ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LX, 2006 SECTIO A 23–30

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Covering domains for the class of typically real odd functions

Abstract. A set $\bigcup_{f\in T^{(2)}} f(D)$ is called the covering domain for the class $T^{(2)}$ of typically real odd functions over some fixed set D. This set is denoted by $L_{T^{(2)}}(D)$. We find sets $L_{T^{(2)}}(\Delta_r)$ and $L_{T^{(2)}}(H)$, where $\Delta_r = \{z \in \mathbf{C} : |z| < r\}$, $r \in (0,1)$ and $H = \left\{z \in \Delta : \left|1+z^2\right| > 2|z|\right\}$ is one of the domains of univalence for $T^{(2)}$.

Let A denote the class of all functions that are analytic in the unit disk $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. For a given domain $D \subset \Delta$, a set $\bigcup_{f\in A} f(D)$ is called the *covering domain* for the class A over the set D, and is denoted by $L_A(D)$. This generalized definition was introduced in [2]. Domains $L_A(D)$ are characterized by the following, easy to prove, properties

1. if all functions of the class A are univalent in Δ and $f \in A \Leftrightarrow e^{-i\varphi} f(ze^{i\varphi}) \in A$ for arbitrary $\varphi \in R$, then $L_A(\Delta_r) = \Delta_{M(r)}$, where $M(r) = \max\{|f(z)| : f \in A,\ z \in \partial\Delta_r\}$;
2. if all functions of the class A have real coefficients, and D is symmetric with respect to the real axis, then $L_A(D)$ is symmetric with respect to the real axis;
3. if all functions of the class A have real coefficients, $f \in A \Leftrightarrow -f(-z) \in A$, and D is symmetric with respect to both axes of

2000 *Mathematics Subject Classification.* Primary 30C45. Secondary 30C75.
Key words and phrases. Typically real functions, covering domain.

the complex plane, then $L_A(D)$ is symmetric with respect to both axes;

4. if $D_1 \subset D_2$, then $L_A(D_1) \subset L_A(D_2)$;
5. if $A_1 \subset A_2 \subset A$, then $L_{A_1}(D) \subset L_{A_2}(D)$.

In this paper we derive some covering domains for the class $T^{(2)}$ consisting of typically real odd functions. Sets $\Delta_r = \{z \in \mathbf{C} : |z| < r\}$, $r \in (0,1)$ and $H = \left\{z \in \Delta : \left|1+z^2\right| > 2|z|\right\}$ are considered. Some related results for the class T of typically real functions the reader can find in [4].

Recall that

$$T^{(2)} = \{f \in A : \operatorname{Im} z \operatorname{Im} f(z) \geq 0,\ f(-z) = -f(z) \text{ for } z \in \Delta\}.$$

It is known (see for example [2]) that

$$f \in T^{(2)} \ \Leftrightarrow\ f(z) \equiv \int_0^1 \frac{z(1+z^2)}{(1+z^2)^2 - 4z^2 t} d\mu(t),$$

where μ is a probability measure on $[0,1]$.

For a given $r \in (0,1)$ we can determine $L_{T^{(2)}}(\Delta_r)$ considering

$$\max\left\{|f(z)| : f \in T^{(2)},\ \operatorname{Arg} f(z) = \alpha,\ |z| = r\right\}, \tag{1}$$

with fixed $\alpha \in [0, 2\pi]$. It is easy to observe that for $z \in \Delta \setminus \{0\}$ the set $\left\{f(z) :\ f \in T^{(2)}\right\}$ coincides with a segment of the disk, whose boundary contains the origin, in case $\operatorname{Re} z \operatorname{Im} z \neq 0$, and coincides with a line segment included in one of the axes of the complex plane, in case $\operatorname{Re} z \operatorname{Im} z = 0$. In both cases $0 \notin \{f(z) :\ f \in T^{(2)}\}$. Therefore, the maximum (1) is equal to

$$\max\{|f_t(z)| : t \in [0,1],\ \operatorname{Arg} f(z) = \alpha,\ |z| = r\}, \tag{2}$$

where the functions f_t are extreme points of $T^{(2)}$, i.e.

$$f_t(z) = \frac{z(1+z^2)}{(1+z^2)^2 - 4z^2 t},\ z \in \Delta,\ t \in [0,1].$$

Throughout the paper we write $2m = r^2 + 1/r^2$, $m > 1$. We also use the notation: ∂D for the boundary of D, $\operatorname{int} D$ for the interior of D, $\operatorname{cl} D$ for the closure of D.

The following properties of f_t will be used to calculate the maximum (2).

Lemma 1. *Let $0 \leq t \leq 1$. Then*

1. $\operatorname{Re} \dfrac{z f_t'(z)}{f_t(z)} \geq 0$ *for* $|z| \leq \sqrt{t+1} - \sqrt{t}$;
2. $\operatorname{Re} \dfrac{z f_t'(z)}{f_t(z)} \geq 0$ *for* $\sqrt{t+1} - \sqrt{t} < |z| < 1$

   and $\cos(2 \arg z) \geq \frac{1}{2}\left(4t - \frac{1}{|z|^2} - |z|^2\right)$.

Proof. Observe that $\operatorname{Re} \dfrac{z f_t'(z)}{f_t(z)} \geq 0$ if and only if

$$(m + \cos 2\varphi)^2 - 4t^2 + 4t - 4t \cos^2 2\varphi \geq 0,$$

where $r = |z|$, $\varphi = \arg z$. Let $-\infty < \alpha < \beta < +\infty$. It is obvious that any real polynomial h of at most second degree such that $h(\alpha) \geq 0$, $h(\beta) \geq 0$ and $h'(\alpha) \geq 0$ or $h'(\beta) \leq 0$ is nonnegative on $[\alpha, \beta]$. Put $x = \cos 2\varphi$ and let $h(x) \equiv (m + x)^2 - 4t^2 + 4t - 4tx^2$.

1. In the case $\rho \leq (\sqrt{t+1} - \sqrt{t})^2$ we get $m \geq 1 + 2t$, $h(-1) \geq 0$, $h(1) \geq 4m > 4$, $h'(-1) = 2(m + 4t - 1) \geq 12t \geq 0$, so $h(x) \geq 0$ for $-1 \leq x \leq 1$.

2. If $(\sqrt{t+1} - \sqrt{t})^2 < \rho < 1$ then $1 < m < 1 + 2t$, $|2t - m| < 1$, $h(2t - m) \geq 8t(1 - t) \geq 0$, $h(1) > 4(1 - t^2) \geq 0$, $2th'(1) - h'(2t - m) = -4tm \leq 0$. Hence $h'(2t - m) \geq 0$ or $h'(1) \leq 0$, i.e. $h(x) \geq 0$ for $2t - m \leq x \leq 1$. The proof is complete. □

Lemma 2. *Let* $0 \leq t \leq 1$. *Then*

1. f_t *is univalent in* Δ_r *if* $r \in (0, \sqrt{t+1} - \sqrt{t}]$, *and is nonunivalent in* Δ_r *if* $r \in (\sqrt{t+1} - \sqrt{t}, 1)$;
2. $f_t(\Delta_r)$ *is a starlike domain for each* $r \in (0, 1)$;
3. *the boundary of* $f_t(\Delta_r)$ *lying in the first quadrant of the complex plane is of the form*
   (i) $\{f_t(re^{i\varphi}) : \varphi \in [0, \frac{\pi}{2}]\}$ *for* $r \in (0, \sqrt{t+1} - \sqrt{t}]$,
   (ii) $\{f_t(re^{i\varphi}) : \varphi \in [0, \varphi(t, r)]\}$ *for* $r \in (\sqrt{t+1} - \sqrt{t}, 1)$,
   where $\varphi(t, r) = \frac{1}{2} \arccos (2t - m)$.

Proof. By Lemma 1, each f_t, $t \in [0, 1]$, is univalent and starlike in Δ_R, $R = \sqrt{t+1} - \sqrt{t}$. Hence $\partial f_t(\Delta_r) = \{f_t(re^{i\varphi}) : \varphi \in [0, 2\pi)\}$ for $0 < r \leq R$.

Let $r \in (R, 1)$. Each function f_t is not univalent in Δ_r because $f_t'(iR) = 0$. Observe that the set

$$\left\{ z \in \mathbf{C} : |z| = r,\ \cos(2 \arg z) \geq \frac{1}{2} \left(4t - \frac{1}{|z|^2} - |z|^2 \right) \right\}$$

consists of two arcs Γ_1, Γ_2, where

$$\Gamma_1 = \left\{ re^{i\varphi} : \varphi \in [-\varphi(t, r), \varphi(t, r)] \right\},$$
$$\Gamma_2 = \left\{ re^{i\varphi} : \varphi \in [\pi - \varphi(t, r), \pi + \varphi(t, r)] \right\}.$$

Moreover, $f_t(\Gamma_1 \cup \Gamma_2)$ is a closed curve without intersection points. Combining it with Lemma 1 we get the point (ii) of 3 and starlikeness of $f_t(\Delta_r)$. □

Lemma 3. *For a fixed* $r \in (0, 1)$,

1. $|f_0(re^{i\varphi})|$ *is an increasing function of* $\varphi \in [0, \frac{\pi}{2}]$,
2. $|f_1(re^{i\varphi})|$ *is a decreasing function of* $\varphi \in [0, \frac{\pi}{2}]$.

Proof. Observe that $|f(re^{i\varphi})|$ is an increasing function of $\varphi \in [0, \frac{\pi}{2}]$ if $\mathrm{Re}\,\dfrac{ire^{i\varphi}f'(re^{i\varphi})}{f(re^{i\varphi})} \geq 0$, and is a decreasing function if $\mathrm{Re}\,\dfrac{ire^{i\varphi}f'(re^{i\varphi})}{f(re^{i\varphi})} \leq 0$.

For this reason and from equalities

$$\mathrm{Im}\, z^2 \cdot \mathrm{Im}\,\frac{zf_0'(z)}{f_0(z)} = \frac{-2(\mathrm{Im}\, z^2)^2}{|1+z^2|^2}$$

and

$$\mathrm{Im}\, z^2 \cdot \mathrm{Im}\,\frac{zf_1'(z)}{f_1(z)} = \frac{2(\mathrm{Im}\, z^2)^2(2+|1+z^2|^2+2|z|^4)}{|1-z^4|^2},$$

the assertion follows. □

With fixed $r \in (0,1)$ let us denote

$$\text{(3)}\qquad \begin{array}{l} \gamma_0 : \varphi \longrightarrow f_0(re^{i\varphi}),\ \varphi \in R, \\ \gamma_1 : \varphi \longrightarrow f_1(re^{i\varphi}),\ \varphi \in R. \end{array}$$

Lemma 4. *The boundary of $f_0(\Delta_r) \cup f_1(\Delta_r)$ in the first quadrant of the complex plane coincides with*

1. $\gamma_1([0,\varphi_1]) \cup \gamma_0([\varphi_0, \frac{\pi}{2}])$ *for* $r \in \big(0, \frac{1}{2}(\sqrt{5}-1)\big]$,
2. $\gamma_1([0, \frac{\pi}{2}])$ *for* $r \in \big(\frac{1}{2}(\sqrt{5}-1), 1\big)$,

where

$$\text{(4)}\qquad \varphi_1 = \frac{1}{2}\arccos\left(\frac{1}{2}\left(m - \sqrt{m^2+4}\right)\right),\ \ \varphi_0 = \pi - 3\varphi_1.$$

Proof. Let $z_0 = re^{i\varphi}$, $z_1 = re^{i\phi}$ and $\varphi, \phi \in [0, \frac{\pi}{2}]$. In order to describe the common points of $\gamma_0([0,\frac{\pi}{2}])$ and $\gamma_1([0,\frac{\pi}{2}])$ we shall solve the equation $f_0(z_0) = f_1(z_1)$, $|z_0| = |z_1| = r$, which is equivalent to

$$\text{(5)}\qquad z_0 + \frac{1}{z_0} = z_1 + \frac{1}{z_1} - \frac{4}{z_1 + \frac{1}{z_1}},$$

and hence to

$$\text{(6)}\qquad \begin{cases} \cos\varphi = \left(1 - \frac{2}{m+\cos 2\phi}\right)\cos\phi \\ \sin\varphi = \left(1 + \frac{2}{m+\cos 2\phi}\right)\sin\phi. \end{cases}$$

From this system we obtain $\cos^2 2\phi + m\cos 2\phi - 1 = 0$. Thus $\cos 2\phi = \frac{1}{2}\Big(-m + \sqrt{m^2+4}\Big) \in \big(0, \frac{1}{2}(\sqrt{5}-1)\big)$ and $\cos 3\phi = -\cos\varphi$. The solution of (6) is

$$\begin{cases} \phi = \frac{1}{2}\arccos\left(\frac{1}{2}\left(m - \sqrt{m^2+4}\right)\right), \\ \varphi = \pi - 3\phi. \end{cases}$$

Observe that $\varphi \in [0, \frac{\pi}{2}]$ if and only if $\phi \in [\frac{\pi}{6}, \frac{\pi}{3}]$, and consequently, if $|\cos 2\phi| \leq \frac{1}{2}$. This inequality holds only for $m \geq \frac{3}{2}$ and hence for $r \in$

$\big(0, \frac{1}{2}(\sqrt{5}-1)\big]$. In the case $r \in \big(\frac{1}{2}(\sqrt{5}-1), 1\big)$ the system (6) has no solutions for $\varphi, \phi \in [0, \frac{\pi}{2}]$.

Additionally, if $r \in (0, \frac{1}{2}(\sqrt{5}-1)]$ then $f_0(r) = \frac{r}{1+r^2} < \frac{r(1+r^2)}{(1-r^2)^2} = f_1(r)$ and $f_0(ir)/i = \frac{r}{1-r^2} > \frac{r(1-r^2)}{(1+r^2)^2} = f_1(ir)/i$. □

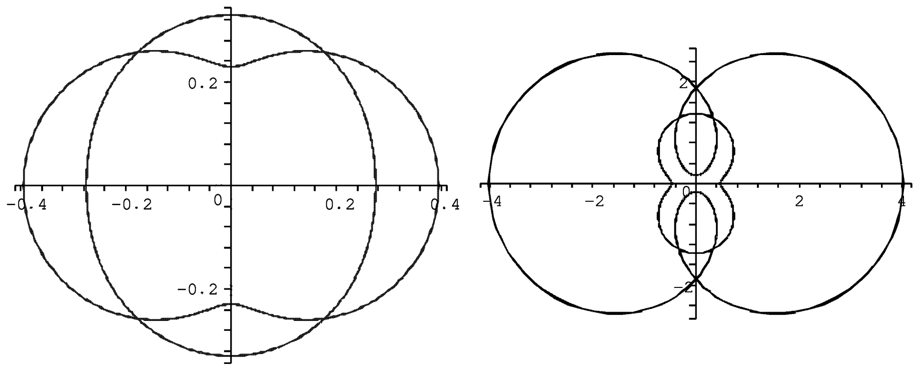

FIGURE 1. $L_{T^{(2)}}(\Delta_r)$ for $r = 0.3$ and $r = 0.7$.

Theorem 1.

$$L_{T^{(2)}}(\Delta_r) = f_0(\Delta_r) \cup f_1(\Delta_r) \textit{ for } r \in \big(0, \tfrac{1}{2}(\sqrt{5}-1)\big),$$
$$L_{T^{(2)}}(\Delta_r) = f_1(\Delta_r) \textit{ for } r \in \big[\tfrac{1}{2}(\sqrt{5}-1), 1\big).$$

Proof. Let $r \in (0,1)$ be fixed and let L denote the set $\bigcup_{0 \le t \le 1} f_t(\Delta_r)$. According to Lemma 1 and Lemma 2, each set $f_t(\Delta_r)$ is starlike with respect to 0, hence L is starlike with respect to the origin. We know that the maxima (1) and (2) are equal. For this reason we shall consider the function

$$F(t,\varphi) \equiv f_t(re^{i\varphi}), \text{ with } t \in [0,1] \text{ and } \varphi \in R. \tag{7}$$

The boundary of L is contained in the set $F([0,1] \times R)$.

Observe that if $(t_0, \varphi_0) \in \operatorname{int}([0,1] \times R)$ and the jacobian $J_F(t_0, \varphi_0)$ is nonzero, then $F(t_0, \varphi_0) \in \operatorname{int} L$. Therefore the set ∂L is included in the set $\{F(t,\varphi) \ : \ (t,\varphi) \in B\}$, where

$$B = \{(t,\varphi) : J_F(t,\varphi) = 0 \text{ or } t(1-t) = 0, \ \varphi \in R\}.$$

The equation $J_F(t,\varphi) = 0$, i.e.

$$\begin{vmatrix} \frac{\partial \operatorname{Re} F}{\partial t} & \frac{\partial \operatorname{Re} F}{\partial \varphi} \\ \frac{\partial \operatorname{Im} F}{\partial t} & \frac{\partial \operatorname{Im} F}{\partial \varphi} \end{vmatrix} (t,\varphi) = 0$$

is equivalent to

$$\operatorname{Im}\left(\overline{\frac{\partial F}{\partial t}}\cdot\frac{\partial F}{\partial \varphi}\right)(t,\varphi)=0. \tag{8}$$

Since

$$\frac{\partial F}{\partial t}=\frac{4r^3e^{3i\varphi}(1+r^2e^{2i\varphi})}{[(1+r^2e^{2i\varphi})^2-4tr^2e^{2i\varphi}]^2},$$
$$\frac{\partial F}{\partial \varphi}=\frac{ire^{i\varphi}[(1+r^2e^{2i\varphi})^2+4tr^2e^{2i\varphi}]}{[(1+r^2e^{2i\varphi})^2-4tr^2e^{2i\varphi}]^2},$$

we can rewrite (8) as follows

$$\operatorname{Re}\left\{r^2e^{-2i\varphi}\left(1+r^2e^{-2i\varphi}\right)\left(1-r^2e^{2i\varphi}\right)\left[\left(1+r^2e^{2i\varphi}\right)^2+4tr^2e^{2i\varphi}\right]\right\}=0.$$

We eventually obtain

$$2t+(m+\cos 2\varphi)\cos 2\varphi=0. \tag{9}$$

The points $(t,\varphi)\in[0,1]\times R$ satisfy (9) only if $\lambda(m)\le\cos 2\varphi\le 0$, where

$$\lambda(m)=\begin{cases}-1 & \text{for} \quad m\le 3,\\ \frac{\sqrt{m^2-8}-m}{2} & \text{for} \quad m>3.\end{cases}$$

Consider the curve

$$\gamma:\varphi\longrightarrow F\left(-\frac{1}{2}(m+\cos 2\varphi)\cos 2\varphi,\varphi\right),\ \varphi\in R \tag{10}$$

and the curves γ_0 and γ_1 defined by (3).

We claim that $\gamma(\{\varphi\in R:\lambda(m)\le\cos 2\varphi\le 0\})$ is included in the closed domain bounded by $\gamma_0(R)$, i.e. in $f_0(\overline{\Delta_r})$.

Indeed, we have

$$\begin{aligned}1/F\left(-\tfrac{1}{2}(m+\cos 2\varphi)\cos 2\varphi,\varphi\right)&=\frac{1}{re^{i\varphi}}+re^{i\varphi}+\frac{2(m+\cos 2\varphi)\cos 2\varphi}{\frac{1}{re^{i\varphi}}+re^{i\varphi}}\\&=2\left(\frac{1}{r}+r\right)\cos^3\varphi-2i\left(\frac{1}{r}-r\right)\sin^3\varphi,\end{aligned}$$

i.e. (10) restricted to $[0,2\pi)$ is a Jordan curve, and

$$1/F(0,\psi)=\frac{1}{re^{i\psi}}+re^{i\psi}=\left(\frac{1}{r}+r\right)\cos\psi-i\left(\frac{1}{r}-r\right)\sin\psi.$$

The equation $F(0,\psi)=F\left(-\frac{1}{2}(m+\cos 2\varphi)\cos 2\varphi,\varphi\right)$ is equivalent to the system

$$\begin{cases}2\cos^3\varphi=\cos\psi\\ 2\sin^3\varphi=\sin\psi.\end{cases}$$

It is easy to check that the only solution of this system for $\varphi,\psi\in[0,\frac{\pi}{2}]$ is $\varphi=\psi=\frac{\pi}{4}$. It means that the sets $\gamma([0,\frac{\pi}{2}])$ and $\gamma_0([0,\frac{\pi}{2}])$ have only one common point.

Moreover,

$$\gamma(0) = \frac{1}{2(\frac{1}{r}+r)} < \frac{1}{\frac{1}{r}+r} = \gamma_0(0)$$

and

$$\gamma(\tfrac{\pi}{2})/i = \frac{1}{2\left(\frac{1}{r}-r\right)} < \frac{1}{\frac{1}{r}-r} = \gamma_0(\tfrac{\pi}{2})/i.$$

Therefore $\gamma([0,\frac{\pi}{2}]) \subset f_0(\overline{\Delta_r})$ and, by symmetry of $\gamma(R)$ with respect to both axes of the complex plane, we have $\gamma(R) \subset f_0(\overline{\Delta_r})$. Consequently, $\partial L \subset \gamma_0(R) \cup \gamma_1(R)$. The assertion of the theorem follows now from Lemma 4. □

From Theorem 1 and Lemma 4 it immediately follows

Corollary 1. *If $r \in \left(0, \frac{1}{2}(\sqrt{5}-1)\right)$ then the boundary of $L_{T^{(2)}}(\Delta_r)$ lying in the first quadrant of the complex plane coincides with $\gamma_1([0,\varphi_1]) \cup \gamma_0([\varphi_0, \frac{\pi}{2}])$, where φ_1 and φ_0 are defined by (4).*

We conclude from Theorem 1 that for $f \in T^{(2)}$ and $|z| = r$ the following sharp bound holds:

$$|\operatorname{Im} f(z)| \le \begin{cases} \max_{z\in\partial\Delta_r} \{\operatorname{Im} f_0(z), \operatorname{Im} f_1(z)\} & \text{for } r \in \left(0, \frac{1}{2}(\sqrt{5}-1)\right), \\ \max_{z\in\partial\Delta_r} \operatorname{Im} f_1(z) & \text{for } r \in \left[\frac{1}{2}(\sqrt{5}-1), 1\right). \end{cases}$$

Denote $x = (\frac{1}{r}+r)^2$,

$$g(r) = \left(\frac{1}{r} - r\right) \frac{(2-x+\sqrt{2x^2+4})(2x-2+\sqrt{2x^2+4})^2\sqrt{3x-2\sqrt{2x^2+4}}}{16x^2(x-4)^2},$$

and

$$h(x) = x^5 - 124x^4 + 4064x^3 - 21632x^2 + 256x + 5120.$$

A simple but extensive calculation leads to

Corollary 2. *For $f \in T^{(2)}$ and $r \in (0,1)$ we have*

$$\left|\operatorname{Im} f\left(re^{i\varphi}\right)\right| \le \begin{cases} \frac{r}{1-r^2} & \textit{for} \quad r \in (0, r_*), \\ g(r) & \textit{for} \quad r \in [r_*, 1), \end{cases}$$

where $r_* = 0.483\ldots$ is the only solution of the equation $h\left((\frac{1}{r}+r)^2\right) = 0$ in $(\sqrt{2}-1, 1)$.

It is interesting to describe the covering domain for the class $T^{(2)}$ over the set H, where $H = \left\{z \in \Delta : \left|1+z^2\right| > 2|z|\right\}$. The lens-shaped set H is the domain of univalence for $T^{(2)}$ ([3], see also [1]). We apply this property of H in the proof of the following theorem.

Theorem 2. *$L_{T^{(2)}}(H) = f_0(H) \cup f_1(H)$.*

Proof. Observe that $f_1(H) = \mathbf{C} \setminus \left\{i\varrho : \varrho \in \left(-\infty, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, \infty\right)\right\}$. The set H is symmetric with respect to both axes as well as each set $f(H)$ for all $f \in T^{(2)}$. From this and from univalence of each $f \in T^{(2)}$ in H we conclude that for $z \in H$ there is $\operatorname{Re} f(z) = 0 \Leftrightarrow \operatorname{Re} z = 0$.

For this reason it suffices to calculate $\max\left\{\frac{1}{i} f(ir_0) : f \in T^{(2)}\right\}$, $r_0 = \sqrt{2} - 1$. We have

$$\max\left\{\frac{1}{i} f(ir_0) : f \in T^{(2)}\right\} = \max\left\{\frac{r_0(1-{r_0}^2)}{(1-{r_0}^2)^2 + 4{r_0}^2 t} : t \in [0,1]\right\}$$
$$= \frac{r_0}{1-{r_0}^2} = \frac{1}{2} = \frac{1}{i} f_0(ir_0) \ .$$

Hence the set $\{i\varrho : \varrho \geq \frac{1}{2}\}$ is disjoint from $L_{T^{(2)}}(H)$. This fact and the symmetry of $L_{T^{(2)}}(H)$ with respect to the real axis completes the proof. □

We get from Theorem 2

Corollary 3. $L_{T^{(2)}}(H) = \mathbf{C} \setminus \left\{i\varrho : \varrho \in \left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)\right\}$.

Corollary 4. *For arbitrary domain $D \supset \operatorname{cl}(H) \setminus \{-1, 1\}$ we have $L_{T^{(2)}}(D) = \mathbf{C}$.*

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Received April 11, 2006