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On a convex class of univalent functions

ABSTRACT. For some $M > 0$ the classes $Q_n(M)$ of all functions $z \mapsto f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$ analytic on the open unit disk Δ, and such that $|f''| \le M$ on Δ, consist only of univalent starlike or convex functions. In the article we get some sharp results in the classes $Q_n(M)$, that improve Theorem 5.2f.1 and Corollary 5.5a.1 from the monograph [2] of S. S. Miller and P. T. Mocanu. Applying our results we construct some not trivial examples of univalent starlike or convex functions.

Let $\mathscr{H}(\Delta)$ denote the class of all analytic functions on the unit disk $\Delta = \{z \in \mathbb{C} \colon |z| < 1\}$ and consider the following convex subsets of $\mathscr{H}(\Delta)$:

$$\mathcal{A}_n = \left\{ f \in \mathscr{H}(\Delta) \colon f(0) = f'(0) - 1 = 0,\ f^{(j)}(0) = 0 \ \text{ for } \ 2 \le j \le n \right\},$$

$$\mathcal{A}_1 = \left\{ f \in \mathscr{H}(\Delta) \colon f(0) = f'(0) - 1 = 0 \right\},$$

$$Q_n(M) = \left\{ f \in \mathcal{A}_n \colon \left| f''(z) \right| \le M \ \text{ for } \ z \in \Delta \right\}, \quad M > 0,\ n = 1, 2, \dots .$$

Clearly, the classes

$$Q_n(M, \alpha) = \left\{ f \in Q_n(M) \colon \frac{f^{(n+1)}(0)}{(n+1)!} = \alpha \right\}, \quad \alpha \in \mathbb{C},$$

are empty for $|\alpha| > M/\big(n^2 + n\big)$. Also

$$\mathcal{A}_n \supset \mathcal{A}_{n+1} \quad \text{and} \quad S^c \subset S^* \subset S \subset \mathcal{A}_1,$$

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where S^c, S^* and S consist of all classically normalized analytic functions on Δ that are *univalent convex*, *univalent starlike* and *univalent*:

$$f \in S^c \text{ if and only if } f \in \mathcal{A}_1 \text{ and } \operatorname{Re}\left(1 + \frac{zf''}{f'}\right) > 0 \text{ on } \Delta,$$

$$f \in S^* \text{ if and only if } f \in \mathcal{A}_1 \text{ and } \operatorname{Re}\left(\frac{zf'}{f}\right) > 0 \text{ on } \Delta,$$

and

$$S = \{f \in \mathcal{A}_1 \colon f \text{ is an injection}\},$$

see [1]. Moreover let $S^c(\beta), S^*(\beta)$ denote subsets of suitable classes S^c, S^*:

$$f \in S^c(\beta) \text{ if and only if } f \in \mathcal{A}_1 \text{ and } \operatorname{Re}\left(1 + \frac{zf''}{f'}\right) > \beta \text{ on } \Delta,$$

$$f \in S^*(\beta) \text{ if and only if } f \in \mathcal{A}_1 \text{ and } \operatorname{Re}\left(\frac{zf'}{f}\right) > \beta \text{ on } \Delta,$$

where $0 \le \beta < 1$.

An extension of the Theorem 5.2f.1 [2] and a generalization of the Corollary 5.5a.1 [2] contains

Theorem 1.

(i) $\max\{M > 0 \colon Q_n(M) \subset S\} = \max\{M > 0 \colon Q_n(M) \subset S^*\} = n,$

(ii) $\max\{M > 0 \colon Q_n(M) \subset S^c\} = \dfrac{n}{n+1}.$

Proof. Consider $f(z) \equiv z + \frac{Mz^{n+1}}{n(n+1)}$. Clearly $f \in Q_n(M)$ and if $M > n$, then $f'(z) \equiv \frac{M}{n}\left(\frac{n}{M} + z^n\right)$ vanishes on Δ, i.e.

$$\max\{M > 0 \colon Q_n(M) \subset S^*\} \le \max\{M > 0 \colon Q_n(M) \subset S\} \le n.$$

Also $zf''(z)/f'(z) \equiv nMz^n/(n + Mz^n)$, and if $M = n/(n+1) + \varepsilon,\ 0 < \varepsilon < n^2/(n+1)$, then

$$\lim_{z^n \to -1} \frac{zf''(z)}{f'(z)} = -\frac{n\,(n + \varepsilon\,(n+1))}{n^2 - \varepsilon\,(n+1)} < -1,$$

i.e.

$$\max\{M > 0 \colon Q_n(M) \subset S^c\} \le \frac{n}{n+1}.$$

Observe now that for $f \in \mathcal{A}_1$ we have

$$z^2 \int_0^1 tf''(tz)dt \equiv \int_0^1 \frac{\partial}{\partial t}\left(f'(tz)\right) tzdt \equiv zf'(z) - f(z) \tag{1}$$

and

$$z^2 \int_0^1 (1-t)\, f''(tz)\, dt + z \equiv \int_0^1 \frac{\partial}{\partial t}\left(f'(tz)\right)(1-t)\, zdt + z \equiv f(z). \tag{2}$$

Let $f \in Q_n(M)$. By the maximum modulus theorem, for $z \in \Delta$ we get $|f''(z)| \leq M|z|^{n-1}$ and hence, according to (1)–(2),

$$\left|f'(z) - \frac{f(z)}{z}\right| \leq M|z|^n \int_0^1 t^n dt = \frac{M|z|^n}{n+1}$$

and

$$\left|\frac{f(z)}{z}\right| \geq 1 - M|z|^n \int_0^1 (1-t)\, t^{n-1} dt = 1 - \frac{M|z|^n}{n^2+n} > 0$$

whenever $M \leq n(n+1)$. Thus for $M = n$ and $z \in \Delta$ we get

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{nM|z|^n}{n(n+1) - M|z|^n} = \frac{n|z|^n}{n+1-|z|^n} < 1,$$

i.e. $f \in S^*$ and

$$\max\{M > 0\colon Q_n(M) \subset S^*\} \geq n.$$

Similarly, take any $f \in Q_n(M)$. Then $|f''(z)| \leq M|z|^{n-1}$ for $z \in \Delta$ and adding (1) and (2) we get

$$\int_0^1 zf''(tz)dt + 1 \equiv f'(z),$$

i.e. $|f'(z)| \geq 1 - M|z|^n \int_0^1 t^{n-1} dt = 1 - M|z|^n/n > 0$ whenever $M \leq n$. Thus for $M = n/(n+1)$ we obtain

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{nM|z|^n}{n - M|z|^n} = \frac{n|z|^n}{n+1-|z|^n} < 1,$$

i.e. $f \in S^c$ and

$$\max\{M > 0\colon Q_n(M) \subset S^c\} \geq \frac{n}{n+1}.$$

□

From the proof of Theorem 1 it follows

Corollary 1.

(i) *If* $f \in Q_n(M),\ 0 < M \leq n(n+1)$, *then*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{nM|z|^n}{n(n+1) - M|z|^n}$$

for all $z \in \Delta$.

(ii) *If* $f \in Q_n(M),\ 0 < M \leq n$, *then*

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{nM|z|^n}{n - M|z|^n}$$

for all $z \in \Delta$.

Example 1. As the first application of Theorem 1 we examine the following function

$$z \longmapsto f(z) = z + \lambda \left(-\sum_{j=1}^{n} a_j z^{2j-1} + \tan z \right),$$

where

$$\tan z = \sum_{j=1}^{\infty} a_j z^{2j-1}.$$

By definition, the tangential function is analytic on $\mathbb{C} \setminus \{\pi/2 + k\pi \colon k = 0, \pm 1, \pm 2, \ldots\}$, so the series $\sum a_j z^{2j-1}$ is convergent on $\{z \in \mathbb{C} \colon |z| < \pi/2\}$. Using induction, we deduce that all the Taylor coefficients a_j are strictly positive. By the identity $\tan z \cos z \equiv \sin z$ we obtain the formula $a_j = T_j/(2j-1)!$, where $T_j - \binom{2j-1}{2} T_{j-1} + \binom{2j-1}{4} T_{j-2} - \ldots + (-1)^{j-1} \binom{2j-1}{2j-2} T_1 = (-1)^{j-1}$, $T_1 = 1$, whence $T_1 = 1$, $T_2 = 2$, $T_3 = 16$, $T_4 = 272$, $T_5 = 7936, \ldots$, i.e.

$$\tan z \equiv z + \frac{1}{3} z^3 + \frac{2}{15} z^5 + \frac{17}{315} z^7 + \frac{62}{2835} z^9 + \ldots.$$

Thus $f \in \mathcal{A}_{2n}$,

$$f''(z) = \lambda \left(-\sum_{j=2}^{n} (2j-1)(2j-2) a_j z^{2j-3} + 2 \tan z \left(1 + \tan^2 z\right) \right)$$

and

$$\begin{aligned} \left|f''(z)\right| &\leq |\lambda| \left(-\sum_{j=2}^{n} (2j-1)(2j-2) a_j + 2 \left(\tan 1 + \tan^3 1\right) \right) \\ &= 2|\lambda| \left(-\sum_{j=2}^{n} (j-1)(2j-1) a_j + \tan 1 + \tan^3 1 \right). \end{aligned}$$

Hence $f \in S^*$ whenever

$$|\lambda| \leq \frac{n}{-\sum_{j=2}^{n} (j-1)(2j-1) a_j + \tan 1 + \tan^3 1}$$

and $f \in S^c$ whenever

$$|\lambda| \leq \frac{n}{(2n+1) \left(-\sum_{j=2}^{n} (j-1)(2j-1) a_j + \tan 1 + \tan^3 1 \right)}.$$

In particular we get the function

$$f(z)=z+\lambda\left(-z-\frac{z^3}{3}-\frac{2}{15}z^5-\frac{17}{315}z^7+\tan z\right)$$

which is starlike for

$$|\lambda|\le 2.141<\frac{4}{\tan 1+\tan^3 1-52/15}$$

and convex for

$$|\lambda|\le 0.237<\frac{4}{9\,(\tan 1+\tan^3 1-52/15)}.$$

Example 2. As the next application we examine the function

$$\begin{aligned}f_1(z)&\equiv z+\lambda\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}+\ldots+(-1)^n\,\frac{z^{2n}}{(2n)!}\right)-\lambda\cos z\\&=z-\lambda\left((-1)^{n+1}\,\frac{z^{2n+2}}{(2n+2)!}+(-1)^{n+2}\,\frac{z^{2n+4}}{(2n+4)!}+\ldots\right)\end{aligned}$$

with $\lambda\in\mathbb{C}$. Then $f_1\in\mathcal{A}_{2n+1}$ and

$$\begin{aligned}f_1''(z)&=\lambda\left(-1+\frac{z^2}{2!}-\frac{z^4}{4!}+\ldots+(-1)^{n-2}\,\frac{z^{2n-2}}{(2n-2)!}\right)+\lambda\cos z\\&=\lambda\left((-1)^n\,\frac{z^{2n}}{(2n)!}+(-1)^{n+1}\,\frac{z^{2n+2}}{(2n+2)!}+\ldots\right).\end{aligned}$$

So for $z\in\Delta$ we have

$$\begin{aligned}\left|f_1''(z)\right|&<|\lambda|\left(\frac{1}{(2n)!}+\frac{1}{(2n+2)!}+\ldots\right)\\&=|\lambda|\left(\cos i-1-\frac{1}{2}-\ldots-\frac{1}{(2n-2)!}\right)\\&=|\lambda|\left(\cosh 1-\sum_{j=0}^{n-1}\frac{1}{(2j)!}\right).\end{aligned}$$

Applying Theorem 1, we deduce that the function $f_1\in S^*$ whenever

$$|\lambda|\le\frac{2n+1}{\cosh 1-\displaystyle\sum_{j=0}^{n-1}\frac{1}{(2j)!}},$$

and also $f_1\in S^c$ if

$$|\lambda|\le\frac{2n+1}{(2n+2)\left(\cosh 1-\displaystyle\sum_{j=0}^{n-1}\frac{1}{(2j)!}\right)}.$$

Example 3. And another example similar to the last one. The function

$$f_2(z) = z + \lambda\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots + (-1)^{n+1}\frac{z^{2n-1}}{(2n-1)!}\right) - \lambda \sin z$$

with $\lambda \in \mathbb{C}$ is in class $\mathcal{A}_{2n}$. Thus $f_2 \in S^*$ whenever

$$|\lambda| \leq \frac{2n}{\sinh 1 - \sum_{j=1}^{n-1} \frac{1}{(2j-1)!}}$$

and $f_2 \in S^c$ whenever

$$|\lambda| \leq \frac{2n}{(2n+1)\left(\sinh 1 - \sum_{j=1}^{n-1} \frac{1}{(2j-1)!}\right)}.$$

Theorem 2.

(i) $\max\{M > 0\colon Q_n(M) \subset S^*(\beta)\} = (1-\beta)\dfrac{n(n+1)}{n+1-\beta}$,

(ii) $\max\{M > 0\colon Q_n(M) \subset S^c(\beta)\} = (1-\beta)\dfrac{n}{n+1-\beta}$.

Proof. Let $f(z) \equiv z + \frac{M}{n(n+1)} z^{n+1}$. It is clear that $f \in Q_n(M)$ and $f'(z) = 1 + Mz^n/n,\ f''(z) = Mz^{n-1}$. For $M < n$ we have

$$\frac{zf''(z)}{f'(z)} = \frac{Mz^n}{1 + Mz^n/n} \overset{z^n \to -1}{\longrightarrow} \frac{-M}{1 - M/n} < \beta - 1$$

whenever $M > \frac{(1-\beta)n}{n+1-\beta}$, so that

$$\max\{M > 0\colon Q_n(M) \subset S^c(\beta)\} \leq (1-\beta)\frac{n}{n+(1-\beta)}.$$

Similarly

$$\frac{zf'(z)}{f(z)} = \frac{z + Mz^{n+1}/n}{z + Mz^{n+1}/\left(n(n+1)\right)} \overset{z^n \to -1}{\longrightarrow} \frac{1 - M/n}{1 - M/\left(n(n+1)\right)} < \beta$$

for $M > (1-\beta)\frac{n(n+1)}{n+1-\beta}$ and therefore

$$\max\{M > 0\colon Q_n(M) \subset S^*(\beta)\} \leq (1-\beta)\frac{n(n+1)}{n+1-\beta}.$$

Similarly to the proof of Theorem 1, if $M = n(1-\beta)/(n+1-\beta)$, then

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{nM|z|^n}{n - M|z|^n} = \frac{n(1-\beta)|z|^n}{n + (1-\beta) - (1-\beta)|z|^n} < 1 - \beta,$$

and from this we see that

$$\max\{M > 0\colon Q_n(M) \subset S^c(\beta)\} \geq \frac{n(1-\beta)}{n+(1-\beta)}.$$

Analogously,

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{nM|z|^n}{n(n+1) - M|z|^n} = \frac{n(1-\beta)\,|z|^n}{n + (1-\beta) - (1-\beta)|z|^n} < 1 - \beta,$$

so we have

$$\max\{M > 0\colon Q_n(M) \subset S^*(\beta)\} \geq (1-\beta)\frac{n(n+1)}{n+1-\beta},$$

the desired result. □

Example 4. Similarly to Example 2 we consider the function

$$k(z) = z + \lambda\left(1 - \frac{z^2}{2} + \frac{z^4}{4}\right) - \lambda\cos z$$

with $\lambda \in \mathbb{C}$, which is in $\mathcal{A}_5$. Since the second derivative of k is

$$k''(z) = \lambda\left(-1 + \frac{z^2}{2}\right) + \lambda\cos z,$$

we get

$$\begin{aligned}\left|k''(z)\right| &= |\lambda|\left|\frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \dots\right| \leq |\lambda|\left(\frac{1}{4!} + \frac{1}{6!} + \frac{1}{8!} + \dots\right)\\ &= |\lambda|\left(\cos i - \frac{3}{2}\right)\end{aligned}$$

for $z \in \Delta$. Hence from Theorem 2 we deduce that if $0 \leq \beta < 1$ and

$$|\lambda| \leq \frac{30\,(1-\beta)}{(\cosh 1 - 3/2)\,(6-\beta)},$$

then $k \in S^*(\beta)$. Also if $0 \leq \beta < 1$ and

$$|\lambda| \leq \frac{5\,(1-\beta)}{(\cosh 1 - 3/2)\,(6-\beta)},$$

then $k \in S^c(\beta)$.

Example 5. For the next application of Theorem 2 assume $\lambda \in \mathbb{C}$ and $0 \leq \beta < 1$, and consider the function

$$h(z) = z + \lambda\left(z - \frac{z^3}{3!} + \frac{z^5}{5!}\right) - \lambda\sin z.$$

Obviously

$$h(z) = z + \lambda\left(-\frac{z^7}{7!} + \frac{z^9}{9!} - \dots\right),$$

i.e. $h \in \mathcal{A}_6$. For the function h we have a sharp bound

$$\left|h''(z)\right| = |\lambda| \left| -\frac{z^5}{5!} + \frac{z^7}{7!} - \dots \right| \le |\lambda| \left(\frac{1}{5!} + \frac{1}{7!} + \frac{1}{9!} + \dots \right)$$
$$= |\lambda| \left(i \sin i - 1 - \frac{1}{3!} \right) = |\lambda| \left(\sinh 1 - \frac{7}{6} \right)$$

for all $z \in \Delta$. Therefore, if

$$|\lambda| \le \frac{42\,(1-\beta)}{(\sinh 1 - 7/6)\,(7-\beta)},$$

then $h \in S^*(\beta)$, and if

$$|\lambda| \le \frac{6\,(1-\beta)}{(\sinh 1 - 7/6)\,(7-\beta)},$$

then $h \in S^c(\beta)$.

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