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Natural affinors on time-dependent higher order cotangent bundles

Abstract. We study natural affinors on time-dependent natural bundles. Then we determine all natural affinors on the time-dependent higher order cotangent bundle $T^{r*}M \times \mathbb{R}$.

1. Introduction. Recently, it has been pointed out that natural tensor fields of type $(1,1)$ (in other words affinors) play an important role in differential geometry. In particular, I. Kolář and M. Modugno have used natural affinors to introduce the general concept of the torsion of a connection, [6]. Using such a point of view, it is useful to classify all natural affinors on some natural bundles. Such an approach has been used e.g. in [3], [4] and [6].

Further, non-autonomous Lagrangian dynamics can be considered as an extension of autonomous Lagrangian dynamics by introducing the additional time coordinate. For example, M. de León and R. P. Rodrigues have introduced the concept of time-dependent (or dynamical) connection, [10]. Quite analogously, one can define dynamical vector fields, affinors, sprays and other structures. M. Doupovec and I. Kolář have classified all natural affinors on time-dependent Weil bundles, [2]. It is well known that Weil algebras and Weil functors generalize many geometric structures and constructions. In particular, there is a complete description of all product

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preserving functors on the category of all smooth manifolds and all smooth maps in terms of Weil functors, [7].

The aim of this paper is twofold. First, we study natural affinors on time-dependent natural bundles from a general point of view. In Example 2 we introduce the new natural affinor on a time-dependent natural bundle, which was not included in [2]. Second, we classify all natural affinors on time-dependent higher order cotangent bundles. We remark that such bundles are used e.g. in higher order mechanics, [14]. In this paper we essentially use the results [8] and [9] of J. Kurek.

All manifolds and maps are assumed to be infinitely differentiable.

2. Natural affinors on time-dependent bundles. In general, an *affinor* on a manifold M is a tensor field of type $(1,1)$ on M, which can be interpreted as a linear morphism $TM \to TM$ over the identity of M. By the Frölicher–Nijenhuis theory, affinors are exactly tangent-valued one-forms on M, i.e. sections from $C^\infty(TM \otimes T^*M)$. Given a fibered manifold $p : Y \to M$, an affinor Q on Y is called *vertical*, if Q has values in the vertical bundle VY, i.e. $Q \in C^\infty(VY \otimes T^*Y)$.

Further, let $T^*M \subset T^*Y$ be the canonical inclusion of cotangent bundles. By [11], vertical affinors of the form $Q \in C^\infty(VY \otimes T^*M)$ are called *soldering forms*. Let F be a natural bundle F on the category $\mathcal{M}f_m$ of all m-dimensional manifolds and their local diffeomorphisms. We recall that a *natural affinor* on a natural bundle F is a system of affinors $Q_M : TFM \to TFM$ for every m-manifold M satisfying $TFf \circ Q_M = Q_N \circ TFf$ for every local diffeomorphism $f : M \to N$. An example of a natural affinor is the classical almost tangent structure on TM.

Definition 1. The *time-dependent natural bundle* $F_\mathbb{R}$ corresponding to the natural bundle F is defined by $F_\mathbb{R}M = FM \times \mathbb{R}$ for every m-dimensional manifold M and by $F_\mathbb{R}f = Ff \times Id_\mathbb{R} : F_\mathbb{R}M \to F_\mathbb{R}N$ for every local diffeomorphism $f : M \to N$.

Clearly, the time-dependent natural bundle $F_\mathbb{R}$ generalizes the well known time-dependent tangent bundle $TM \times \mathbb{R}$ and also the time-dependent Weil bundle $T^A_\mathbb{R}$ from [2], if we restrict $T^A_\mathbb{R}$ to the category $\mathcal{M}f_m$.

In what follows we introduce some examples of natural affinors on time--dependent bundles.

Example 1. For any natural bundle F we have three simple constructions of natural affinors on $F_\mathbb{R}$. First, every natural affinor Q on F induces a natural affinor $\widetilde{Q}$ on $F_\mathbb{R}$ by means of the product structure $FM \times \mathbb{R}$. Quite analogously, the identity $Id_{T\mathbb{R}}$ of $T\mathbb{R}$ determines another affinor $\widetilde{Id_{T\mathbb{R}}}$ on $F_\mathbb{R}$. The third type of natural affinors on $F_\mathbb{R}$ can be defined by tensor products $X \otimes dt$ of absolute vector fields on FM with the canonical one-form dt on $\mathbb{R}$.

We recall that an absolute vector field can be interpreted as an absolute natural operator transforming vector fields on M into vector fields on FM, [7]. Clearly, absolute vector fields are natural in the following sense.

Definition 2. A *natural vector field* X on natural bundle F is a system of vector fields $X_M : FM \to TFM$ for every m-manifold M satisfying $TFf \circ X_M = X_N \circ Ff$ for all local diffeomorphisms $f : M \to N$.

If F is a natural vector bundle, then the classical Liouville vector field L_{FM} on FM is natural. Clearly, L_{FM} is generated by the one-parameter family of homotheties. More generally, let $\Phi(t)$ be a smooth one-parameter family of natural transformations $F \to F$, where smoothness means that the map $\Phi(t)_M : FM \times \mathbb{R} \to FM$ is smooth for every manifold M. Then the formula $X_M = \frac{d}{dt}\big|_0 \Phi(t)_M$ defines a natural vector field $X_M : FM \to TFM$.

By [7], every natural vector field X on F is vertical. This yields that natural affinors $X \otimes dt$ on $F_{\mathbb{R}}$ from Example 1 are soldering forms.

Example 2. Let F be a natural vector bundle and let f be a natural function on TF. We recall that this is a system of functions $f_M : TFM \to \mathbb{R}$ for every m-dimensional manifold M satisfying $f_M = f_N \circ TF\varphi$ for all local diffeomorphisms $\varphi : M \to N$. Denote by $\pi_M : FM \to M$ the bundle projection and by $p_M : TM \to M$ the tangent bundle projection. For any $X \in TF_{\mathbb{R}}M = TFM \times T\mathbb{R}$ we have $p_{F_{\mathbb{R}}M}(X) \in F_{\mathbb{R}}M$, $pr_1(p_{F_{\mathbb{R}}M}(X)) \in FM$ and $x := \pi_M(pr_1(p_{F_{\mathbb{R}}M}(X))) \in M$. Let $s : M \to FM$ be a zero section. Then the cartesian product of $s(x)$ with $f_M(pr_1(X))$ defines an element

$$R(X) := s(x) \times f_M(pr_1(X)) \in FM \times \mathbb{R} = F_{\mathbb{R}}M.$$

As FM is a vector bundle, $F_{\mathbb{R}}M$ is a vector bundle too. For $X \in TF_{\mathbb{R}}M$ we have

$$P(X) := (p_{F_{\mathbb{R}}M}(X), R(X)) \in (F_{\mathbb{R}}M \oplus F_{\mathbb{R}}M) \cong VF_{\mathbb{R}}M \subset TF_{\mathbb{R}}M.$$

This defines a natural affinor P on $F_{\mathbb{R}}M$.

We remark that natural affinors from Example 2 did not appear in the description of all natural affinors on time-dependent Weil bundles, [2]. We also point out that the classical Liouville one-form of the cotangent bundle T^*M is the simplest example of a natural function on TT^*.

It is well known that natural affinors play a significant role in the theory of torsions of connections. In particular, if we interpret a general connection $\Gamma : FM \to J^1FM$ as its horizontal projection (denoted by the same symbol) $\Gamma : TFM \to TFM$, we obtain an affinor on F. Further, I. Kolář and M. Modugno introduced the generalized torsion of Γ as the Frölicher–Nijenhuis bracket $[\Gamma, Q]$ of Γ with some natural affinor Q on F, [6]. Such an approach has been used e.g. in [3], [4] and [6]. There are also many papers which classify all natural affinors on some natural bundles, see [5], [8], [12] and [13].

Denote by T^A the Weil functor corresponding to a Weil algebra A, [2]. By the general theory, every product preserving functor F on the category $\mathcal{M}f$ of all smooth manifolds and all smooth maps is the Weil functor $F = T^A$, where $A = F\mathbb{R}$. M. Doupovec and I. Kolář have determined all natural affinors on the time-dependent Weil bundle $T^A_{\mathbb{R}}M$, [2]. It is interesting to point out that all natural affinors on $T^A_{\mathbb{R}}M$ are generated only by affinors from Example 1. Using this result, M. Doupovec has described torsions of dynamical connections on time-dependent Weil bundles, [1].

Further, natural affinors on time-dependent higher order tangent bundles were determined by I. Kolář and J. Gancarzewicz, [5]. Such affinors are also generated only by three affinors from Example 1.

3. Natural affinors on time-dependent higher order cotangent bundles. Let M be a smooth m-dimensional manifold and denote by $T^{r*}M = J^r(M,\mathbb{R})_0$ the space of all r-jets from M into $\mathbb{R}$ with target 0. Every local diffeomorphism $f : M \to N$ can be extended to a vector bundle morphism $T^{r*}f : T^{r*}M \to T^{r*}N$ by $j^r_x\varphi \mapsto j^r_{f(x)}(\varphi \circ f^{-1})$, where f^{-1} is constructed locally. Then $\pi_M : T^{r*}M \to M$ is a natural vector bundle which is called the r-th order cotangent bundle. Clearly, $T^{1*}M = T^*M$ is the classical cotangent bundle.

Denote by

$$q_M : T^{r*}M \to T^*M$$

the bundle projection defined by $q_M\left(j^r_x f\right) = j^1_x f$. If $X \in TT^{r*}M$, then $T\pi_M(X) \in TM$ and $q_M(p_{T^{r*}M}(X)) \in T^*M$. So we can define a map

$$\lambda_M : TT^{r*}M \to \mathbb{R}, \;\; \lambda_M(X) = \langle q_M(p_{T^{r*}M}(X)), T\pi_M(X)\rangle,$$

which is called the generalized Liouville form on $T^{r*}M$.

Further, let $A^r_s : T^{r*}M \to T^{r*}M$ be the s-th power natural transformation defined by $A^r_s(j^r_x f) = j^r_x(f)^s$, where $(f)^s$ denotes s-th power of f. Since $\pi_M : T^{r*}M \to M$ is a vector bundle, the vertical bundle $VT^{r*}M$ can be identified with the Whitney sum $T^{r*}M \oplus T^{r*}M$. Using this identification we can define natural affinors $Q^s_M : TT^{r*}M \to VT^{r*}M$ by

$$Q^s_M(X) = (p_{T^{r*}M}(X), \lambda_M(X)A^r_s(p_{T^{r*}M}(X))).$$

In what follows we will use the following results, which were proved by J. Kurek.

Lemma 1 ([8]). *All natural affinors on the r-th order cotangent bundle $T^{r*}M$ are of the form*

$$k_0 Id_{T^{r*}M} + k_1 Q^1_M + \cdots + k_r Q^r_M, \quad k_i \in \mathbb{R}.$$

Lemma 2 ([9]). *All natural transformations $T^{r*}M \to T^{r*}M$ are of the form*

$$k_1 A_1^r + \cdots + k_r A_r^r, \quad k_i \in \mathbb{R}.$$

Multiplying the s-th power transformation A_s^r by a real number t, we obtain a smooth one-parameter family of natural transformations $(tA_s^r) : T^{r*}M \to T^{r*}M$. This generates a vector field $L_s : T^{r*}M \to VT^{r*}M$ by

$$L_s(u) = \left.\frac{d}{dt}\right|_0 (u + tA_s^r(u)).$$

Clearly, L_1 is the classical Liouville vector field on T^{r*} and L_s can be also defined by $L_s(u) = (u, A_s^r(u))$.

Using Example 1 and Example 2, we have four types of natural affinors on the time-dependent bundle $T_{\mathbb{R}}^{r*}M = T^{r*}M \times \mathbb{R}$:

I) Each natural affinor on $T^{r*}M$ from Lemma 1 induces a natural affinor on $T_{\mathbb{R}}^{r*}M$ by means of the product structure. In this way we obtain natural affinors $\widetilde{Q}_M^1, \ldots, \widetilde{Q}_M^r$ and $\widetilde{Id}_{T^{r*}M}$.

II) The identity of $T\mathbb{R}$ induces a natural affinor $\widetilde{Id}_{\mathbb{R}}$ on $T_{\mathbb{R}}^{r*}M$.

III) Natural vector fields $L_s : T^{r*}M \to TT^{r*}M$ induce natural affinors $(L_s \otimes dt)$ on $T_{\mathbb{R}}^{r*}M$.

IV) Clearly, the generalized Liouville form $\lambda_M : TT^{r*}M \to \mathbb{R}$ is a natural function on $TT^{r*}M$. By Example 2, this natural function determines a natural affinor P on $T_{\mathbb{R}}^{r*}M$.

In the rest of this paper we prove that natural affinors from I–IV generate all natural affinors on $T_{\mathbb{R}}^{r*}M$. We first introduce the coordinate form of affinors from I–IV.

The canonical coordinates (x^i) on M induce the additional fiber coordinates $(u_i, u_{ij}, \ldots, u_{i_1 \ldots i_r})$ on $T^{r*}M$, which are symmetric in all indices, [4]. Denoting by t the coordinate on $\mathbb{R}$, the coordinates on $TT_{\mathbb{R}}^{r*}M$ are of the form

$$(x^i, t, u_i, \ldots, u_{i_1 \cdots i_r}, X^i = dx^i, T = dt, U_i = du_i, \ldots, U_{i_1 \cdots i_r} = du_{i_1 \cdots i_r}).$$

Clearly, we have

$$\begin{aligned} \widetilde{Id}_{T^{r*}M}(dx^i, dt, du_i, \ldots, du_{i_1 \cdots i_r}) =& (dx^i, 0, du_i, \ldots, du_{i_1 \cdots i_r}) \\ \widetilde{Id}_{\mathbb{R}}(dx^i, dt, du_i, \ldots, du_{i_1 \cdots i_r}) =& (0, dt, 0, \ldots, 0). \end{aligned}$$

Obviously, the generalized Liouville form λ_M has the coordinate expression $u_i dx^i$. Then

$$P(dx^i, dt, du_i, \ldots, du_{i_1 \cdots i_r}) = (0, u_j dx^j, 0, \ldots, 0).$$

J. Kurek has computed the coordinate form of affinors $Q^1_M, \dots, Q^r_M$ on $T^{r*}M$. Using [8], we have

$$\begin{aligned}
\widetilde{Q}^1_M(dx^i, dt, du_i, \dots, du_{i_1\cdots i_r}) &= (0, 0, u_i u_j dx^j, \dots, u_{i_1\cdots i_r} u_j dx^j) \\
&\vdots \\
\widetilde{Q}^s_M(dx^i, dt, du_i, \dots, du_{i_1\cdots i_r}) &= \big(0, 0, 0, \dots, u_{i_1\cdots i_s} u_j dx^j\ , \\
&\quad \frac{(s+1)!}{(s-1)!2!}\, u_{(i_1} \cdots u_{i_{s-1}} u_{i_s i_{s+1})} u_j dx^j, \dots, \\
&\quad \frac{r!}{(s-1)!(r-s+1)!}\ u_{(i_1} \cdots u_{i_{s-1}} u_{i_s\cdots i_r)} u_j dx^j\big) \\
&\vdots \\
\widetilde{Q}^r_M(dx^i, dt, du_i, \dots, du_{i_1\cdots i_r}) &= (0, 0, 0, \dots, 0, u_{i_1} \cdots u_{i_r} u_j dx^j),
\end{aligned}$$

where $(i_i \cdots i_r)$ denotes the symmetrization.

Finally, the natural vector field L_s is of the form

$$L_s = u_{i_1} \cdots u_{i_s} \frac{\partial}{\partial u_{i_1\cdots i_s}} + \cdots + \frac{r!}{(s-1)!(r-s+1)!} u_{(i_1} \cdots u_{i_{s-1}} u_{i_s\cdots i_r)} \frac{\partial}{\partial u_{i_1\cdots i_r}},$$

see [4]. So we have

$$\begin{aligned}
(L_1 \otimes dt)(dx^i, dt, du_i, \dots, du_{i_1\cdots i_r}) &= (0, 0, u_i dt, \dots, u_{i_1\cdots i_r} dt) \\
&\vdots \\
(L_r \otimes dt)(dx^i, dt, du_i, \dots, du_{i_1\cdots i_r}) &= (0, 0, 0, \dots, 0, u_{i_1} u_{i_2} \cdots u_{i_r} dt).
\end{aligned}$$

Proposition 1. *All natural affinors $F^r : TT^{r*}_{\mathbb{R}}M \to TT^{r*}_{\mathbb{R}}M$ are of the form*

$$\begin{aligned}
F^r = a(t)\widetilde{\widetilde{Id}}_{T^{r*}M} + b(t)\widetilde{\widetilde{Id}}_{\mathbb{R}} + a_1(t)\widetilde{Q}^1_M + \cdots + a_r(t)\widetilde{Q}^r_M \\
+ b_1(t) L_1 \otimes dt + \cdots + b_r(t) L_r \otimes dt + c(t)P,
\end{aligned}$$

where $a(t), \dots, c(t)$ are arbitrary smooth functions of $\mathbb{R}$.

Proof. Denote by G^r_m the group of all invertible r-jets of $\mathbb{R}^m$ into $\mathbb{R}^m$ with the source and the target zero. By the general theory, [7], it suffices to find all G^{r+1}_m-equivariant linear maps $T(T^{r*}_{\mathbb{R}}\mathbb{R}^m)_0 \to T(T^{r*}_{\mathbb{R}}\mathbb{R}^m)_0$ of standard fibers.

Let $(a^i_j, a^i_{jk}, \dots, a^i_{j_1 j_2\cdots j_r})$ be the coordinates on G^r_m and denote by a tilde the inverse element. By standard evaluations we find the action of G^{r+1}_m on the standard fibre $T(T^{r*}_{\mathbb{R}}\mathbb{R}^m)_0$

$$(1) \qquad \overline{u}_i = \widetilde{a}_i^j u_j$$

$$(2) \qquad \overline{u}_{i_1 i_2} = \widetilde{a}_{i_1}^{j_1} \widetilde{a}_{i_2}^{j_2} u_{j_1 j_2} + \widetilde{a}_{i_1 i_2}^{j_1} u_{j_1}$$

$$\vdots$$

$$(3) \qquad \begin{aligned} \overline{u}_{i_1 \dots i_r} &= \widetilde{a}_{i_1}^{j_1} \dots \widetilde{a}_{i_r}^{j_r} u_{j_1 \dots j_r} \\ &\quad + \frac{r!}{(r-2)!2!} \widetilde{a}_{(i_1}^{j_1} \dots \widetilde{a}_{i_{r-2}}^{j_{r-2}} \widetilde{a}_{i_{r-1} i_r)}^{j_{r-1}} u_{j_1 \dots j_{r-1}} \\ &\quad + \dots + \left[\frac{r!}{(r-1)!1!} \widetilde{a}_{(i_1}^{j_1} \widetilde{a}_{i_2 \dots i_r)}^{j_2} + \cdots \right] u_{j_1 j_2} + \widetilde{a}_{i_1 \dots i_r}^{j} u_j \end{aligned}$$

$$(4) \qquad \overline{X}^i = a_j^i X^j$$

$$(5) \qquad \overline{T} = T$$

$$(6) \qquad \overline{U}_i = \widetilde{a}_i^j U_j + \widetilde{a}_{ik}^j a_l^k X^l u_j$$

$$(7) \qquad \begin{aligned} \overline{U}_{i_1 i_2} &= \widetilde{a}_{i_1}^{j_1} \widetilde{a}_{i_2}^{j_2} U_{j_1 j_2} + \widetilde{a}_{i_1 i_2}^{j_1} U_{j_1} \\ &\quad + \left(\widetilde{a}_{i_1}^{j_1} \widetilde{a}_{i_2 k}^{j_2} a_l^k X^l + \widetilde{a}_{i_2}^{j_2} \widetilde{a}_{i_1 k}^{j_1} a_l^k X^l \right) u_{j_1 j_2} + \widetilde{a}_{i_1 i_2 k}^{j_1} a_l^k X^l u_{j_1} \end{aligned}$$

$$\vdots$$

$$(8) \qquad \begin{aligned} \overline{U}_{i_1 \cdots i_r} &= \widetilde{a}_{i_1}^{j_1} \cdots \widetilde{a}_{i_r}^{j_r} U_{i_1 \cdots i_r} + \frac{r!}{(r-2!)2!} \widetilde{a}_{(i_1}^{j_1} \cdots \widetilde{a}_{i_{r-2}}^{j_{r-2}} \widetilde{a}_{i_{r-1} i_r)}^{j_{r-1}} U_{j_1 \cdots j_{r-1}} \\ &\quad + \dots + \left[\frac{r!}{(r-1)!1!} \widetilde{a}_{(i_1}^{j_1} \widetilde{a}_{i_2 \cdots i_r)}^{j_2} + \cdots \right] U_{j_1 j_2} \\ &\quad + \widetilde{a}_{i_1 \dots}^{j_1} i_r U_{j_1} + \left[\widetilde{a}_{i_1 k}^{j_1} \cdots \widetilde{a}_{i_r}^{j_r} a_l^k X^l + \cdots \right] u_{j_1 \cdots j_r} \\ &\quad + \left[\frac{r!}{(r-2)!2!} \widetilde{a}_{(i_1 k}^{j_1} \cdots \widetilde{a}_{i_{r-2}}^{j_{r-2}} \widetilde{a}_{i_{r-1} i_r)}^{j_{r-1}} a_l^k X^l + \cdots \right] u_{j_1 \cdots j_{r-1}} + \cdots \\ &\quad + \left[\frac{r!}{(r-1)!1!} \widetilde{a}_{(i_1 k}^{j_1} \widetilde{a}_{i_2 \cdots i_r)}^{j_1} a_l^k X^l + \cdots \right] u_{j_1 j_2} + \widetilde{a}_{i_1 \cdots i_r k}^{j_1} a_l^k X^l u_{j_1}. \end{aligned}$$

Write $u = (u_i, u_{ij}, \dots, u_{i_1 \cdots i_r})$. Any linear map of the standard fibre into itself has the form

$$(9) \qquad \overline{T} = \alpha_j(t,u) X^j + \beta(t,u) T + A^j(t,u) U_j + \cdots + A^{j_1 \cdots j_r}(t,u) U_{j_1 \cdots j_r}$$

$$(10) \qquad \overline{X}^i = \gamma_j^i(t,u) X^j + \delta^i(t,u) T + B^{ij}(t,u) U_j + \cdots + B^{i j_1 \cdots j_r}(t,u) U_{j_1 \cdots j_r}$$

$$\overline{U}_i = \eta_{ij}(t,u)X^j + \zeta_i(t,u)T + C_i^j(t,u)U_j + \cdots + C_i^{j_1\cdots j_r}(t,u)U_{j_1\cdots j_r} \tag{11}$$

$$\vdots$$

$$\overline{U}_{i_1\cdots i_r} = \eta_{i_1\cdots i_r j}(t,u)X^j + \zeta_{i_1\cdots i_r}(t,u)T + C^j_{i_1\cdots i_r}(t,u)U_j + \cdots + C^{j_1\cdots j_r}_{i_1\cdots i_r}(t,u)U_{j_1\cdots j_r}\ . \tag{12}$$

Considering equivariancy of (9) with respect to the homotheties $a^i_j = k\delta^i_j$ we obtain

$$\begin{aligned}
\frac{1}{k}\,\alpha_j(t,u_i,\dots,u_{i_1\cdots i_r}) &= \alpha_j\left(t,\frac{1}{k}u_i,\dots,\frac{1}{k^r}u_{i_1\cdots i_r}\right)\\
\beta(t,u_i,\dots,u_{i_1\cdots i_r}) &= \beta\left(t,\frac{1}{k}u_i,\dots,\frac{1}{k^r}u_{i_1\cdots i_r}\right)\\
A^j(t,u_i,\dots,u_{i_1\cdots i_r}) &= \frac{1}{k}A^j\left(t,\frac{1}{k}u_i,\dots,\frac{1}{k^r}u_{i_1\cdots i_r}\right)\\
&\ \,\vdots\\
A^{j_1\cdots j_r}(t,u_i,\dots,u_{i_1\cdots i_r}) &= \frac{1}{k^r}A^{j_1\cdots j_r}\left(t,\frac{1}{k}u_i,\dots,\frac{1}{k^r}u_{i_1\cdots i_r}\right).
\end{aligned}$$

By the homogenous function theorem from [7] we compute $\alpha_j(t,u) = \alpha(t)u_j$, $\beta(t,u) = \beta(t)$, $A^j(t,u) = 0$, $A^{j_1\cdots j_r}(t,u) = 0$. Thus (9) can be written in the form

$$\overline{T} = \alpha(t)u_jX^j + \beta(t)T. \tag{13}$$

Quite analogously we prove

$$\overline{X}^i = \gamma(t)X^i. \tag{14}$$

Further, equivariancy of (11) implies

$$\frac{1}{k^2}\,\eta_{ij}(t,u_i,\dots,u_{i_1\cdots i_r}) = \eta_{ij}\left(t,\frac{1}{k}u_i,\dots,\frac{1}{k^r}u_{i_1\cdots i_r}\right)$$

$$\frac{1}{k}\,\zeta_i(t,u_i,\dots,u_{i_1\cdots i_r}) = \zeta_i\left(t,\frac{1}{k}u_i,\dots,\frac{1}{k^r}u_{i_1\cdots i_r}\right)$$

$$
\begin{aligned}
C_i^j(t,u_i,\dots,u_{i_1\cdots i_r}) &= C_i^j\left(t,\frac{1}{k}u_i,\dots,\frac{1}{k^r}u_{i_1\cdots i_r}\right)\\
C_i^{j_1j_2}(t,u_i,\dots,u_{i_1\cdots i_r}) &= \frac{1}{k}C_i^{j_1j_2}\left(t,\frac{1}{k}u_i,\dots,\frac{1}{k^r}u_{i_1\cdots i_r}\right)\\
&\vdots\\
C_{i_1\cdots i_r}^{j_1\cdots j_r}(t,u_i,\dots,u_{i_1\cdots i_r}) &= \frac{1}{k^{r-1}}C_{i_1\cdots i_r}^{j_1\cdots j_r}\left(t,\frac{1}{k}u_i,\dots,\frac{1}{k^r}u_{i_1\cdots i_r}\right).
\end{aligned}
$$

Using the homogenous function theorem we obtain

$$
\overline{U}_i = \left({}_1\eta_{ij}(t)u_{ij} + {}_2\eta_{ij}(t)u_iu_j\right)X^j + \zeta(t)u_iT + C(t)\,U_i\ . \tag{15}
$$

Finally, the equivariancy of (12) leads to following relations:

$$
\begin{aligned}
\frac{1}{k^{r+1}}\,\eta_{i_1\cdots i_rj}(t,u_i,\dots,u_{i_1\cdots i_r}) &= \eta_{i_1\cdots i_rj}\left(t,\frac{1}{k}u_i,\dots,\frac{1}{k^r}u_{i_1\cdots i_r}\right)\\
\frac{1}{k^{r}}\,\zeta_{i_1\cdots i_r}(t,u_i,\dots,u_{i_1\cdots i_r}) &= \zeta_{i_1\cdots i_r}\left(t,\frac{1}{k}u_i,\dots,\frac{1}{k^r}u_{i_1\cdots i_r}\right)\\
\frac{1}{k^{r-1}}\,C^j_{i_1\cdots i_r}(t,u_i,\dots,u_{i_1\cdots i_r}) &= C^j_{i_1\cdots i_r}\left(t,\frac{1}{k}u_i,\dots,\frac{1}{k^r}u_{i_1\cdots i_r}\right)\\
\frac{1}{k^{r-2}}\,C^{j_1j_2}_{i_1\cdots i_r}(t,u_i,\dots,u_{i_1\cdots i_r}) &= C^{j_1j_2}_{i_1\cdots i_r}\left(t,\frac{1}{k}u_i,\dots,\frac{1}{k^r}u_{i_1\cdots i_r}\right)\\
&\vdots\\
C^{j_1\cdots j_r}_{i_1\cdots i_r}(t,u_i,\dots,u_{i_1\cdots i_r}) &= C^{j_1\cdots j_r}_{i_1\cdots i_r}\left(t,\frac{1}{k}u_i,\dots,\frac{1}{k^r}u_{i_1\cdots i_r}\right).
\end{aligned}
$$

By the homogenous function theorem, the function $\eta_{i_1\cdots i_rj}$ is a sum of the polynomials of degree a_s in $u_{i_1\cdots i_s}$ satisfying the relation

$$
r+1 = a_1 + 2a_2 + \cdots + ra_r.
$$

This has the following solutions:

$$
\begin{aligned}
&a_1 = r+1, a_2 = \cdots = a_r = 0\\
&a_1 = r-1, a_2 = 1, a_3 = \cdots = a_r = 0\\
&a_1 = r-2, a_3 = 1, a_2 = \cdots = a_r = 0\\
&\quad\vdots\\
&a_1 = 1, a_r = 1, a_2 = \cdots = a_{r-1} = 0
\end{aligned}
$$

so that $\eta_{i_1\cdots i_r j}$ can be written in the form

$$\begin{aligned}\eta_{i_1\cdots i_r j}(t,u) = {}& {}_r\eta_{i_1\cdots i_r j}(t)u_{i_1}u_{i_2}\cdots u_{i_r}u_j\\ &+ {}_{r-1,1}\eta_{i_1\cdots i_r j}(t)u_{(i_1 i_2}\cdots u_{i_{r-2}}u_{i_{r-1}i_r)}u_j\\ &+ {}_{r-1,2}\eta_{i_1\cdots i_r j}(t)u_{(i_1 i_2}\cdots u_{i_{r-2}}u_{i_{r-1}}u_{i_r)j}\\ &+\cdots+ {}_1\eta_{i_1\cdots i_r j}(t)u_{i_1\cdots i_r j}\ .\end{aligned}$$

By similar computations we find the expression of $\zeta_{i_1\cdots i_r}, C^j_{i_1\cdots i_r}, \ldots, C^{j_1\cdots j_r}_{i_1\cdots i_r}$ and we obtain

$$\begin{aligned}U_{i_1\cdots i_r} = {}& \big[{}_r\eta_{i_1\cdots i_r j}(t)u_{i_1}u_{i_2}\cdots u_{i_r}u_j\\ &\quad + {}_{r-1,1}\eta_{i_1\cdots i_r j}(t)u_{(i_1 i_2}\cdots u_{i_{r-2}}u_{i_{r-1}i_r)}u_j\\ &\quad + {}_{r-1,2}\eta_{i_1\cdots i_r j}(t)u_{(i_1 i_2}\cdots u_{i_{r-2}}u_{i_{r-1}}u_{i_r)jk}\\ &\quad +\cdots+ {}_1\eta_{i_1\cdots i_r j}(t)u_{i_1\cdots i_r j}\big]X^j\\ &+\big[{}_r\zeta_{i_1\cdots i_r}(t)u_{i_1}\cdots u_{i_r} + {}_{r-1}\zeta_{i_1\cdots i_r}(t)u_{(i_1}\cdots u_{i_{r-2}}u_{i_{r-1}i_r)}\\ &\quad +\cdots+ {}_1\zeta_{i_1\cdots i_r}(t)u_{i_1\cdots i_r}\big]T\\ &+\Big[{}_{r-1}C^j_{i_1\cdots i_r}(t)\delta^{j_1}_{(i_1}u_{i_2}\cdots u_{i_r)}\\ &\quad +\cdots+ {}_1C^j_{i_1\cdots i_r}(t)\delta^{j_1}_{(i_1}u_{i_2\cdots i_r)}\Big]U_{j_1}\\ &+\cdots+ C^{j_1\cdots j_r}_{i_1\cdots i_r}(t)\delta^{j_1}_{(i_1}\cdots\delta^{j_r}_{i_r)}U_{i_1\cdots i_r}\ .\end{aligned}\tag{16}$$

We first prove our assertion for $r=2$. Formulas (13)–(16) for $r=2$ are of the form

$$\overline{T} = \alpha(t)u_kX^k + \beta(t)T \tag{17}$$

$$\overline{X}^{\,i} = \gamma(t)X^i \tag{18}$$

$$\overline{U}_i = \big({}_1\eta_{ik}(t)u_{ik} + {}_2\eta_{ik}(t)u_iu_k\big)X^k + \zeta(t)u_iT + C(t)\,U_i \tag{19}$$

$$\begin{aligned}\overline{U}_{ij} = {}& \big({}_2\eta_{ijk}(t)u_iu_ju_k + {}_{1,1}\eta_{ijk}(t)u_{ij}u_k + {}_{1,2}\eta_{ijk}(t)u_{(i}u_{j)k}\\ &\quad + {}_1\eta_{ijk}(t)u_{ijk}\big)X^k + \big({}_2\zeta_{ij}(t)u_iu_j + {}_1\zeta_{ij}(t)u_{ij}\big)T\\ &+ C^i(t)\delta^k_{(i}u_{j)}U_k + C^{ij}(t)U_{ij}\ .\end{aligned}\tag{20}$$

The equivariancy of (19) with respect to the kernel of the jet projection $G^2_m \to G^1_m$ given by $a^i_j = \delta^i_j$ and a^i_{jk} arbitrary leads to relations

$$C(t) = \gamma(t),\ {}_1\eta_{ik} = 0$$

so that (19) is of the form

$$\overline{U}_i = \eta_i(t)u_iu_kX^k + \zeta_i(t)u_iT + \gamma(t)\,U_i\ . \tag{21}$$

Finally, the equivariancy of (20) with respect to the kernel of the jet projection $G^3_m \to G^1_m$ given by $a^i_j = \delta^i_j$ and a^i_{jk}, a^i_{jkl} arbitrary leads to relations

$$C^i = 0,\ C^{ij}(t) = \gamma(t),\ {}_1\zeta_{ij}(t) = \zeta_i(t),$$

$${}_{1,1}\eta_{ijk}(t) = \eta_i(t),\ {}_1\eta_{ijk} = 0,\ {}_{1,2}\eta_{ijk} = 0,$$

so that (20) is of the form

$$\begin{aligned} \overline{U}_{ij} = \eta_{ij}(t)u_iu_ju_kX^k + \eta_i(t)u_{ij}u_kX^k + \zeta_{ij}(t)u_iu_jT \\ + \zeta(t)u_{ij}T + \gamma(t)U_{ij}\ . \end{aligned} \tag{22}$$

Hence we have proved

$$\begin{aligned} F^2 = a(t)\widetilde{Id}_{TT^{2*}M} + b(t)\widetilde{Id}_{T\mathbb{R}} + a_1(t)\widetilde{Q}^1_M + a_2(t)\widetilde{Q}^2_M \\ + b_1(t)(L_1 \otimes dt) + b_2(t)(L_2 \otimes dt) + c(t)P, \end{aligned} \tag{23}$$

where

$$\begin{aligned} a(t) = \gamma(t),\ b(t) = \beta(t),\ a_1(t) = \eta_i(t),\ a_2(t) = \eta_{ij}(t) \\ b_1(t) = \zeta_i(t),\ b_2(t) = \zeta_{ij}(t),\ c(t) = \alpha(t). \end{aligned}$$

This proves our proposition for $r = 2$. To finish the proof, we will use the induction with respect to r. Suppose now, that our proposition is true for $r-1$, i.e.

$$\begin{aligned} F^{r-1} = a(t)\widetilde{Id}_{TT^{(r-1)*}M} + b(t)\widetilde{Id}_{T\mathbb{R}} + a_1(t)\widetilde{Q}^1_M + \cdots a_{r-1}(t)\widetilde{Q}^{r-1}_M \\ + b_1(t)(L_1 \otimes dt) + \cdots + b_{r-1}(t)(L_{r-1} \otimes dt) + c(t)P. \end{aligned} \tag{24}$$

Using the homogenous function theorem we deduce easily that the components of F^r at $T, X^i, U_i, \ldots, U_{i_1\cdots i_r}$ are exactly the corresponding components of F^{r-1}. That is why it suffices to determine the last $(r+2)$-th component of F^r, which is given by (16). The equivariancy with respect to the kernel of the projection $G^{r+1}_m \to G^1_m$ leads to the relations

$C^{j_1\cdots j_r}_{i_1\cdots i_r}(t) = a(t),\ \ C^{j}_{i_1\cdots i_r}(t) = \cdots = C^{j_1\cdots j_{r-1}}_{i_1\cdots i_r}(t) = 0,$

${}_1\eta_{i_1\cdots i_r j}(t) = a_1(t),\ \ {}_{s-1,1}\eta_{i_1\cdots i_r j}(t) = a_{s-1}(t)$ where $s = 2, \ldots, r,$

${}_{s-1,2}\eta_{i_1\cdots i_r j}(t) = 0$ where $s = 2, \ldots, r,$

${}_r\eta_{i_1\cdots i_r j}(t) = a_r(t)$ is a new function,

${}_1\zeta_{i_1\cdots i_r}(t) = b_1(t),\ \ {}_{s-1}\zeta_{i_1\cdots i_r}(t) = b_{s-1}(t)$ where $s = 2, \ldots, r,$

${}_r\zeta_{i_1\cdots i_r}(t) = b_r(t)$ is a new function.

This completes the proof. □

Corollary 1. *All natural affinors on the time-dependent cotangent bundle* $T^*_{\mathbb{R}}M$ *are of the form*

$$\overline{X}^i = a(t)X^i \tag{25}$$

$$\overline{U}_i = a_1(t)u_iu_kX^k + b_1(t)u_iT + a(t)\,U_i \tag{26}$$

$$\overline{T} = c(t)u_kX^k + b(t)T. \tag{27}$$

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