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Luecking's condition for zeros of analytic functions

Abstract. Let $A(\sigma)$ denote the class of functions f analytic in the unit disk $\mathbb{D}$ and such that $|f(z)| \le C\sigma(|z|)+C_1$, where C, C_1 are some positive constants and σ is a nonnegative, nondecreasing function on $[0,1)$. We characterize zero-sets of $f \in A(\sigma)$ in terms of a subharmonic function introduced by D. Luecking in [7]. Using this characterization we obtain new necessary conditions for $A(\sigma)$ zero-sets provided $\log\sigma$ satisfies the Dini condition $1/(1-r)\int_0^1 \log\sigma(t)dt \le C\log\sigma(r)$. This generalizes the known results obtained, e.g., in [4] and [1].

1. Introduction. Let σ be a nonnegative and nondecreasing function on $[0,1)$. A measurable function f defined in the unit disk $\mathbb{D}$ is said to be in the space $L(\sigma)$ if there are positive constants C, C_1 such that

$$|f(z)| \le C\sigma(|z|) + C_1, \quad z \in \mathbb{D}.$$

Throughout the paper we shall say that $\sigma : [0,1) \to [1,\infty)$ is an admissible weight if σ is nondecreasing and $\log(\sigma) \in L^1(0,1)$. In the case σ is an admissible weight we define $L(\sigma)$ to be the space of all measurable functions in $\mathbb{D}$ which satisfy

$$|f(z)| \le C\sigma(|z|), \quad z \in \mathbb{D},$$

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with some positive C. Let $H(\mathbb{D})$ denotes the space of functions analytic in the unit disk $\mathbb{D}$. We set $A(\sigma) = H(\mathbb{D}) \cap L(\sigma)$.

In the case when $\sigma(t) = \frac{1}{(1-t)^\alpha}$, $\alpha > 0$, and $\sigma(t) = \log \frac{e}{1-t}$ the corresponding spaces will be denoted by $L^{-\alpha}$ and L^0, respectively. We also put $A^{-\alpha} = H(\mathbb{D}) \cap L^{-\alpha}$ and $A^0 = H(\mathbb{D}) \cap L^0$.

The Bergman space A^p, $0 < p < \infty$, consists of the functions $f \in H(\mathbb{D})$ that belong to the space $L^p(\mathbb{D})$, that is, the integral $\int_{\mathbb{D}} |f(z)|^p dA(z)$ with respect to the normalized area measure dA is finite. The inclusion $A^p \subset A^{-2/p}$, $0 < p < \infty$, is well known, see, e.g., [3, p. 53].

If $X \subset H(\mathbb{D})$, then a sequence of points $\{z_n\} \subset \mathbb{D}$ is called X zero-set if there is a function $f \in X$ that vanishes precisely on this set. A^p zero-sets were studied e.g. in [4], [5] and [8]. In [7] D. Luecking gave a characterization for $A^{-\alpha}$ zero-sets and for A^p zero-sets in terms of the subharmonic function k defined by

$$k(z) = \frac{|z|^2}{2} \sum_{n=1}^{\infty} \frac{(1 - |z_n|^2)^2}{|1 - \bar{z}_n z|^2}, \quad z \in \mathbb{D}. \tag{1}$$

He proved that $\{z_n\}$ is an A^p zero-set if and only if there is a harmonic function h such that $e^{pk+h} \in L^1(\mathbb{D})$, or equivalently there is a non-zero analytic function F such that $F(z)e^{k(z)}$ is in $L^p(\mathbb{D})$. He also obtained a similar characterization for the growth spaces $A^{-\alpha}$: a sequence $\{z_n\}$ of points in $\mathbb{D}$ is a zero-set for $A^{-\alpha}$ if and only if the function $k(z) - \alpha \log \frac{1}{1-|z|^2}$ has a harmonic majorant.

Here we prove an analogous condition for $A(\sigma)$ zero-sets provided $\log \sigma$ satisfies the following Dini condition: there exits $C \geq 1$ such that

$$\log(\sigma(t)) \leq \frac{1}{1-t} \int_t^1 \log(\sigma(s)) ds \leq C \log(\sigma(t)), \quad 0 < t < 1.$$

As a special case we obtain that $\{z_n\}$ is a zero-set for A^0 space, if and only if there is a function h harmonic in $\mathbb{D}$ and such

$$k(z) - \log\log \frac{e}{1 - |z|} \leq h(z), \quad |z| < 1, \tag{2}$$

where k is given by (1).

A function $f \in H(\mathbb{D})$ is said to be a Bloch function if

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty\,.$$

Since the space of Bloch functions is contained in A^0, the condition stated above is necessary for zeros of Bloch functions. In the last section we show how some necessary conditions for $A(\sigma)$ zero-sets can be derived from their Luecking's characterizations.

Results on $A(\sigma)$ zero-sets with some σ have been obtained for example in [9], [6], [2] and [1].

Let A^0_α, $-1 < \alpha < \infty$, denote the Bergman–Nevalinna space consisting of functions $f \in H(\mathbb{D})$ satisfying the condition

$$\int_{\mathbb{D}} \log^+ |f(z)|(1-|z|)^\alpha dA(z) < \infty.$$

It is known that a sequence $\{z_n\}$ is an A^0_α zero-set if and only if

$$\sum_{n=1}^{\infty}(1-|z_n|)^{2+\alpha} < \infty, \quad \text{see, e.g., [3, p. 131].} \tag{3}$$

Note that our assumption on the weight σ implies that $A(\sigma) \subset A^0_0$. Therefore, if $\{z_n\}$ is $A(\sigma)$ zero-set, then $\sum\limits_{n=1}^{\infty}(1-|z_n|)^2 < \infty$.

2. Results on weights.

Definition 1. Let σ be a nondecreasing and nonnegative function on $[0,1)$, and let $0 < p < \infty$.

We say that σ satisfies the Dini condition D_p, in short $\sigma \in D_p$, if $\sigma \in L^p(0,1)$ and there exists $C \geq 1$,

$$\left(\frac{1}{1-t}\int_t^1 \sigma^p(s)ds\right)^{1/p} \leq C\sigma(t) + O(1) \qquad (t \to 1).$$

We denote by $C(p,\sigma)$ the infimum of all possible values of such C.

We say that an admissible weight σ satisfies the Dini condition D_0, in short $\sigma \in D_0$, if $\log(\sigma) \in D_1$, that is $\log(\sigma) \in L^1(0,1)$ and there exists $C \geq 1$,

$$\frac{1}{1-t}\int_t^1 \log(\sigma(s))ds \leq C\log(\sigma(t)) + O(1) \qquad (t \to 1).$$

We denote $C(0,\sigma)$ the infimum of all possible values of such C.

Note that if $\sigma(t) \geq 1$ for $t \in [0,1)$, then σ satisfies D_p condition , $0 < p < \infty$, if and only if there is a constant $C \geq 1$ such that

$$\left(\frac{1}{1-t}\int_t^1 \sigma^p(s)ds\right)^{1/p} \leq C\sigma(t), \quad 0 \leq t < 1.$$

Proposition 1. *Let σ be a nondecreasing and nonnegative function on $[0,1)$, and let $0 < p < \infty$.*

Then $\sigma \in D_p$ if and only if $\sigma^p \in D_1$, and

$$\min\{2^{1-\frac{1}{p}}, 1\}C(1, \sigma^p)^{1/p} \le C(p, \sigma) \le \max\{2^{\frac{1}{p}-1}, 1\}C(1, \sigma^p)^{1/p}.$$

Proof. Assume $\sigma \in D_p$. Then

$$\begin{aligned} \frac{1}{1-t}\int_t^1 \sigma^p(s)ds &\le (C(p,\sigma)\sigma(t) + O(1))^p \\ &\le \max\{2^{p-1}, 1\}C^p(p,\sigma)\sigma^p(t) + O(1). \end{aligned}$$

Hence

$$C(1, \sigma^p) \le \max\{2^{p-1}, 1\}C^p(p, \sigma),$$

or equivalently,

$$\min\{2^{1-\frac{1}{p}}, 1\}C(1, \sigma^p)^{1/p} \le C(p, \sigma).$$

Assume now $\sigma^p \in D_1$. Then

$$\begin{aligned} \left(\frac{1}{1-t}\int_t^1 \sigma^p(s)ds\right)^{1/p} &\le (C(1,\sigma^p)\sigma^p(t) + O(1))^{1/p} \\ &\le \max\{2^{(1/p)-1}, 1\}C(1,\sigma^p)^{1/p}\sigma(t) + O(1). \end{aligned}$$

Therefore

$$C(p, \sigma) \le \max\{2^{\frac{1}{p}-1}, 1\}C(1, \sigma^p)^{1/p}.$$

□

Proposition 2. *For $0 < p \le q < \infty$,*

(i) *$D_p \subset D_q$ and $C(p, \sigma) \le C(q, \sigma)$ for any $\sigma \in D_p$.*

(ii) *$\bigcup_{p>0} D_p \subset D_0$ and $C(0, \sigma) \le 1$ for any $\sigma \in \bigcup_{p>0} D_p$.*

Proof. (i) Note that

$$\left(\frac{1}{1-t}\int_t^1 \sigma^p(s)ds\right)^{1/p} \le \left(\frac{1}{1-t}\int_t^1 \sigma^q(s)ds\right)^{1/q} \le C(q,\sigma)\sigma(t) + O(1).$$

(ii) Assume $\sigma \in D_p$ and use Jensen's inequality to write

$$\begin{aligned} \exp\left[\frac{1}{1-t}\int_t^1 \log(\sigma(s))ds\right] &= \left(\exp\left(\frac{1}{1-t}\int_t^1 \log(\sigma^p(s))ds\right)\right)^{1/p} \\ &\le \left(\frac{1}{1-t}\int_t^1 \sigma^p(s)ds\right)^{1/p} \\ &\le C(p,\sigma)\sigma(t) + O(1) \\ &\le \exp[\log(C(p,\sigma)) + \log(\sigma(t))] + O(1). \end{aligned}$$

Hence using the inequality $\exp(A-B)-1 \le \exp(A)-\exp(B)$ for $A, B > 0$, we obtain

$$\begin{aligned}
&\exp\left[\left(\frac{1}{1-t}\int_t^1 \log(\sigma(s))ds\right) - \log(C(p,\sigma) - \log(\sigma(t)))\right] \\
&\quad \le \exp\left[\frac{1}{1-t}\int_t^1 \log(\sigma(s))ds\right] - \exp[\log(C(p,\sigma)) + \log(\sigma(t))] + 1 \\
&\quad \le O(1),
\end{aligned}$$

which gives

$$\frac{1}{1-t}\int_t^1 \log(\sigma(s))ds - \log(\sigma(t)) \le \log(C(p,\sigma)) + O(1) = O(1).$$

□

Lemma 1. *Let $\rho : [0,1) \to [1,\infty)$ be nondecreasing and satisfy the following Dini condition*

$$(D) \qquad \frac{1}{1-t}\int_t^1 \rho(s)ds \le C\rho(t),$$

where $C \ge 1$. Then

(a) $\displaystyle \frac{1}{1-t}\int_t^1 \log\left(\frac{e}{1-s}\right)\rho(s)ds \le C^2 \log\left(\frac{e}{1-t}\right)\rho(t)$.

(b) $\displaystyle \frac{1}{(1-t)m!}\int_t^1 \left(\log\left(\frac{1-t}{1-s}\right)\right)^m \rho(s)ds \le C^{m+1}\rho(t)$.

(c) $\displaystyle \frac{\rho(t)}{(1-t)^a}$ *is integrable and for any $0 < a < \frac{1}{C}$ satisfies condition (D).*

Proof. (a) Integrating condition (D) we obtain

$$\begin{aligned}
C\int_u^1 \rho(t)dt &\ge \int_u^1 \left(\frac{1}{1-t}\int_t^1 \rho(s)ds\right)dt \\
&\ge \int_u^1 \left(\int_u^s \frac{1}{1-t}dt\right)\rho(s)ds \\
&= \int_u^1 \log\left(\frac{1-u}{1-s}\right)\rho(s)ds \\
&= \int_u^1 \log\left(\frac{e}{1-s}\right)\rho(s)ds - \log\left(\frac{e}{1-u}\right)\int_u^1 \rho(s)ds \\
&\ge \int_u^1 \log\left(\frac{e}{1-s}\right)\rho(s)ds - C\log\left(\frac{e}{1-u}\right)(1-u)\rho(u).
\end{aligned}$$

Applying again Dini condition (D) we get

$$\begin{aligned}\frac{1}{1-u}\int_u^1 \log\left(\frac{e}{1-s}\right)\rho(s)ds &\le C\log\frac{e}{1-u}\rho(u)+C^2\rho(u)\\ &\le C^2\log\frac{e}{1-u}\rho(u).\end{aligned}$$

(b) The case $m=0$ is Dini condition (D). We will use induction over m. Assume the result holds for m and integrate again

$$\begin{aligned}C^{m+1}m!\int_u^1\rho(t)dt &\ge \int_u^1\left(\frac{1}{1-t}\int_t^1\left(\log\left(\frac{1-t}{1-s}\right)\right)^m\rho(s)ds\right)dt\\ &\ge \int_u^1\left(\int_u^s\frac{1}{1-t}\left(\log\left(\frac{1-t}{1-s}\right)\right)^m dt\right)\rho(s)ds\\ &= \frac{1}{m+1}\int_u^1\left(\log\left(\frac{1-u}{1-s}\right)\right)^{m+1}\rho(s)ds.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{(1-u)(m+1)!}\int_u^1\left(\log\left(\frac{1-u}{1-s}\right)\right)^{m+1}\rho(s)ds &\le \frac{1}{(1-u)}C^{m+1}\int_u^1\rho(t)dt\\ &\le C^{m+2}\rho(u).\end{aligned}$$

(c) Take $0<a<\frac{1}{C}$. Using (b) we obtain

$$\sum_{m=0}^{\infty}\frac{1}{(1-t)m!}\int_t^1\left(a\log\left(\frac{1-t}{1-s}\right)\right)^m\rho(s)ds \le C\sum_{m=0}^{\infty}(aC)^m\rho(t).$$

Since

$$\frac{1}{(1-t)}\int_t^1\sum_{m=0}^{\infty}\frac{1}{m!}\left(\log\left(\frac{(1-t)^a}{(1-s)^a}\right)\right)^m\rho(s)ds = \frac{1}{(1-t)}\int_t^1\frac{(1-t)^a}{(1-s)^a}\rho(s)ds,$$

we see that

$$\frac{1}{(1-t)}\int_t^1\frac{\rho(s)}{(1-s)^a}ds \le \frac{C}{1-aC}\frac{\rho(t)}{(1-t)^a}.$$

□

3. Main results. One of the most important facts used in the proof of the Luecking characterization of A^p zero-sets is that for $1 < p \leq \infty$ the Berezin transform R defined by

$$Rf(z) = \int_{\mathbb{D}} f(w) \frac{(1-|z|^2)^2}{|1-\bar{z}w|^4} dA(w) \tag{4}$$

is bounded from $L^p(\mathbb{D})$ to itself (see also [3]). It has been also proved in [7] that

if $0 < \alpha < 1$, *then* R *is a bounded operator from* $L^{-\alpha}$ *to* $L^{-\alpha}$.

We now present a different proof of this fact. Assume that $|f(z)| \leq M(1-|z|^2)^{-\alpha}$, $0 < \alpha < 1$. Then we have

$$\begin{aligned}
|Rf(re^{i\theta})| &= \left| \frac{1}{\pi} \int_0^1 \int_0^{2\pi} f(\rho e^{it}) \frac{(1-r^2)^2}{|1-r\rho e^{i(t-\theta)}|^4} dt \rho d\rho \right| \\
&\leq \frac{1}{\pi} \int_0^1 \sup_t |f(\rho e^{it})| \int_0^{2\pi} \frac{(1-r^2)^2}{|1-r\rho e^{it}|^4} dt \rho d\rho \\
&\leq CM \int_0^1 \frac{(1-r^2)^2 \rho d\rho}{(1-\rho^2)^\alpha (1-r^2\rho^2)^3} \\
&\leq CM \int_0^1 \frac{\rho d\rho}{(1-\rho^2)^\alpha (1-r^2\rho^2)} \\
&\leq \frac{K}{(1-r)^\alpha},
\end{aligned}$$

where we have used subsequently the known estimates:

$$\int_0^{2\pi} \frac{dt}{|1-re^{it}|^b} \leq \frac{C}{(1-r^2)^{b-1}}, \quad b > 1,$$

and

$$I(r) = \int_0^1 \frac{d\rho}{(1-\rho)^\alpha (1-r\rho)} \sim \frac{1}{(1-r)^\alpha}$$

(see, e.g., [10]). □

We now include a direct proof for the case $\sigma(t) = \log(\frac{1}{1-t})$.

Proposition 3. *The operator* R*, defined by (4), is bounded on* L^0*, that is, there is a positive constant* M *such that if* $|f(z)| \leq C \log \frac{1}{1-|z|} + O(1)$*, then*

$$|Rf(z)| \leq CM \log \frac{1}{1-|z|} + O(1), \quad z \in \mathbb{D}.$$

Proof. For $|z| = r$ we get

$$\begin{aligned}
|Rf(z)| &\leq \frac{C}{\pi}\int_0^1 \log\frac{1}{1-\rho}\int_0^{2\pi}\frac{(1-r^2)^2}{|1-r\rho e^{it}|^4}dt\rho d\rho \\
&\leq \frac{2C}{\pi}(1-r^2)\int_0^1 \log\frac{1}{1-\rho}\int_0^{2\pi}\frac{1}{|1-r\rho e^{it}|^3}dt\rho d\rho \\
&\leq CM(1-r)\int_0^1 \log\frac{1}{1-\rho}\frac{\rho d\rho}{(1-\rho r)^2} \\
&= CM(1-r)\sum_{n=1}^{\infty} n r^{n-1}\int_0^1 \rho^n \log\frac{1}{1-\rho}d\rho \\
&= CM(1-r)\sum_{n=1}^{\infty} n r^{n-1}\int_0^1 \sum_{k=1}^{\infty}\frac{\rho^{k+n}}{k}d\rho \\
&= CM(1-r)\sum_{n=1}^{\infty}\left(n r^{n-1}\sum_{k=1}^{\infty}\frac{1}{k(k+n+1)}\right) \\
&= CM(1-r)\sum_{n=1}^{\infty}\left(\frac{n r^{n-1}}{n+1}\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+n+1}\right)\right) \\
&= CM(1-r)\sum_{n=1}^{\infty}\frac{n r^{n-1}}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n+1}\right).
\end{aligned}$$

Putting $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, we have

$$\begin{aligned}
|Rf(z)| &\leq CM\sum_{n=1}^{\infty} H_{n+1}(r^{n-1}-r^n) \\
&= CM\left(\frac{3}{2}+\sum_{n=1}^{\infty}(H_{n+2}-H_{n+1})r^n\right) \\
&= CM\left(\frac{3}{2}+\sum_{n=1}^{\infty}\frac{r^n}{n+2}\right) \\
&\leq CM\left(\frac{3}{2}+\log\left(\frac{1}{1-r}\right)\right).
\end{aligned}$$

□

Actually one can show the following general principle.

Theorem 1. *Let σ be a nondecreasing and nonnegative function integrable on $[0,1)$. The following statements are equivalent:*

(i) *the operator R defined by (4) maps boundedly $L(\sigma)$ into $L(\sigma)$,*

(ii) $\sigma \in D_1$.

Moreover, $\|R\| \approx C(1, \sigma)$.

Proof. Assume that R defined by (4) is a bounded operator from $L(\sigma)$ into $L(\sigma)$. Define $f(z) = \sigma(|z|)$ for $|z| < 1$. Clearly $f \in L(\sigma)$ and $\|f\| = C(1, \sigma) = 1$.

Hence

$$\begin{aligned}
\|R\|\sigma(|z|) &\geq |Rf(z)| + O(1) \\
&= (1-|z|^2)^2 \int_{\mathbb{D}} \frac{\sigma(|w|)}{|1-\bar{z}w|^4} dA(w) + O(1) \\
&\geq (1-|z|^2)^2 \int_{|w|>|z|} \frac{\sigma(|w|)}{|1-\bar{z}w|^4} dA(w) + O(1) \\
&\geq K(1-|z|^2)^2 \int_{|z|}^1 \frac{\sigma(r)}{(1-|z|r)^3} dr + O(1) \\
&\geq K \frac{1}{(1-|z|)} \int_{|z|}^1 \sigma(r) dr + O(1).
\end{aligned}$$

Assume now that σ satisfies the Dini condition. If $f \in L(\sigma)$, then we get

$$\begin{aligned}
|Rf(z)| &\leq (1-|z|^2)^2 \int_{\mathbb{D}} \frac{|f(w)|}{|1-\bar{z}w|^4} dA(w) \\
&\leq C(1-|z|)^2 \int_{\mathbb{D}} \frac{\sigma(|w|)}{|1-\bar{z}w|^4} dA(w) + O(1) \\
&\leq C(1-|z|)^2 \int_0^1 \frac{\sigma(r)}{(1-|z|r)^3} dr + O(1) \\
&\leq C(1-|z|)^2 \left(\int_0^{|z|} \frac{\sigma(r)}{(1-r)^3} dr + \frac{1}{(1-|z|)^3} \int_{|z|}^1 \sigma(r) dr \right) + O(1).
\end{aligned}$$

Since σ is a nondecreasing function on $[0, 1)$, we see that

$$\int_0^{|z|} \frac{\sigma(r)}{(1-r)^3} dr \leq \sigma(|z|) \int_0^{|z|} \frac{1}{(1-r)^3} dr \leq \frac{\sigma(|z|)}{2(1-|z|)^2},$$

and consequently, using Dini condition, $|Rf(z)| \leq C\sigma(|z|) + O(1)$. □

Observe that Theorem 1 implies that R is bounded on $L^{-\alpha}$, $0 < \alpha < 1$, and on L^0.

We can now state the analogue of Theorem 2 in [7].

Theorem 2. *Let $\{z_n\}$ be a zero sequence of $f \in A(\sigma)$.*
If $\sigma \in D_0$, then there exists $\alpha \geq 1$ and $K > 0$ such that

$$\frac{|f(z)|}{\prod_{n=1}^{\infty}\left\{\left|\frac{z_n-z}{1-\bar{z}_n z}\right|\exp\left[\frac{1}{2}\left(1-\left|\frac{z_n-z}{1-\bar{z}_n z}\right|^2\right)\right]\right\}} \leq K\sigma^{\alpha}(|z|).$$

If $\sigma \in \bigcup_{p>0} D_p$ then there exists $K > 0$ such that

$$\frac{|f(z)|}{\prod_{n=1}^{\infty}\left\{\left|\frac{z_n-z}{1-\bar{z}_n z}\right|\exp\left[\frac{1}{2}\left(1-\left|\frac{z_n-z}{1-\bar{z}_n z}\right|^2\right)\right]\right\}} \leq K\sigma(|z|) + O(1).$$

Proof. Assume first that $\sigma \in D_0$. If $f \in A(\sigma)$, then there is a positive constant A such that

$$|f(z)| \leq A\sigma(|z|), \quad z \in \mathbb{D}.$$

It follows from formula (3) in [7] that

$$\frac{|f(z)|}{\prod_{n=1}^{\infty}\left\{\left|\frac{z_n-z}{1-\bar{z}_n z}\right|\exp\left[\frac{1}{2}\left(1-\left|\frac{z_n-z}{1-\bar{z}_n z}\right|^2\right)\right]\right\}} = \exp\Big(R(\log|f|)(z)\Big).$$

Since $\log|f|$ satisfies the Dini condition D_1 with some $C \geq 1$, Theorem 1 implies

$$R(\log(|f|)(z) \leq C\log(\sigma(|z|) + O(1),$$

and the result follows with $\alpha = C$.

Under the stronger assumption that $\sigma \in D_p$ for some $p > 0$ one can apply Jensen's inequality and obtain,

$$\frac{|f(z)|}{\prod_{n=1}^{\infty}\left\{\left|\frac{z_n-z}{1-\bar{z}_n z}\right|\exp\left[\frac{1}{2}\left(1-\left|\frac{z_n-z}{1-\bar{z}_n z}\right|^2\right)\right]\right\}} \leq \left(R(|f|^p)(z)\right)^{1/p}.$$

Since $\sigma^p \in D_1$, Theorem 1 yields

$$\left(R(|f|^p)(z)\right)^{1/p} \leq \left(C\sigma(|z|)^p + O(1)\right)^{1/p} \leq K\sigma(|z|) + O(1).$$

□

Now reasoning similar to that used in [7] gives

Theorem 3. *Let σ be an admissible weight in D_0 and let k be the subharmonic function defined by (1). Then the following statements are equivalent*

(a) *$\{z_n\}$ is an $A(\sigma)$ zero-set,*

(b) *there are* $\alpha \geq 1$ *and a nonzero analytic function* F *such that*

$$F(z)e^{k(z)} = O\left(\sigma^{\alpha}(|z|)\right) \quad \text{as } |z| \to 1,$$

(c) *there is a real valued harmonic function* h *such that*

$$e^{h(z)+k(z)} = O\left(\sigma^{\alpha}(|z|)\right) \quad \text{as } |z| \to 1.$$

In particular condition (c) means that $\{z_n\}$ is a zero-set of $f \in A(\sigma)$ if and only if there are a real valued harmonic function h such that

$$k(z) - \alpha \log \sigma(|z|) \leq h(z) \quad \text{for } |z| < 1. \tag{5}$$

4. Necessary conditions for $A(\sigma)$ zero-sets. We now take the advantage of Dini condition to get necessary conditions for $A(\sigma)$ zero-sets.

Corollary 1. *Assume that* σ *is an admissible weight and* $\log \sigma$ *satisfies Dini condition (D) stated in Lemma 1. If* $\{z_n\}$ *is an* $A(\sigma)$ *zero-set, then for* $0 < a < 1/C$,

$$\sum_{n=1}^{\infty} (1 - |z_n|^2)^{2-a} < \infty.$$

Proof. It suffices to use (c) in Lemma 1 to see that $A(\sigma) \subset A^0_\alpha$ with $\alpha = -a$. Now the result follows from (3). ☐

Theorem 4. *Assume that* σ *is an admissible weight and* $\log \sigma$ *satisfies condition (D) in Lemma 1. If* $\{z_n\}$ *is an* $A(\sigma)$ *zero-set, then there exists* $0 < a < 1/C$ *such that*

$$\sum_{n=1}^{\infty} (1 - |z_n|) F_a\left(\frac{1-s}{1-|z_n|}\right) \leq C_a \log(\sigma(s)), \tag{6}$$

where $F_a : (0, \infty) \to (0, \infty)$ *is given by* $F_a(t) = t^{a-1} \int_0^t \frac{du}{u^a(1+u)}$. *Moreover,*

$$\frac{1}{(1-r)^{1-a}} \int_r^1 \frac{\varphi(t)}{(1-t)^a} dt = O(\log \sigma(r)), \tag{7}$$

where $\varphi(r) = \sum_{|z_n| \leq r} (1 - |z_n|), \quad 0 \leq r < 1$*; and*

$$n(r) = O\left(\frac{1}{1-r} \log \sigma(r)\right), \tag{8}$$

where $n(r)$ *stands for the number of zeros of* f *in* $\{z : |z| \leq r\}$.

Proof. In (5) replacing k by k_1, given by

$$k_1(z) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(1-|z_n|^2)^2}{|1-\bar{z}_n z|^2}, \quad \text{(see [7, p. 354])},$$

we can write

$$\frac{1}{2}\sum_{n=1}^{\infty} \frac{(1-|z_n|^2)^2}{|1-\bar{z}_n z|^2} \leq \alpha \log \sigma(|z|) + h(z) \quad \text{for } |z|<1.$$

Integrating over the circle of radius r gives

$$\frac{1}{2}\sum_{n=1}^{\infty} \frac{(1-|z_n|^2)^2}{(1-|z_n|^2 r^2)} dr \leq \alpha \log \sigma(r) + h(0).$$

Hence for any $0<s<1$ and $0<a<1/C$,

$$\frac{1}{2}\int_s^1 \sum_{n=1}^{\infty} \frac{(1-|z_n|^2)^2}{(1-r)^a(1-|z_n|^2 r^2)} dr \leq \alpha \int_s^1 \frac{\log \sigma(r)}{(1-r)^a} dr + h(0)\int_s^1 \frac{1}{(1-r)^a} dr.$$

Since

$$\begin{aligned}
\int_s^1 \frac{dr}{(1-r)^a((1-|z_n|^2 r^2)} &\approx \int_s^1 \frac{dr}{(1-r)^a((1-|z_n| r)} \\
&\approx \int_s^1 \frac{dr}{(1-r)^a((1-|z_n|)+(1-r))} \\
&\approx \int_0^{1-s} \frac{1}{t^a((1-|z_n|)+t)} dt \\
&\approx \frac{1}{(1-|z_n|)^a}\int_0^{\frac{1-s}{1-|z_n|}} \frac{1}{u^a(1+u)} du
\end{aligned}$$

we have, due to the fact that $\frac{\log \sigma(r)}{(1-r)^a}$ satisfies Dini condition (D) by (c) in Lemma 1,

$$\begin{aligned}
&\sum_{n=1}^{\infty}(1-|z_n|)^{2-a}\left(\int_0^{\frac{1-s}{1-|z_n|}} \frac{1}{u^a(1+u)} du\right) \\
&\qquad \leq K\left(C\log(\sigma(s))(1-s)^{1-a} + \frac{h(0)}{1-a}(1-s)^{1-a}\right).
\end{aligned}$$

Hence

$$\sum_{n=1}^{\infty}(1-|z_n|)\left(\frac{1-|z_n|}{1-s}\right)^{1-a}\left(\int_0^{\frac{1-s}{1-|z_n|}}\frac{1}{u^a(1+u)}du\right) \leq K\left(C\log(\sigma(s))+\frac{h(0)}{1-a}\right).$$

We split the sum as follows:

$$\begin{aligned}
\sum_{|z_n|\leq s}(1-|z_n|)\left(\frac{1-|z_n|}{1-s}\right)^{1-a}&\left(\int_0^{\frac{1-s}{1-|z_n|}}\frac{du}{u^a(1+u)}\right)\\
&+\sum_{|z_n|>s}(1-|z_n|)\left(\frac{1-|z_n|}{1-s}\right)^{1-a}\left(\int_0^{1}\frac{du}{u^a(1+u)}\right)\\
&+\sum_{|z_n|>s}(1-|z_n|)\left(\frac{1-|z_n|}{1-s}\right)^{1-a}\left(\int_1^{\frac{1-s}{1-|z_n|}}\frac{du}{u^a(1+u)}\right)\\
&\approx \sum_{|z_n|\leq s}(1-|z_n|)\\
&\quad+\frac{1}{(1-s)^{1-a}}\sum_{|z_n|>s}(1-|z_n|)^{2-a}\\
&\quad+\sum_{|z_n|>s}(1-|z_n|)\left(\frac{1-|z_n|}{1-s}\right)^{1-a}\left(\int_1^{\frac{1-s}{1-|z_n|}}\frac{du}{u^a(1+u)}\right).
\end{aligned}$$

Note that the third sum is bounded by the second one, hence we get the estimates

$$\sum_{|z_n|\leq s}(1-|z_n|)\leq C\log(\sigma(s))+O(1), \tag{9}$$

and

$$\sum_{|z_n|>s}(1-|z_n|)^{2-a}\leq C(1-s)^{1-a}\log(\sigma(s)).$$

Finally (7) follows from (9) by Dini condition (D), and (8) is a simple consequence of (9). □

Theorem 5. *Assume that σ is a strictly increasing and continuously differentiable admissible weight such that $\log\sigma$ satisfies condition (D) in Lemma 1. If $\{z_n\}$, $z_n\neq 0$, is an $A(\sigma)$ zero-set, then*

$$\sum_{n=1}^{\infty}(1-|z_n|)\left(\int_{\sigma(|z_n|)}^{\infty}\frac{F(u)}{\log(u)}du\right)<\infty \tag{10}$$

for every nonnegative function $F \in L^1([1,\infty))$.

Proof. We may assume additionally that $\lim_{r\to 1}\sigma(r)=\infty$, because in the case when σ is bounded, the Blaschke condition $\sum(1-|z_n|)<\infty$ is satisfied. Under this assumption we have

$$\sum_{n=1}^{\infty}(1-|z_n|)\left(\int_{\sigma(|z_n|)}^{\infty}\frac{F(u)}{\log(u)}du\right)=\sum_{n=1}^{\infty}(1-|z_n|)\left(\int_{|z_n|}^{1}\frac{F(\sigma(r))}{\log(\sigma(r))}\sigma'(r)dr\right)$$
$$=\int_0^1\varphi(r)\frac{F(\sigma(r))}{\log(\sigma(r))}\sigma'(r)dr.$$

Now using the inequality $\varphi(t)\le C\log(\sigma(t))$, for all $t_0<t<1$, we obtain

$$\sum_{n=1}^{\infty}(1-|z_n|)\left(\int_{\sigma(|z_n|)}^{\infty}\frac{F(u)}{\log(u)}du\right)\le C\int_0^1 F(\sigma(r))\sigma'(r)dr$$
$$=C\int_1^{\infty}F(u)du<\infty.$$

□

Corollary 2. *Under the assumption of Theorem 5,*

$$\sum_{n=1}^{\infty}(1-|z_n|)\left(\log\sigma(|z_n|)\right)^{-1-\varepsilon}<\infty \tag{11}$$

for every $\varepsilon>0$.

Proof. Apply Theorem 5 with $F(u)=\frac{(\log(u))^{-(1+\varepsilon)}}{u}$ and observe that

$$\int_{\sigma(|z_n|)}^{\infty}\frac{du}{u(\log(u))^{2+\varepsilon}}du\approx\frac{1}{(\log(\sigma(|z_n|)))^{1+\varepsilon}}.$$

□

In the case of $A^{-\alpha}$, $\alpha>0$, and A^0 condition (11) was known, see, e.g. [3] and [1]. In this case this condition is the best in the sense that $\varepsilon>0$ cannot be omitted.

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