ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LIX, 2005 SECTIO A 77–84

JAN KUREK and WŁODZIMIERZ M. MIKULSKI

The natural transformations $TT^{(r),a} \to TT^{(r),a}$

Abstract. For integers $r \geq 1$ and $n \geq 2$ and a real number $a < 0$ all natural endomorphisms of the tangent bundle $TT^{(r),a}$ of generalized higher order tangent bundle $T^{(r),a}$ over n-manifolds are completely described.

0. Let us recall the following definitions (see for ex. [3], [8]).

Let $F : \mathcal{M}f_n \to \mathcal{FM}$ be a functor from the category $\mathcal{M}f_n$ of all n-dimensional manifolds and their local diffeomorphisms into the category $\mathcal{FM}$ of fibered manifolds and their fiber maps. Let B be the base functor from the category of fibered manifolds to the category of manifolds.

A *natural bundle* over n-manifolds is a functor F satisfying $B \circ F = id$ and the localization condition: for every inclusion of an open subset $i_U : U \to M$, FU is the restriction $p_M^{-1}(U)$ of $p_M : FM \to M$ over U and Fi_U is the inclusion $p_M^{-1}(U) \to FM$.

A *natural transformation* $A : F \to G$ from a natural bundle F into a natural bundle G is a system of maps $A : FM \to GM$ for every n-manifold M satisfying $Gf \circ A = A \circ Ff$ for every local diffeomorphism $f : M \to N$ between n-manifolds. (Then $A : FM \to GM$ is a fibered map covering id_M for any M.)

In other words, natural transformations are morphisms in the category of natural bundles. That is why, they are intensively studied, see [3].

2000 *Mathematics Subject Classification.* 58A20, 53A55.
Key words and phrases. Natural bundle, natural transformation.

Some special natural transformations $TF \to TF$ called natural affinors on F are very important. A *natural affinor* on a natural bundle F is a natural transformation $A : TF \to TF$ such that $A : TFM \to TFM$ is a tensor field of type $(1,1)$ on FM for any n-manifold M. Natural affinors play an important role in the theory of generalized connections $\Gamma : TFM \to TFM$ on FM. The Frolicher–Nijenhuis bracket $[\Gamma, A]$ of a connection Γ on FM with a natural affinor A on FM is a generalized torsion of Γ. That is why, natural affinors have been studied in many papers: [2], [6], etc.

All natural transformations $A : F \to G$ for some natural bundles are classified, see e.g. [1], [3], [4], [7], etc. For example, in [7] the second author classified all natural endomorphisms $A : TT^{(r)} \to TT^{(r)}$, where $T^{(r)} = (J^r(., \mathbf{R})_0)^*$ is the vector r-tangent natural bundle, and reobtained a result from [2] about natural affinors on $T^{(r)}$ saying that the vector space of all natural affinors on $T^{(r)}$ is 2-dimensional.

In [5], the second author extended the concept of vector r-tangent bundles and introduced the concept of generalized higher order tangent bundles. In [6], the second author extended the result from [2]. He proved that for every $a < 0$ and every natural numbers r and $n \geq 2$ every natural affinor on generalized higher order tangent bundle $T^{(r),a}$ is a constant multiple of the identity affinor.

In the present note we generalize the results of [2], [6] and [7]. We prove that for natural numbers r and $n \geq 2$ and a negative real number a every natural transformation $\tilde{A} : TT^{(r),a} \to T^{(r),a}$ is a constant multiple of the tangent bundle projection $p^T : TT^{(r),a} \to T^{(r),a}$. Next we prove that for n, r, a as above the vector space of all natural transformations $A : TT^{(r),a} \to TT^{(r),a}$ over $\tilde{A} : TT^{(r),a} \to T^{(r),a}$ is 2-dimensional and we construct the basis of this vector space. In other words, for integers $r \geq 1$ and $n \geq 2$ and a negative real number $a < 0$ we classify all natural endomorphisms $A : TT^{(r),a} \to TT^{(r),a}$ over n-manifolds. In particular, we reobtain the result of [6].

The usual coordinates on $\mathbf{R}^n$ are denoted by x^i and $\partial_i = \frac{\partial}{\partial x^i}$, $i = 1, \dots, n$. All manifolds and maps are assumed to be of class C^∞.

1. Let us cite the notion of $T^{(r),a}M$, [5].

The linear action $\alpha^{(a)} : GL(n, \mathbf{R}) \times \mathbf{R} \to \mathbf{R}$, $\alpha^{(a)}(B, x) = |\det(B)|^a x$ defines the natural vector bundle $T^{(0,0),a}M = LM \times_{\alpha^{(a)}} \mathbf{R}$ (associated to the principal bundle LM of linear frames). Every embedding $\varphi : M \to N$ of n-manifolds induces a vector bundle mapping $T^{(0,0),a}\varphi = L\varphi \times_{\alpha^{(a)}} id_{\mathbf{R}} : T^{(0,0),a}M \to T^{(0,0),a}N$. Let

$$T^{r*,a}M = \{j^r_x\sigma \mid \sigma \text{ is a local section of } T^{(0,0),a}M,\ \sigma(x) = 0,\ x \in M\}$$

be the vector bundle over M of all r-jets of local sections of $T^{(0,0),a}M$ with target 0 with respect to the source projection. We set $T^{(r),a}M =$

$(T^{r*,a}M)^*$, the dual vector bundle. Every embedding $\varphi : M \to N$ of n-manifolds induces a vector bundle mapping $T^{r*,a}\varphi : T^{r*,a}M \to T^{r*,a}N$, $j^r_x\sigma \to j^r_{\varphi(x)}(T^{(0,0),a}\varphi \circ \sigma \circ \varphi^{-1})$, and (next) it induces a vector bundle mapping $T^{(r),a}\varphi = ((T^{r*,a}\varphi)^*)^{-1} : T^{(r),a}M \to T^{(r),a}N$ over φ, and we obtain a natural vector bundle $T^{(r),a}$ over n-manifolds. (For $a = 0$ we get the r-th order vector tangent bundle $T^{(r)}$. That is why $T^{(r),a}M$ is called the generalized higher order tangent bundle.)

$T^{(0,0),a}M$ is the bundle of densities with weight a. $T^{(r),a}M$ appears if we consider linear differential operators $D \in \mathit{Diff}^r(C^\infty_x(T^{(0,0),a}M)_0, \mathbf{R})$ of order $\le r$ on the $C^\infty_x(M)$-module $C^\infty_x(T^{(0,0),a}M)_0$ of germs at $x \in M$ of fields of densities on M with weight a vanishing at x. These operators are in bijection with elements $I(D) \in T^{(r),a}_x M$. This bijection is given by $I(D)(j^r_x\sigma) = D(germ_x(\sigma))$, σ is a field of densities of weight a on M vanishing at x. Thus $T^{(r),a}M$ is the vector bundle of such operators.

2. In this section we study natural transformations $C : TT^{(r),a} \to T^{(r),a}$ over n-manifolds. An example of such a transformation is the tangent projection $p^T : TT^{(r),a}M \to T^{(r),a}M$ for any n-manifold M.

Proposition 1. *For natural numbers r and $n \ge 2$ and a real number $a < 0$ every natural transformation $C : TT^{(r),a} \to T^{(r),a}$ over n-manifolds is a constant multiple of the tangent projection $p^T : TT^{(r),a} \to T^{(r),a}$.*

Proof. We modify the proof of Proposition 1 in [6] as follows.

From now on the set of all $\alpha \in (\mathbf{N} \cup \{0\})^n$ with $1 \le |\alpha| \le r$ will be denoted by $P(r, n)$.

Clearly, every section of $T^{(0,0),a}\mathbf{R}^n = L\mathbf{R}^n \times_{\alpha(a)} \mathbf{R}$ can be considered as a real valued function f on $\mathbf{R}^n$ satisfying the transformation rule

$$\varphi_* f(x) = |\det(d_0(\tau_{-x} \circ \varphi \circ \tau_{\varphi^{-1}(x)}))|^a \cdot f \circ \varphi^{-1}(x)$$

for every local diffeomorphism $\varphi : \mathbf{R}^n \to \mathbf{R}^n$, where $\tau_y : \mathbf{R}^n \to \mathbf{R}^n$ is the translation by $y \in \mathbf{R}^n$. Then any element v from the fibre $T^{(r),a}_0\mathbf{R}^n$ of $T^{(r),a}\mathbf{R}^n$ over 0 is a linear combination of the $(j^r_0 x^\alpha)^*$ for all $\alpha \in P(r, n)$, where the $(j^r_0 x^\alpha)^*$ form the basis dual to the basis $j^r_0 x^\alpha \in T^{r*,a}_0\mathbf{R}^n$. From now on we denote the coefficient of v corresponding to $(j^r_0 x^\alpha)^*$ by $[v]_\alpha$.

Of course, any natural transformation C as above is (fully) determined by the contractions $\langle C(u), j^r_0 x^\alpha\rangle \in \mathbf{R}$ for $u \in (TT^{(r),a})_0\mathbf{R}^n = \mathbf{R}^n \times (VT^{(r),a})_0\mathbf{R}^n = \mathbf{R}^n \times T^{(r),a}_0\mathbf{R}^n \times T^{(r),a}_0\mathbf{R}^n$ and $\alpha \in P(r, n)$, $j^r_0 x^\alpha \in T^{r*,a}_0\mathbf{R}^n$.

We are going to prove that C is determined by the values $\langle C(u), j^r_0(x^1)\rangle \in \mathbf{R}$ for $u \in (TT^{(r),a})_0\mathbf{R}^n$, where $j^r_0(x^1) \in T^{r*,a}_0\mathbf{R}^n$.

If $\alpha = (\alpha_1, \dots, \alpha_n) \in P(r, n)$ with $\alpha_1 + \dots + \alpha_{n-1} \ge 1$ and $\tau \in \mathbf{R}$, then the diffeomorphism $\varphi_{\alpha,\tau} = (x^1, \dots, x^{n-1}, x^n - \tau(x^1)^{\alpha_1} \cdot \ldots \cdot (x^{n-1})^{\alpha_{n-1}})$ sends $j^r_0((x^n)^{\alpha_n+1}) \in T^{r*,a}_0\mathbf{R}^n$ into $j^r_0((x^n + \tau(x^1)^{\alpha_1} \cdot \ldots \cdot (x^{n-1})^{\alpha_{n-1}})^{\alpha_n+1})$ (as $\varphi^{-1}_{\alpha,\tau} = (x^1, \dots, x^{n-1}, x^n + \tau(x^1)^{\alpha_1} \cdot \ldots \cdot (x^{n-1})^{\alpha_{n-1}})$ and $\det(d_0(\tau_{-\varphi_{\alpha,\tau}(y)} \circ$

$\varphi_{\alpha,\tau} \circ \tau_y)) = 1$ for any $y \in \mathbf{R}^n$). Then by the naturality of C with respect to the diffeomorphisms $\varphi_{\alpha,\tau}$, the values $\langle C(u), j^r_0((x^n + \tau(x^1)^{\alpha_1} \cdot \ldots \cdot (x^{n-1})^{\alpha_{n-1}})^{\alpha_n+1})\rangle$ for $u \in (TT^{(r),a})_0\mathbf{R}^n$ and $\tau \in \mathbf{R}$ are determined by the values $\langle C(u), j^r_0((x^n)^{\alpha_n+1})\rangle$ for $u \in (TT^{(r),a})_0\mathbf{R}^n$. On the other hand, given $u \in (TT^{(r),a})_0\mathbf{R}^n$ the value $\frac{1}{\alpha_n+1}\langle C(u), j^r_0 x^\alpha\rangle$ is the coefficient on τ of the polynomial $\langle C(u), j^r_0((x^n + \tau(x^1)^{\alpha_1} \cdot \ldots \cdot (x^{n-1})^{\alpha_{n-1}})^{\alpha_n+1})\rangle$ with respect to τ. Therefore the values $\langle C(u), j^r_0 x^\alpha\rangle$ for $u \in (TT^{(r),a})_0\mathbf{R}^n$ are determined by the values $\langle C(u), j^r_0((x^n)^{\alpha_n+1})\rangle$ for $u \in (TT^{(r),a})_0\mathbf{R}^n$. Then C is fully determined by the values $\langle C(u), j^r_0((x^n)^i)\rangle$ for $u \in (TT^{(r),a})_0\mathbf{R}^n$ and $i = 1,\ldots,r$. For $i \in \{1,\ldots,r\}$ the diffeomorphism $\varphi_i = (x^1 - (x^n)^i, x^2, \ldots, x^n)$ sends $j^r_0(x^1)$ into $j^r_0(x^1 + (x^n)^i)$ (as $\varphi_i^{-1} = (x^1 + (x^n)^i, x^2, \ldots, x^n)$ and $\det(d_0(\tau_{-\varphi_i(y)} \circ \varphi_i \circ \tau_y)) = 1$ for any $y \in \mathbf{R}^n$). Then by the naturality of C with respect to φ_i, the values $\langle C(u), j^r_0((x^n)^i)\rangle$ for $u \in (TT^{(r),a})_0\mathbf{R}^n$ are fully determined by the values $\langle C(u), j^r_0(x^1)\rangle$ for $u \in (TT^{(r),a})_0\mathbf{R}^n$. That is why C is fully determined by the values $\langle C(u), j^r_0(x^1)\rangle \in \mathbf{R}$ for $u \in (TT^{(r),a})_0\mathbf{R}^n = \mathbf{R}^n \times T^{(r),a}_0\mathbf{R}^n \times T^{(r),a}_0\mathbf{R}^n$.

We continue the proof of the proposition. For any $t \in \mathbf{R}_+$ and any $\alpha \in P(r,n)$ the homothety $a_t = (tx^1,\ldots,tx^n)$ sends $j^r_0 x^\alpha \in T^{r*,a}_0\mathbf{R}^n$ into $t^{na-|\alpha|} j^r_0 x^\alpha$, i.e. $(j^r_0 x^\alpha)^*$ into $t^{|\alpha|-na} \cdot (j^r_0 x^\alpha)^*$. Then (since $a < 0$) by the naturality of C with respect to a_t and the homogeneous function theorem [3] we deduce that given $u = (u_1,u_2,u_3) \in (TT^{(r),a})_0\mathbf{R}^n = \mathbf{R}^n \times T^{(r),a}_0\mathbf{R}^n \times T^{(r),a}_0\mathbf{R}^n$, $u_1 = (u_1^1,\ldots,u_1^n) \in \mathbf{R}^n$, $u_2,u_3 \in T^{(r),a}_0\mathbf{R}^n$ we have $\langle C(u), j^r_0(x^1)\rangle = \sum_{i=1}^n \lambda_i [u_2]_{e_i} + \sum_{i=1}^n \mu_i [u_3]_{e_i} + \cdots$, where λ_i, μ_i are the reals, the dots denote the linear combination of monomials in $u_1^1,\ldots,u_1^n$ of degree $1 - na$ and $e_i = (0,\ldots,1,\ldots,0) \in P(r,n)$, 1 in the i-th position.

For any $t \in \mathbf{R}_+$ and $k = 1,\ldots,n$ the homothety $b^k_t = (x^1,\ldots,tx^k,\ldots,x^n)$ (only the k-th position is exceptional) sends $(j^r_0(x^i))^* \in T^{r*,a}_0\mathbf{R}^n$ into $t^{\delta^i_k - a}(j^r_0(x^i))^*$ for $i = 1,\ldots,n$. Then, by the naturality of C with respect to b^k_t and $a < 0$,

$$\langle C(u), j^r_0(x^1)\rangle = \lambda[u_2]_{e_1} + \mu[u_3]_{e_1} + \rho(u_1^1)^{1-a}(u_1^2)^{-a}\ldots(u_1^n)^{-a}$$

for real numbers λ, μ and ρ.

Using the invariance of A with respect to $\psi = (x^1, x^2 + x^1, x^3, \ldots, x^n)$ (only the second position is exceptional) we get that $\rho = 0$.

On replacing C by $C - \lambda p^T$ we can assume that $\lambda = 0$, i.e.

$$(*) \qquad \langle C(u), j^r_0(x^1)\rangle = \mu[u_3]_{e_1}$$

for real number μ. In particular, if $n \geq 2$,

$$(**) \qquad \langle C(\partial_2^C{}_{|\omega}), j^r_0(x^1)\rangle = \langle C(e_2,\omega,0), j^r_0(x^1)\rangle = 0$$

for any $\omega \in T^{(r),a}_0\mathbf{R}^n$, where $(\ )^C$ is the complete lift of vector fields to $T^{(r),a}$.

Clearly, the proof of the proposition will be complete after proving that $\mu = 0$, i.e. $\langle C(0,0,(j_0^r(x^1))^*), j_0^r(x^1)\rangle = 0$. But (if $n \geq 2$) we have

$$(***) \qquad \begin{aligned} 0 &= \langle C(((x^2)^r\partial_1)^C{}_{|\omega}), j_0^r(x^1)\rangle \\ &= \langle C(0,\omega,(j_0^r(x^1))^*), j_0^r(x^1)\rangle \\ &= \langle C(0,0,(j_0^r(x^1))^*), j_0^r(x^1)\rangle\ , \end{aligned}$$

where $\omega = (j_0^r((x^2)^r))^*$.

Let us explain $(***)$. The equality

$$\langle C(0,\omega,(j_0^r(x^1))^*), j_0^r(x^1)\rangle = \langle C(0,0,(j_0^r(x^1))^*), j_0^r(x^1)\rangle$$

is an immediate consequence of the formula $(*)$.

We prove that $0 = \langle C(((x^2)^r\partial_1)^C{}_{|\omega}), j_0^r(x^1)\rangle$. Let us consider the diffeomorphism $\psi = (x^1 + \frac{1}{r+1}(x^2)^{r+1}, x^2, \dots, x^n)$. Clearly, ψ sends ∂_2 into $\partial_2 + (x^2)^r\partial_1$. It is easily seen that $\det(d_0(\tau_{-\psi(y)} \circ \psi \circ \tau_y)) = 1$ for any $y \in \mathbf{R}^n$ and $j_0^r\psi = id$. Hence ψ preserves $j_0^r(x^1) \in T_0^{r*,a}\mathbf{R}^n$. Then using the naturality of C with respect to ψ from $(**)$ it follows that $\langle C((\partial_2 + (x^2)^r\partial_1)^C{}_{|\omega}), j_0^r(x^1)\rangle = 0$ for any $\omega \in T_0^{(r),a}\mathbf{R}^n$. Now, by $(*)$ we obtain $\langle C(((x^2)^r\partial_1)^C{}_{|\omega}), j_0^r(x^1)\rangle = \langle C((\partial_2 + (x^2)^r\partial_1)^C{}_{|\omega}), j_0^r(x^1)\rangle = 0$.

The flow of $(x^2)^r\partial_1$ is $\varphi_t = (x^1 + t(x^2)^r, x^2, \dots, x^n)$ and $\det(d_0(\tau_{-\varphi_t(y)} \circ \varphi_t \circ \tau_y)) = 1$ for any $y \in \mathbf{R}^n$. Then

$$\begin{aligned} \left\langle ((x^2)^r\partial_1)^C{}_{|\omega}, j_0^r(x^1)\right\rangle &= \left\langle \frac{d}{dt}_{|t=0} T^{(r),a}(\varphi_t)(\omega), j_0^r(x^1)\right\rangle \\ &= \frac{d}{dt}_{|t=0} \langle T^{(r),a}(\varphi_t)(\omega), j_0^r(x^1)\rangle \\ &= \frac{d}{dt}_{|t=0} \left\langle \omega, j_0^r(x^1 \circ \varphi_t)\right\rangle \\ &= \left\langle \omega, j_0^r\left(\frac{d}{dt}_{|t=0}(x^1 \circ \varphi_t)\right)\right\rangle \\ &= \langle \omega, j_0^r((x^2)^r\partial_1 x^1)\rangle \\ &= \langle \omega, j_0^r((x^2)^r)\rangle \\ &= 1\ . \end{aligned}$$

Then $((x^2)^r\partial_1)^C{}_{|\omega} = (j_0^r(x^1))^* + \beta$ under the isomorphism $V_\omega T^{(r),a}\mathbf{R}^n = T_0^{(r),a}\mathbf{R}^n$, where β is a linear combination of the $(j_0^r(x^\alpha))^* \neq (j_0^r(x^1))^*$. Now, by $(*)$

$$\begin{aligned} \langle C(((x^2)^r\partial_1)^C{}_{|\omega}), j_0^r(x^1)\rangle &= \langle C(0,\omega,(j_0^r(x^1))^* + \cdots), j_0^r(x^1)\rangle \\ &= \langle C(0,\omega,(j_0^r(x^1))^*), j_0^r(x^1)\rangle. \end{aligned}$$

□

3. The tangent map $Tp : TT^{(r),a}M \to TM$ of the bundle projection $p : T^{(r),a}M \to M$ defines a natural transformation over n-manifolds.

Proposition 2. *For natural numbers r and n and for a real number $a < 0$ every natural transformation $B : TT^{(r),a} \to T$ over n-manifolds is a constant multiple of Tp.*

Proof. Clearly, every natural transformation B as in the proposition is uniquely determined by the contractions $\langle B(u), d_0x^1\rangle$ for $u = (u_1, u_2, u_3) \in (TT^{(r),a})_0\mathbf{R}^n = \mathbf{R}^n \times T_0^{(r),a}\mathbf{R}^n \times T_0^{(r),a}\mathbf{R}^n$. Using the invariance of B with respect to the homotheties $a_t = (tx^1, \dots, tx^n)$ for $t \in \mathbf{R}_+$ and the homogeneous function theorem we deduce (similarly as in the proof of Proposition 1) that $\langle B(u), d_0x^1\rangle$ for $u = (u_1, u_2, u_3) \in (TT^{(r),a})_0\mathbf{R}^n = \mathbf{R}^n \times T_0^{(r),a}\mathbf{R}^n \times T_0^{(r),a}\mathbf{R}^n$ is the linear combination (with real coefficients) of the $u_1^1, \dots, u_1^n$ and it is independent of u_2 and u_3, where $u_1 = (u_1^1, \dots, u_1^n) \in \mathbf{R}^n$. Next, using the invariance of B with respect to the homotheties $b_t = (x^1, tx^2, \dots, tx^n)$ we see that $\langle B(u), d_0x^1\rangle$ is proportional (by a real number) to $u_1^1 = \langle Tp(u), d_0x^1\rangle$. □

4. Let $\underline{A} : TT^{(r),a}M \to T^{(r),a}M$ be a natural transformation over n-manifolds. We say that a natural transformation $A : TT^{(r),a}M \to TT^{(r),a}M$ over n-manifolds is *over* $\underline{A}$ if $p^T \circ A = \underline{A}$.

If $B : TT^{(r),a}M \to T^{(r),a}M$ is another natural transformation over n-manifolds, we define a natural transformation

$$\underline{A}^B := (\underline{A}, B) : TT^{(r),a}M \to T^{(r),a}M \times_M T^{(r),a}M \tilde{=} VT^{(r),a}M \subset TT^{(r),a}M \ .$$

Clearly, $\underline{A}^B$ is over $\underline{A}$. We call $\underline{A}^B$ *the B-vertical lift of* $\underline{A}$.

In particular, considering $p^T : TT^{(r),a}M \to T^{(r),a}M$ we produce natural transformation $\underline{A}^{p^T} : TT^{(r),a}M \to TT^{(r),a}M$ over $\underline{A}$. The above natural transformations $\underline{A}^B$ are of vertical type, i.e. they have values in $VT^{(r),a}M$.

If $A : TT^{(r),a}M \to VT^{(r),a}M \tilde{=} T^{(r),a}M \times_M T^{(r),a}M$ is a natural transformation of vertical type over $\underline{A}$, then $A = (\underline{A}, B)$ for natural transformation $B = pr_2 \circ A : TT^{(r),a}M \to T^{(r),a}M$, i.e. $A = \underline{A}^B$ for some B.

Then applying Proposition 1 we obtain the following proposition.

Proposition 3. *Let r and $n \geq 2$ be natural numbers and a be a negative real number. Let $\underline{A} : TT^{(r),a}M \to T^{(r),a}M$ be a natural transformation over n-manifolds. Then every natural transformation $A : TT^{(r),a}M \to VT^{(r),a}M$ over n-manifolds of vertical type over $\underline{A}$ is a constant multiple of $\underline{A}^{p^T}$.*

5. Let $\lambda \in \mathbf{R}$. For every n-manifold M we define $A^{(\lambda)} : TT^{(r),a}M \to TT^{(r),a}M$ by

$$A^{(\lambda)}(v) = T(\lambda id_{T^{(r),a}M})(v), \ v \in TT^{(r),a}M \ .$$

Clearly $A^{(\lambda)} : TT^{(r),a} \to TT^{(r),a}$ is a natural transformation over $\tilde{A} = \lambda p^T : TT^{(r),a} \to T^{(r),a}$.

Proposition 4. *Let $\lambda \in \mathbf{R}$. If r and $n \geq 2$ are natural numbers and a is a negative real number, then every natural transformation $A : TT^{(r),a} \to TT^{(r),a}$ over n-manifolds over $\underline{A} = \lambda p^T$ is a linear combination of $\underline{A}^{p^T}$ and $A^{(\lambda)}$ with real coefficients.*

Proof. Let $A : TT^{(r),a}M \to TT^{(r),a}M$ be a natural transformation over n-manifolds over $\underline{A}$. The composition $Tp \circ A : TT^{(r),a}M \to TM$ is a natural transformation. By Proposition 2, there exists the real number ρ such that $Tp \circ A = \rho Tp$. Clearly, $Tp \circ A^{(\lambda)} = Tp$. Then $A - \rho A^{(\lambda)} : TT^{(r),a}M \to TT^{(r),a}M$ is of vertical type. Then Proposition 3 ends the proof. □

Remark. Every natural transformation $A : TT^{(r),a}M \to TT^{(r),a}M$ over n-manifolds is over $\underline{A} = p^T \circ A : TT^{(r),a}M \to T^{(r),a}M$. So, Proposition 4 together with Proposition 1 gives a complete description of all natural transformations $TT^{(r),a}M \to TT^{(r),a}M$ over n-manifolds in the case where $a < 0$, $r \geq 1$ and $n \geq 2$.

6. As a corollary of Proposition 4 we get immediately the following fact.

Corollary 1 ([6])**.** *If r and $n \geq 2$ are natural numbers and a is a negative real number, then every natural affinor $A : TT^{(r),a} \to TT^{(r),a}$ on $T^{(r),a}$ over n-manifolds is a constant multiple of the identity affinor.*

7. Similarly as $T^{(r),a}$ starting from the action $GL(n, \mathbf{R}) \times \mathbf{R} \to \mathbf{R}$ given by $(B, x) \to \mathrm{sgn}(\det(B))|\det(B)|^a x$ instead of $\alpha^{(a)} : GL(n, \mathbf{R}) \times \mathbf{R} \to \mathbf{R}$, we can define natural vector bundles $\tilde{T}^{(r),a}$ over n-manifolds. Using obviously modified arguments as in Items 3–6 we obtain the following facts.

Proposition 1'. *For natural numbers r and $n \geq 2$ and a real number $a < 0$ every natural transformation $C : T\tilde{T}^{(r),a} \to \tilde{T}^{(r),a}$ over n-manifolds is a constant multiple of the tangent projection $p^T : T\tilde{T}^{(r),a} \to \tilde{T}^{(r),a}$.*

Proposition 2'. *For natural numbers r and n and for a real number $a < 0$ every natural transformation $B : T\tilde{T}^{(r),a} \to T$ over n-manifolds is a constant multiple of Tp, where $p : \tilde{T}^{(r),a}M \to M$ is the bundle projection.*

Similarly as in Items 4 and 5 we define $\underline{A}^{p^T} : T\tilde{T}^{(r),a} \to V\tilde{T}^{(r),a}$ and $A^{(\lambda)} : T\tilde{T}^{(r),a} \to T\tilde{T}^{(r),a}$.

Proposition 3'. *Let r and $n \geq 2$ be natural numbers and a be a negative real number. Let $\underline{A} : T\tilde{T}^{(r),a}M \to \tilde{T}^{(r),a}M$ be a natural transformation over n-manifolds. Then every natural transformation $A : T\tilde{T}^{(r),a}M \to V\tilde{T}^{(r),a}M$ over n-manifolds of vertical type over $\underline{A}$ is a constant multiple of $\underline{A}^{p^T}$.*

Proposition 4'. *Let* $\lambda \in \mathbf{R}$. *If* r *and* $n \geq 2$ *are natural numbers and* a *is a negative real number, then every natural transformation* $A : T\tilde{T}^{(r),a} \to T\tilde{T}^{(r),a}$ *over* n*-manifolds over* $\underline{A} = \lambda p^T$ *is a linear combination of* $\underline{A}^{p^T}$ *and* $A^{(\lambda)}$ *with real coefficients.*

Corollary 1' ([6]). *If* r *and* $n \geq 2$ *are natural numbers and* a *is a negative real number, then every natural affinor* $A : T\tilde{T}^{(r),a} \to T\tilde{T}^{(r),a}$ *on* $\tilde{T}^{(r),a}$ *over* n*-manifolds is a constant multiple of the identity affinor.*

References

[1] Doupovec, M., *Natural transformations between* TTT^*M *and* TT^*TM, Czechoslovak Math. J. **43**(118) (1993), 599–613.

[2] Gancarzewicz, J., I. Kolář, *Natural affinors on the extended r-th order tangent bundles*, Rend. Circ. Mat. Palermo (2) Suppl. **30** (1993), 95–100.

[3] Kolář, I., P. W. Michor and J. Slovák , *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin 1993.

[4] Kurek, J., *Natural transformations of higher order cotangent bundle functors*, Ann. Polon. Math. **58** (1993), 29–33.

[5] Mikulski, W. M., *The natural operators lifting vector fields to generalized higher order tangent bundles*, Arch. Math. Brno **36**(III) (2000), 207–212.

[6] Mikulski, W. M., *The natural affinors on generalized higher order tangent bundles*, Rend. Mat. Roma **21**(VII) (2001), 339–349.

[7] Mikulski, W. M., *The natural transformations* $TT^{(r)} \to TT^{(r)}$, Arch. Math. Brno **36**(1) (2000), 71–75.

[8] Paluszny, M., A. Zajtz, *Foundation of Differential Geometry of Natural Bundles*, Lect. Notes Univ. Caracas, 1984.

Jan Kurek
Institute of Mathematics
M. Curie-Skłodowska University
pl. Marii Curie-Skłodowskiej 1
20-031 Lublin, Poland
e-mail: kurek@golem.umcs.lublin.pl

Włodzimierz M. Mikulski
Institute of Mathematics
Jagiellonian University
ul. Reymonta 4
30-059 Kraków, Poland
e-mail: mikulski@im.uj.edu.pl

Received February 21, 2005