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Convergent infinite products and the minimization of convex functions

Dedicated to W.A. Kirk on the occasion of his receiving an Honorary Doctorate from Maria Curie-Skłodowska University

Abstract. We consider a metric space of sequences of uniformly continuous mappings, acting on a bounded, closed and convex subset of a Banach space, which share a common convex and lower semicontinuous Lyapunov function f. We show that for a generic sequence taken from this space, the corresponding infinite product tends to the unique point where f attains its minimum.

1. Introduction. In a series of recent papers [2], [6], [7], [11] we studied certain minimization methods for convex functions from the point of view of the theory of dynamical systems, and obtained several results regarding the convergence of these methods under the assumption that the function to be minimized is either uniformly continuous [2], [6], [7] or at least continuous [11]. In the present paper we consider the case where the objective function is merely lower semicontinuous. In our treatment the convergence of the minimization algorithms is cast in the language of (random) infinite products of operators. The convergence of such products is known to be of interest in many areas of mathematics and its applications [1], [8], [10].

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We also use the generic approach the aim of which is to show that a typical (in the sense of Baire category) element of an appropriate complete metric space has the relevant convergence property. For other applications of this approach to fixed point theory, nonlinear analysis and optimization see, for example, [8], [9], [10].

Assume that $(X, \|\cdot\|)$ is a Banach space, $K \subset X$ is a nonempty, bounded, closed and convex subset of X, and that $f : K \to R^1$ is a convex, bounded and lower semicontinuous function. Set

$$\inf(f) = \inf\{f(x) : \ x \in K\}.$$

Assume further that there exists $x_* \in K$ such that

$$f(x_*) = \inf(f), \tag{1.1}$$

and that the following conditions hold:

(i) f is continuous at x_*;

(ii) if $\{x_i\}_{i=1}^\infty \subset K$ and $\lim_{i\to\infty} f(x_i) = \inf(f)$, then $\|x_i - x_*\| \to 0$ as $i \to \infty$.

Denote by $\mathfrak{A}$ the set of all uniformly continuous mappings $A : K \to K$ (for each $\epsilon > 0$, there exists $\delta > 0$ such that if $x_1, x_2 \in K$ satisfy $\|x_1 - x_2\| \le \delta$, then $\|Ax_1 - Ax_2\| \le \epsilon$) such that

$$f(Ax) \le f(x) \text{ for all } x \in K. \tag{1.2}$$

For the set $\mathfrak{A}$ we define a metric $\rho : \mathfrak{A} \times \mathfrak{A} \to R^1$ by

$$\rho(A, B) = \sup\{\|Ax - Bx\| : \ x \in K\}, \ A, B \in \mathfrak{A}. \tag{1.3}$$

Clearly, the metric space $(\mathfrak{A}, \rho)$ is complete. Denote by $\mathfrak{M}$ the set of all sequences $\{A_t\}_{t=1}^\infty \subset \mathfrak{A}$. Members $\{A_t\}_{t=1}^\infty$, $\{B_t\}_{t=1}^\infty$ and $\{C_t\}_{t=1}^\infty$ of $\mathfrak{M}$ will occasionally be denoted by boldface $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, respectively.

For each $A \in \mathfrak{A}$ set $\widehat{\mathbf{A}} = \{\widehat{A}_t\}_{t=1}^\infty$, where $\widehat{A}_t = A$, $t = 1, 2, \dots$.

For the set $\mathfrak{M}$ we will consider two uniformities and the topologies induced by them. The first uniformity is determined by the following base:

$$\begin{aligned} E_w(N, \epsilon) = \{(\{A_t\}_{t=1}^\infty, \{B_t\}_{t=1}^\infty) \in \mathfrak{M} \times \mathfrak{M} : \rho(A_t, B_t) \le \epsilon,& \\ t = 1, \dots, N\},& \end{aligned} \tag{1.4}$$

where N is a natural number and $\epsilon > 0$. Clearly, the uniform space $\mathfrak{M}$ with this uniformity is metrizable (by a metric $\rho_w : \mathfrak{M} \times \mathfrak{M} \to R^1$) and complete. We equip the set $\mathfrak{M}$ with the topology induced by this uniformity. This topology will be called weak and denoted by τ_w.

The second uniformity is determined by the following base:

$$E_s(\epsilon) = \{(\{A_t\}_{t=1}^\infty, \{B_t\}_{t=1}^\infty) \in \mathfrak{M} \times \mathfrak{M} : \rho(A_t, B_t) \le \epsilon, \ t \ge 1\}, \tag{1.5}$$

where $\epsilon > 0$. The uniform space $\mathfrak{M}$ with this uniformity is also metrizable (by a metric $\rho_s : \mathfrak{M} \times \mathfrak{M} \to R^1$) and complete. We equip the set $\mathfrak{M}$ with the topology induced by this uniformity and denote this topology by τ_s. Since τ_s is obviously stronger than τ_w, it will be called strong.

From the point of view of the theory of dynamical systems, each element of $\mathfrak{M}$ describes a nonstationary dynamical system with a Lyapunov function f. Also, some optimization procedures in Hilbert and Banach spaces can be represented by elements of $\mathfrak{M}$ (see [3], [4]).

Denote by $\mathfrak{M}_u$ the set of all $\{A_t\}_{t=1}^{\infty} \in \mathfrak{M}$ which have the following property:

(iii) For each $\epsilon > 0$, there exists $\delta > 0$ such that for each $x, y \in K$ satisfying $\|x - y\| \le \delta$ and each integer $t \ge 1$,

$$\|A_t x - A_t y\| \le \epsilon.$$

It is clear that $\mathfrak{M}_u$ is a closed subset of $(\mathfrak{M}, \rho_s)$. We consider the metric space $(\mathfrak{M}_u, \rho_s)$ with the topology induced by the metric ρ_s.

A sequence $\{A_t\}_{t=1}^{\infty} \in \mathfrak{M}$ is called convergent if

$$A_n \cdots A_1 x \to x_* \text{ as } n \to \infty, \text{ uniformly on } K.$$

A sequence $\{A_t\}_{t=1}^{\infty} \in \mathfrak{M}$ is called strictly convergent if for each $\epsilon > 0$, there exists a natural number n such that for each integer $T \ge n$, each mapping $r : \{1, \dots, T\} \to \{1, 2, \dots\}$, and each $x \in K$,

$$\|A_{r(T)} \cdots A_{r(1)} x - x_*\| \le \epsilon.$$

A mapping $A \in \mathfrak{A}$ is called convergent if $\widehat{\mathbf{A}} = \{\widehat{A}_t\}_{t=1}^{\infty}$ is (strictly) convergent.

Our goal in this paper is to establish the following five results. The first two are convergence theorems while the last three illustrate the generic approach.

Theorem 1.1. *Let $\{A_t\}_{t=1}^{\infty} \in \mathfrak{M}$ be convergent and let ϵ be a positive number. Then there exist a natural number n and a neighborhood $\mathcal{U}$ of $\{A_t\}_{t=1}^{\infty}$ in $\mathfrak{M}$ with the weak topology such that the following property holds:*

For each $\{B_t\}_{t=1}^{\infty} \in \mathcal{U}$, each $x \in K$, and each integer $T \ge n$,

$$\|B_T \cdots B_1 x - x_*\| \le \epsilon.$$

Theorem 1.2. *Let $\{A_t\}_{t=1}^{\infty} \in \mathfrak{M}_u$ be strictly convergent and let ϵ be a positive number. Then there exist a natural number n and a neighborhood $\mathcal{U}$ of $\{A_t\}_{t=1}^{\infty}$ in $\mathfrak{M}_u$ with the strong topology such that the following property holds:*

For each $\{B_t\}_{t=1}^{\infty} \in \mathcal{U}$, each $x \in K$, each integer $T \ge n$, and each mapping $r : \{1, \dots, T\} \to \{1, 2, \dots\}$,

$$\|B_{r(T)} \cdots B_{r(1)} x - x_*\| \le \epsilon.$$

Theorem 1.3. *There exists a set $\mathcal{F} \subset \mathfrak{M}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathfrak{M}$ such that each element of $\mathcal{F}$ is convergent.*

Theorem 1.4. *There exists a set $\mathcal{F}_u$ which is a countable intersection of open everywhere dense subsets of $(\mathfrak{M}_u, \rho_s)$ such that each element of $\mathcal{F}_u$ is strictly convergent.*

Theorem 1.5. *There exists a set $F \subset \mathfrak{A}$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{A}$ such that each element of F is convergent.*

Our paper is organized as follows: the next section contains two lemmata on (random) infinite products. The first two theorems are established in Section 3. The last section is devoted to the proofs of our last three theorems.

2. Auxiliary results.

Lemma 2.1. *Let $\{A_t\}_{t=1}^{\infty} \in \mathfrak{M}$, n be a natural number and $\epsilon > 0$. Then there exists $\delta > 0$ such that for each $\{B_t\}_{t=1}^{\infty} \in \mathfrak{M}$ satisfying $(\{B_t\}_{t=1}^{\infty}, \{A_t\}_{t=1}^{\infty}) \in E_w(n, \delta)$, the following inequality holds for all $x \in K$:*

$$\|B_n \cdots B_1 x - A_n \cdots A_1 x\| \le \epsilon.$$

Proof. We prove this lemma by induction on n. It is clear that for $n = 1$, the assertion of the lemma holds for any $\epsilon > 0$.

Assume that k is a natural number and that the assertion of the lemma holds for any $\epsilon > 0$ with $n = k$. We intend to show that the assertion of the lemma holds with $n = k + 1$ for any $\epsilon > 0$. Indeed, given $\epsilon > 0$, there exists

$$\epsilon_0 \in \left(0, \frac{\epsilon}{8}\right) \tag{2.1}$$

such that

$$\|A_{k+1} x - A_{k+1} y\| \le \frac{\epsilon}{8} \tag{2.2}$$

for all $x, y \in K$ satisfying $\|x - y\| \le \epsilon_0$.

Using the inductive assumption with $n = k$ and $\epsilon = \epsilon_0$, we see that there exists

$$\delta \in (0, \epsilon_0) \tag{2.3}$$

such that the following property holds:

(iv) For each $\{B_t\}_{t=1}^{\infty} \in \mathfrak{M}$ satisfying $(\{B_t\}_{t=1}^{\infty}, \{A_t\}_{t=1}^{\infty}) \in E_w(k, \delta)$,

$$\|B_k \cdots B_1 x - A_k \cdots A_1 x\| \le \epsilon_0 \text{ for all } x \in K.$$

Assume now that $\{B_t\}_{t=1} \in \mathfrak{M}$,

$$(\{B_t\}_{t=1}^{\infty}, \{A_t\}_{t=1}^{\infty}) \in E_w(k + 1, \delta), \tag{2.4}$$

and that $x \in K$.

By (2.4) and property (iv),

$$\|B_k \cdots B_1 x - A_k \cdots A_1 x\| \le \epsilon_0. \tag{2.5}$$

It follows from (2.5) and the choice of ϵ_0 (see (2.1) and (2.2)) that

$$\|A_{k+1}B_k \cdots B_1 x - A_{k+1}A_k \cdots A_1 x\| \le \frac{\epsilon}{8}. \tag{2.6}$$

Relations (2.3), (2.4), (2.1) and (1.4) imply that

$$\|B_{k+1}(B_k \cdots B_1 x) - A_{k+1}(B_k \cdots B_1 x)\| \le \delta < \epsilon_0 < \frac{\epsilon}{8}.$$

When combined with (2.6), this inequality implies that

$$\begin{aligned}
&\|B_{k+1}B_k \cdots B_1 x - A_{k+1}A_k \cdots A_1 x\| \\
&\qquad \le \|B_{k+1}B_k \cdots B_1 x - A_{k+1}B_k \cdots B_1 x\| \\
&\qquad\quad + \|A_{k+1}B_k \cdots B_1 x - A_{k+1}A_k \cdots A_1 x\| \\
&\qquad \le \frac{\epsilon}{8} + \frac{\epsilon}{8} < \epsilon.
\end{aligned}$$

Thus the assertion of the lemma holds with $n = k+1$ for any $\epsilon > 0$. This completes the proof of the Lemma 2.1. □

Lemma 2.2. *Let $\{A_t\}_{t=1}^{\infty} \in \mathfrak{M}_u$, n be a natural number and $\epsilon > 0$. Then there exists $\delta > 0$ such that for each $\{B_t\}_{t=1}^{\infty} \in \mathfrak{M}_u$ satisfying $(\{B_t\}_{t=1}^{\infty}, \{A_t\}_{t=1}^{\infty}) \in E_s(\delta)$ and each $r : \{1, \dots, n\} \to \{1, 2, \dots\}$, the following inequality holds for all $x \in K$:*

$$\|B_{r(n)} \cdots B_{r(1)} x - A_{r(n)} \cdots A_{r(1)} x\| \le \epsilon.$$

Proof. Once again we use induction on n. It is clear that for $n = 1$, the assertion of the lemma holds for any $\epsilon > 0$.

Assume now that k is a natural number and that the lemma holds for any $\epsilon > 0$ when $n = k$. We intend to show that the assertion of the lemma holds with $n = k+1$ for any $\epsilon > 0$. Since $\{A_t\}_{t=1}^{\infty} \in \mathfrak{M}_u$, there exists

$$\epsilon_0 \in \left(0, \frac{\epsilon}{8}\right) \tag{2.7}$$

such that

$$\|A_i x - A_i y\| \le \frac{\epsilon}{8} \tag{2.8}$$

for each natural number i and all $x, y \in K$ satisfying $\|x - y\| \le \epsilon_0$.

Using the inductive assumption with $n = k$ and $\epsilon = \epsilon_0$, we see that there exists

$$\delta \in (0, \epsilon_0) \tag{2.9}$$

such that the following property holds:

(v) For each $\{B_t\}_{t=1}^{\infty} \in \mathfrak{M}$ satisfying

$$(\{B_t\}_{t=1}^{\infty}, \{A_t\}_{t=1}^{\infty}) \in E_s(\delta), \tag{2.10}$$

each $r : \{1, \dots, k\} \to \{1, 2, \dots\}$, and all $x \in K$,

$$\|B_{r(k)} \cdots B_{r(1)} x - A_{r(k)} \cdots A_{r(1)} x\| \le \epsilon_0. \tag{2.11}$$

Assume that $\{B_t\}_{t=1} \in \mathfrak{M}$ satisfies (2.10), $x \in K$, and $r : \{1, \dots, k+1\} \to \{1, 2, \dots\}$. Inclusion (2.10) and property (v) imply (2.11).

Therefore the definition of ϵ_0 (see (2.7) and (2.8)) implies that

$$\|A_{r(k+1)}B_{r(k)} \cdots B_{r(1)}x - A_{r(k+1)}A_{r(k)} \cdots A_{r(1)}x\| \le \frac{\epsilon}{8}. \tag{2.12}$$

Relations (2.10), (1.5), (2.9) and (2.7) yield

$$\|B_{r(k+1)}B_{r(k)} \cdots B_{r(1)}x - A_{r(k+1)}(B_{r(k)} \cdots B_{r(1)})x\| \le \delta < \epsilon_0 < \frac{\epsilon}{8}.$$

When combined with (2.12), this inequality implies that

$$\begin{aligned}
&\|B_{r(k+1)}B_{r(k)} \cdots B_{r(1)}x - A_{r(k+1)}A_{r(k)} \cdots A_{r(1)}x\| \\
&\qquad \le \|B_{r(k+1)}B_{r(k)} \cdots B_{r(1)}x - A_{r(k+1)}B_{r(k)} \cdots B_{r(1)}x\| \\
&\qquad\quad + \|A_{r(k+1)}B_{r(k)} \cdots B_{r(1)}x - A_{r(k+1)}A_{r(k)} \cdots A_{r(1)}x\| \\
&\qquad \le \frac{\epsilon}{8} + \frac{\epsilon}{8} < \epsilon.
\end{aligned}$$

Thus the assertion of the lemma indeed holds with $n = k+1$ for any $\epsilon > 0$. This completes the proof of Lemma 2.2. □

3. Proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. By property (ii) there is

$$\epsilon_0 \in (0, \epsilon) \tag{3.1}$$

such that

$$\text{if } z \in K \text{ and } f(z) \le \inf(f) + \epsilon_0, \text{ then } \|z - x_*\| \le \epsilon. \tag{3.2}$$

By property (i) there is

$$\epsilon_1 \in (0, \epsilon_0) \tag{3.3}$$

such that

$$\text{if } z \in K \text{ and } \|z - x_*\| \le \epsilon_1, \text{ then } |f(z) - f(x_*)| \le \epsilon_0. \tag{3.4}$$

Since $\{A_t\}_{t=1}^{\infty}$ is convergent, there is a natural number $n \ge 4$ such that

$$\|A_n \cdots A_1 x - x_*\| \le \frac{\epsilon_1}{4} \text{ for all } x \in K. \tag{3.5}$$

By Lemma 2.1, there exists a neighborhood $\mathcal{U}$ of $\{A_t\}_{t=1}^{\infty}$ in $\mathfrak{M}$ with the weak topology such that for each $\{B_t\}_{t=1}^{\infty} \in \mathcal{U}$,

$$\|B_n \cdots B_1 x - A_n \cdots A_1 x\| \le \frac{\epsilon_1}{4} \text{ for all } x \in K. \tag{3.6}$$

Let

$$\{B_t\}_{t=1}^{\infty} \in \mathcal{U}. \tag{3.7}$$

Then (3.6) holds. Inequalities (3.5) and (3.6) imply that for all $x \in K$,

$$\begin{aligned}\|B_n \cdots B_1 x - x_*\| &\le \|B_n \cdots B_1 x - A_n \cdots A_1 x\| + \|A_n \cdots A_1 x - x_*\| \\ &\le \frac{\epsilon_1}{4} + \frac{\epsilon_1}{4} < \epsilon_1.\end{aligned} \tag{3.8}$$

Let $x \in K$ and let $T \ge n$ be an integer. Relations (3.8) and (3.4) imply that

$$f(B_n \cdots B_1 x) \le \inf(f) + \epsilon_0.$$

This inequality implies, in its turn, that

$$f(B_T \cdots B_1 x) \le f(B_n \cdots B_1 x) \le \inf(f) + \epsilon_0.$$

When combined with (3.2), this last inequality implies that

$$\|B_T \cdots B_1 x - x_*\| \le \epsilon.$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. By property (ii), there is $\epsilon_0 \in (0, \epsilon)$ such that (3.2) holds. By property (i), there is $\epsilon_1 \in (0, \epsilon_0)$ such that (3.4) holds. Since $\{A_t\}_{t=1}^{\infty}$ is strictly convergent, there is a natural number $n \ge 4$ such that the following property holds:

(vi) For each mapping $r : \{1, \dots, n\} \to \{1, 2, \dots\}$ and each $x \in K$,

$$\|A_{r(n)} \cdots A_{r(1)} - x_*\| \le \frac{\epsilon_1}{4}.$$

By Lemma 2.2, there exists a neighborhood $\mathcal{U}$ of $\{A_t\}_{t=1}^{\infty}$ in $\mathfrak{A}$ with the strong topology such that for each $\{B_t\}_{t=1}^{\infty} \in \mathcal{U}$, each $r : \{1, \dots, n\} \to \{1, 2, \dots\}$, and each $x \in K$,

$$\|B_{r(n)} \cdots B_{r(1)} x - A_{r(n)} \cdots A_{r(1)} x\| \le \frac{\epsilon_1}{4}. \tag{3.9}$$

Now assume that

$$\{B_t\}_{t=1}^{\infty} \in \mathcal{U},\ x \in K,\ T \ge n \text{ is an integer, and } r : \{1, \dots, T\} \to \{1, 2, \dots\}. \tag{3.10}$$

Then (3.9) holds. By property (vi),

$$\|A_{r(n)} \cdots A_{r(1)} x - x_*\| \le \frac{\epsilon_1}{4}.$$

When combined with (3.9), this inequality implies that

$$\begin{aligned}&\|B_{r(n)} \cdots B_{r(1)} x - x_*\| \\ &\quad \le \|B_{r(n)} \cdots B_{r(1)} x - A_{r(n)} \cdots A_{r(1)} x\| + \|A_{r(n)} \cdots A_{r(1)} x - x_*\| \\ &\quad \le \frac{\epsilon_1}{4} + \frac{\epsilon_1}{4} < \epsilon_1.\end{aligned}$$

When combined with (3.4), this last inequality implies that

$$f(B_{r(n)} \cdots B_{r(1)} x) \le \inf(f) + \epsilon_0$$

and

$$f(B_{r(T)} \cdots B_{r(1)}x) \le f(B_{r(n)} \cdots B_{r(1)}x) \le \inf(f) + \epsilon_0.$$

Together with (3.2) this implies that

$$\|B_{r(T)} \cdots B_{r(1)}x - x_*\| \le \epsilon,$$

as asserted. The proof of Theorem 1.2 is complete. □

4. Proofs of Theorems 1.3–1.5. Let $A \in \mathfrak{A}$ and $\gamma \in (0,1)$. Define $A_\gamma : K \to K$ by

$$A_\gamma x = (1-\gamma)Ax + \gamma x_*, \ x \in K. \tag{4.1}$$

Clearly, $A_\gamma \in \mathfrak{A}$.

Let $\mathbf{A} = \{A_t\}_{t=1}^\infty \in \mathfrak{M}$ and $\gamma \in (0,1)$. Define $\mathbf{A}_\gamma = \{A_{t\gamma}\}_{t=1}^\infty$, where for each natural number t,

$$A_{t\gamma}x = (1-\gamma)A_t x + \gamma x_*, \ x \in K. \tag{4.2}$$

It is clear that $\mathbf{A}_\gamma \in \mathfrak{M}$, and if $\mathbf{A} \in \mathfrak{M}_u$, then $\mathbf{A}_\gamma \in \mathfrak{M}_u$.

We precede the proofs of Theorems 1.3–1.5 with another lemma.

Lemma 4.1. *Let* $\mathbf{A} = \{A_t\}_{t=1}^\infty \in \mathfrak{M}$ *and* $\gamma \in (0,1)$. *Then* $\mathbf{A}_\gamma = \{A_{t\gamma}\}_{t=1}^\infty$ *is strictly convergent.*

Proof. Let $\epsilon > 0$. Choose $M > 0$ such that

$$|f(x)| \le M \text{ for all } x \in K. \tag{4.3}$$

It follows from (4.2) and the convexity of f that for each natural number t and each $x \in K$,

$$\begin{aligned} f(A_{t\gamma}x) &= f((1-\gamma)A_t x + \gamma x_*) \\ &\le (1-\gamma)f(A_t x) + \gamma f(x_*) \le (1-\gamma)f(x) + \gamma(x_*) \\ &= (1-\gamma)f(x) + \gamma \inf(f). \end{aligned} \tag{4.4}$$

We claim that for each natural number m, the following property holds:

(vii) For each $r : \{1, \dots, m\} \to \{1, 2, \dots\}$ and each $x \in K$,

$$f(A_{r(m)\gamma} \cdots A_{r(1)\gamma}x) \le (1-\gamma)^m f(x) + [1 - (1-\gamma)^m] \inf(f). \tag{4.5}$$

We prove this claim by using induction on m. Clearly, for $m = 1$ property (vii) does hold. Let k be a natural number and assume that property (vii) holds for $m = k$. We will show that property (vii) holds with $m = k+1$.

To this end, assume that $x \in K$ and $r : \{1, \dots, k+1\} \to \{1, 2, \dots\}$. Since property (vii) holds with $m = k$, we have

$$f(A_{r(k)\gamma} \dots A_{r(1)\gamma}x) \le (1-\gamma)^k f(x) + [1 - (1-\gamma)^k] \inf(f). \tag{4.6}$$

By (4.2), the convexity of f, (1.2) and (4.6),

$$\begin{aligned} & f(A_{r(k+1)\gamma}A_{r(k)\gamma}\cdots A_{r(1)\gamma}x) \\ & \qquad = f((1-\gamma)A_{r(k+1)}A_{r(k)\gamma}\ldots A_{r(1)\gamma}x+\gamma x_*) \\ & \qquad \le (1-\gamma)f(A_{r(k+1)\gamma}A_{r(k)\gamma}\cdots A_{r(1)\gamma}x)+\gamma f(x_*) \\ & \qquad \le (1-\gamma)f(A_{r(k)\gamma}\cdots A_{r(1)\gamma}x)+\gamma\inf(f) \\ & \qquad \le (1-\gamma)[(1-\gamma)^k f(x)+(1-(1-\gamma)^k)\inf(f)]+\gamma\inf(f) \\ & \qquad = (1-\gamma)^{k+1}f(x)+[(1-\gamma)-(1-\gamma)^{k+1}+\gamma]\inf(f) \\ & \qquad = (1-\gamma)^{k+1}f(x)+[1-(1-\gamma)^{k+1}]\inf(f). \end{aligned}$$

This means that property (vii) holds for $m=k+1$. Thus we have shown that property (vii) holds for all natural numbers m. By property (ii), there is $\epsilon_0 \in (0,\epsilon)$ such that

$$\text{if } z\in K \text{ and } f(z)\le \inf(f)+\epsilon_0, \text{ then } \|z-x_*\|\le\epsilon. \tag{4.7}$$

Choose an integer $n\ge 4$ such that

$$(1-\gamma)^n M+[1-(1-\gamma)^n]\inf(f)\le \inf(f)+\epsilon_0. \tag{4.8}$$

Assume that $T\ge n$ is an integer, $r:\{1,\ldots,T\}\to\{1,2,\ldots\}$, and $x\in K$. By property (vii), (4.8), (1.2) and (4.3),

$$\begin{aligned} f(A_{r(T)\gamma}\cdots A_{r(1)\gamma}x) &\le f(A_{r(n)\gamma}\cdots A_{r(1)\gamma}x) \\ &\le (1-\gamma)^n f(x)+[1-(1-\gamma)^n]\inf(f) \\ &\le (1-\gamma)^n M+[1-(1-\gamma)^n]\inf(f)\le\inf(f)+\epsilon_0. \end{aligned}$$

When combined with (4.7), this inequality implies that

$$\|A_{r(T)\gamma}\cdots A_{r(1)\gamma}x-x_*\|\le\epsilon.$$

Lemma 4.1 is proved. □

Proof of Theorem 1.3. It is easy to see that for each $\mathbf{A}\in\mathfrak{M}$, $\mathbf{A}_\gamma\to\mathbf{A}$ as $\gamma\to 0^+$ in the strong topology. Thus the set

$$\{\mathbf{A}_\gamma:\ \mathbf{A}\in\mathfrak{M},\ \gamma\in(0,1)\}$$

is an everywhere dense subset of $\mathfrak{M}$ with the strong topology.

Let $\mathbf{A}=\{A_t\}_{t=1}^\infty\in\mathfrak{M}$, $\gamma\in(0,1)$, and let $i\ge 1$ be a natural number. By Lemma 4.1, $\mathbf{A}_\gamma=\{A_{t\gamma}\}_{t=1}^\infty$ is strictly convergent. By Theorem 1.1, there exist a natural number $n(\mathbf{A},\gamma,i)$ and an open neighborhood $\mathcal{U}(\mathbf{A},\gamma,i)$ of $\mathbf{A}_\gamma$ in $\mathfrak{M}$ with the weak topology such that the following property holds:

(viii) For each $\{B_t\}_{t=1}^\infty\in\mathcal{U}(\mathbf{A},\gamma,i)$, each $x\in K$, and each integer $T\ge n(\mathbf{A},\gamma,i)$,

$$\|B_T\cdots B_1x-x_*\|\le\frac{1}{i}.$$

Define

$$\mathcal{F} = \bigcap_{i=1}^{\infty} \bigcup \{\mathcal{U}(\mathbf{A}, \gamma, i) : \ \mathbf{A} \in \mathfrak{M}, \ \gamma \in (0,1)\}.$$

Clearly, $\mathcal{F}$ is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathfrak{M}$.

Assume that $\mathbf{C} = \{C_t\}_{t=1}^{\infty} \in \mathcal{F}$. We claim that $\mathbf{C}$ is convergent. To prove this claim, we take an arbitrary $\epsilon > 0$ and then choose a natural number $i > 1/\epsilon$.

There exist $\mathbf{A} \in \mathfrak{M}$ and $\gamma \in (0,1)$ such that

$$\mathbf{C} \in \mathcal{U}(\mathbf{A}, \gamma, i).$$

By property (viii), for each $x \in K$ and each integer $T \geq n(\mathbf{A}, \gamma, i)$,

$$\|C_T \cdots C_1 x - x_*\| \leq \frac{1}{i} < \epsilon.$$

Thus $\mathbf{C}$ is indeed convergent and Theorem 1.3 is proved. □

Proof of Theorem 1.4. It is clear that the set

$$\{\mathbf{A}_\gamma : \ \mathbf{A} \in \mathfrak{M}_u, \ \gamma \in (0,1)\}$$

is an everywhere dense subset of $\mathfrak{M}_u$ in the strong topology.

Let $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathfrak{M}_u$, $\gamma \in (0,1)$, and let $i \geq 1$ be a natural number. Then $\mathbf{A}_\gamma \in \mathfrak{M}_u$. By Lemma 4.1, $\mathbf{A}_\gamma = \{A_{t\gamma}\}_{t=1}^{\infty}$ is strictly convergent. By Theorem 1.2, there exist a natural number $n(\mathbf{A}, \gamma, i)$ and an open neighborhood $\mathcal{U}(\mathbf{A}, \gamma, i)$ of $\mathbf{A}_\gamma$ in $(\mathfrak{M}_u, \rho_s)$ such that the following property holds:

(ix) For each $\{B_t\}_{t=1}^{\infty} \in \mathcal{U}(\mathbf{A}, \gamma, i)$, each $x \in K$, each integer $T \geq n(\mathbf{A}, \gamma, i)$, and each mapping $r : \{1, \ldots, T\} \to \{1, 2, \ldots\}$,

$$\|B_{r(T)} \cdots B_{r(1)} x - x_*\| \leq \frac{1}{i}.$$

Define

$$\mathcal{F}_u = \bigcap_{i=1}^{\infty} \bigcup \{\mathcal{U}(\mathbf{A}, \gamma, i) : \ \mathbf{A} \in \mathfrak{M}_u, \ \gamma \in (0,1)\}.$$

Clearly, $\mathcal{F}_u$ is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $(\mathfrak{M}_u, \rho_s)$.

Assume that $\mathbf{C} = \{C_t\}_{t=1}^{\infty} \in \mathcal{F}_u$. To show that $\mathbf{C}$ is strictly convergent, let $\epsilon > 0$ and choose a natural number $i > 1/\epsilon$.

There exist $\mathbf{A} \in \mathfrak{M}_u$ and $\gamma \in (0,1)$ such that

$$\mathbf{C} \in \mathcal{U}(\mathbf{A}, \gamma, i).$$

By property (ix), for each $x \in K$, each integer $T \geq n(\mathbf{A}, \gamma, i)$, and each mapping $r : \{1, \ldots, T\} \to \{1, 2, \ldots\}$,

$$\|C_{r(T)} \cdots C_{r(1)} x - x_*\| \leq \frac{1}{i} < \epsilon.$$

Thus $\mathbf{C}$ is strictly convergent and Theorem 1.4 is proved. □

The proof of Theorem 1.5 is analogous to that of Theorem 1.4.

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