\frac{\partial}{\partial\theta}f(re^{i\theta})\right) \geq 0.$$

The corresponding definition for harmonic convex functions leads to analogous coefficient bounds and extreme points.

4. Convolution and convex combination. For harmonic functions

$$f(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}| z^{n+m-1} + \sum_{n=1}^{\infty} |b_{n+m-1}| \bar{z}^{n+m-1} \tag{10}$$

and

$$F(z) = z^m - \sum_{n=2}^{\infty} |A_{n+m-1}| z^{n+m-1} + \sum_{n=1}^{\infty} |B_{n+m-1}| \bar{z}^{n+m-1} \tag{11}$$

we define the convolution of f and F as

$$(f*F)(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}A_{n+m-1}| z^{n+m-1} + \sum_{n=1}^{\infty} |b_{n+m-1}B_{n+m-1}| \bar{z}^{n+m-1}. \tag{12}$$

In the following theorem we examine the convolution properties of the class $\mathcal{TH}(m)$.

Theorem 5. *If f and F are in $\mathcal{TH}(m)$, so is $f * F$.*

Proof. Let f and F of the forms (10) and (11) belong to $\mathcal{TH}(m)$. Then the convolution of f and F is given by (12). Note that $|A_{n+m-1}| \leq 1$ and $|B_{n+m-1}| \leq 1$ since $F \in \mathcal{TH}(m)$. Then we can write

$$\begin{aligned}\sum_{n=1}^{\infty}&(n+m-1)\left(|a_{n+m-1}||A_{n+m-1}|+|b_{n+m-1}||B_{n+m-1}|\right)\\ &\leq \sum_{n=1}^{\infty}(n+m-1)\left(|a_{n+m-1}|+|b_{n+m-1}|\right).\end{aligned}$$

The right hand side of the above inequality is bounded by $2m$ because $f \in \mathcal{TH}(m)$. Therefore $f * F \in \mathcal{TH}(m)$. $\square$

Our next result is on the convex combinations of the members of the family $\mathcal{TH}(m)$.

Theorem 6. *The class $\mathcal{TH}(m)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, ...$ suppose that $f_i(z) \in \mathcal{TH}(m)$ where f_i is given by

$$f_i(z) = z^m - \sum_{n=2}^{\infty} |a_{i_{n+m-1}}| z^{n+m-1} + \sum_{n=1}^{\infty} |b_{i_{n+m-1}}| \bar{z}^{n+m-1}.$$

Then, by (6),

$$\sum_{n=1}^{\infty} (n+m-1) \left(|a_{i_{n+m-1}}| + |b_{i_{n+m-1}}| \right) \leq 2m. \tag{13}$$

For $\sum_{i=1}^{\infty} t_i = 1,\ 0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z^m - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{i_{n+m-1}}| \right) z^{n+m-1} + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{i_{n+m-1}}| \right) \bar{z}^{n+m-1}.$$

Then, by (13),

$$\begin{aligned} &\sum_{n=1}^{\infty} (n+m-1) \left(\left| \sum_{i=1}^{\infty} t_i |a_{i_{n+m-1}}| \right| + \left| \sum_{i=1}^{\infty} t_i |b_{i_{n+m-1}}| \right| \right) \\ &\quad = \sum_{i=1}^{\infty} t_i \left\{ \sum_{n=1}^{\infty} (n+m-1) \left(|a_{i_{n+m-1}}| + |b_{i_{n+m-1}}| \right) \right\} \leq \sum_{i=1}^{\infty} t_i (2m) = 2m, \end{aligned}$$

and so $\sum_{i=1}^{\infty} t_i f_i(z) \in \mathcal{TH}(m)$.

5. Positive order. We say that f of the form (1) is harmonic starlike of order $\alpha,\ 0 \leq \alpha < 1$, in $\mathcal{D}$ if

$$\frac{\partial}{\partial \theta} \arg(f(re^{i\theta})) \geq m\alpha$$

for each $z,\ |z| = r < 1$.

Denote by $\mathcal{SH}(m, \alpha)$ and $\mathcal{TH}(m, \alpha)$ the subclasses of $\mathcal{SH}(m)$ and $\mathcal{TH}(m)$, respectively, that are starlike of order $\alpha,\ 0 \leq \alpha < 1$.

Note that $\mathcal{SH}(m, 0) \equiv \mathcal{SH}(m)$ and $\mathcal{TH}(m, 0) \equiv \mathcal{TH}(m)$.

Many of our results can be generalized to multivalent harmonic starlike functions of positive order. For instance, using arguments similar to those already given for Theorems 1 and 2 leads to the following results.

Theorem 7. *Let $f = h + \bar{g}$ be given by (1). If*

$$(14) \qquad \sum_{n=1}^{\infty} \left[\frac{n+m(1-\alpha)-1}{m-\alpha} |a_{n+m-1}| + \frac{n+m(1+\alpha)-1}{m-\alpha} |b_{n+m-1}| \right] \leq 2$$

where $a_m = 1$ and $m \geq 1$ then the harmonic function f is sense-preserving, m-valent, and $f \in \mathcal{SH}(m, \alpha)$.

As an outline for the proof of Theorem 7, first note that

$$-m\alpha + \frac{\partial}{\partial\theta} \arg(f(re^{i\theta})) = \operatorname{Re}\left(\frac{m(1-\alpha) + A(z) - m\alpha B(z)}{1 + B(z)} \right) \geq 0$$

where $A(z)$ and $B(z)$ are described in the proof of Theorem 1. Now, following a similar line of proof which is used for Theorem 1, yields the required coefficient condition (14).

Theorem 8. *For $f \in \mathcal{TH}(m, \alpha)$ the coefficient condition (14) is both necessary and sufficient.*

As in Theorem 3, these necessary and sufficient coefficient conditions for $\mathcal{TH}(m, \alpha)$ lead to the extreme points, namely,

$$h_m(z) = z^m,$$

$$h_{n+m-1}(z) = z^m - \frac{m-\alpha}{n+m(1-\alpha)-1} z^{n+m-1} \ (n = 2, 3, ...),$$

and

$$g_{n+m-1}(z) = z^m + \frac{m-\alpha}{n+m(1+\alpha)-1} \bar{z}^{n+m-1} \ (n = 1, 2, 3, ...).$$

Remark 2. The corresponding definition for harmonic functions convex of order α, $0 \leq \alpha < 1$ leads to analogous coefficient bounds and extreme points.

Remark 3. The results of this paper, for $m = 1$ and $0 \leq \alpha < 1$, coincide with those given by the second author in [4] and [5].

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