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Optimal investment and consumption in the presence of default on a financial market driven by a Lévy noise

ABSTRACT. In this paper we investigate a problem of optimal investment and consumption. We consider a financial market consisting of a risk-free asset with a deterministic force of interest and a risky asset whose price is driven by a time-inhomogeneous Lévy process. We also take into account a possibility of default, which is an unpredictable event of exiting a financial market, and we model a default intensity as a diffusion process. The classical verification theorem for the Hamilton–Jacobi–Bellman equation is proved and explicit results are derived for HARA utility functions.

1. Introduction. When dealing with investment and consumption decisions one should take into account two types of risks. The first one is market risk which arises due to unpredictable changes in asset prices. The second is timing risk which arises due to the uncertainty over an investment time-horizon. In most cases, an investor when entering a financial market does not know with certainty the time of exiting this market. Some events may occur, which are commonly called defaults, which can force an investor to leave a market liquidating or not his/her assets. What is more important, a rate of arrival of a default event at future dates, which is called a default

intensity, is also unknown and should not be treated deterministically but in a stochastic way. An example of such default event is a death of an agent.

In this paper we are dealing with an optimal investment and consumption problem taking into account both types of risks together with the uncertainty over a default intensity. A closely-related problem was considered by Blanchet-Scalliet et al. [4] on a Black–Scholes market. In this work, a deterministic distribution function of default time was assumed and dynamic programming, as well as martingale methods were applied. Explicit results were derived in the case of HARA utility functions for two separate problems of an optimal investment when maximizing an expected utility of a wealth at default time and an optimal consumption/investment when maximizing a total expected utility of consumption rates up to default without a bequest motive. They also tried to take into account the uncertainty of future default intensities and considered a stochastic density process of default time, correlated with a financial market, modelled as a geometric Brownian motion. In this particular case they were able to derive the Hamilton–Jacobi–Bellman equation and found a solution for HARA utility functions. Also worth mentioning is another paper of Blanchet-Scalliet et al. [5] in which they developed the asset pricing theory with uncertain time-horizon.

Recently, Øksendal [15] solved an optimal consumption problem with a bequest motive in a financial market driven by a Lévy noise for an insider in the case of $\mathcal{F}_\infty$-measurable default time. It was also showed that an optimal consumption/investment problem could be split into a consumption problem and an investment problem which was solved in the same setting in Di Nunno et al. [11]. In both papers logarithmic utility was applied.

Bouchard and Pham [6] studied a wealth-path dependent utility maximization problem in a general incomplete semimartingale model by applying martingale duality methods. A special case is an optimal investment problem of maximizing an expected utility of a wealth at a random date by taking into account a stochastic density process of default time. For an optimal investment problem with a certain terminal time in a financial market driven by a Lévy process we refer the reader to Choulli and Hurd [8].

This paper is structured as follows. In Section 2 we introduce a model of a financial market together with a default intensity process and default time. An optimal investment and consumption problem is discussed in Section 3. In Section 4 we derive the Hamilton–Jacobi–Bellman equation for our optimization problem and prove the verification theorem. In the case of power and logarithmic utility preferences, classical solutions are found. The insurance applications of our model are presented in Section 5.

2. The model. Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$, where T is fixed, finite time horizon. The filtration satisfies the usual hypotheses of completeness ($\mathcal{F}_0$ contains all sets of $\mathbb{P}$-measure zero) and right continuity ($\mathcal{F}_t = \mathcal{F}_{t+}$). The filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ consists

of three subfiltrations. We set $\mathcal{F}_t = \mathcal{F}_t^F \vee \mathcal{F}_t^\tau \vee \mathcal{F}_t^D$ for all $t \in [0, T]$, where $\mathcal{F}_t^F$ contains information about a financial market, $\mathcal{F}_t^\tau$ contains information whether a default event in a financial market has already occurred or not, and $\mathcal{F}_t^D$ contains information about a default intensity. We assume that the subfiltrations $\mathcal{F}_t^F$ and $(\mathcal{F}_t^\tau, \mathcal{F}_t^D)$ are independent. In the following sub-sections we introduce a financial market and a stochastic default intensity process.

2.1. The financial market. We consider a financial market consisting of two assets. One of the assets is risk-free (a bank account) and its price $(B(t), 0 \le t \le T)$ is described by an ordinary differential equation

$$(2.1) \qquad \frac{dB(t)}{B(t)} = r(t)dt, \quad B(0) = 1,$$

where $r(t)$ denotes a rate of interest. The second tradeable financial instrument is a risky asset (a stock) and its price $(S(t), 0 \le t \le T)$ is modelled as a geometric Lévy process. The dynamics of the stock price is given by a stochastic differential equation

$$(2.2) \qquad \frac{dS(t)}{S(t-)} = \mu(t)dt + \xi(t)dL(t), \quad S(0) = s_0 > 0,$$

where $\mu(t)$ and $\xi(t)$ denote a drift and a volatility, $(L(t), 0 \le t \le T)$ denotes a zero-mean time-inhomogeneous Lévy process (an additive process), $\mathcal{F}_t^F$-adapted with càdlàg sample paths (continuous on the right and having limits on the left).

Let us recall the definition of an additive process.

Definition 2.1. A stochastic process $(L(t))_{t \ge 0}$ is an additive process if it has the following properties:

    1. $L(0) = 0$ (a.s.),
    2. $(L(t))_{t \ge 0}$ has independent increments,
    3. $(L(t))_{t \ge 0}$ is stochastically continuous,

$$\forall \varepsilon > 0, \ \lim_{t \to s} \mathbb{P}(|L(t) - L(s)| > \varepsilon) = 0.$$

The zero-mean additive process $(L(t), 0 \le t \le T)$ is assumed to satisfy the Lévy–Itô decomposition

$$(2.3) \qquad L(t) = \int_0^t \sigma(s)dW(s) + \int_{(0,t]} \int_{\mathbb{R}} z \big(M(ds \times dz) - \nu_s(dz)ds \big),$$

where $(W(t), 0 \le t \le T)$ is a $\mathbb{P}$-Brownian motion and $M((s,t] \times A) = \#\{s < u \le t : (L(u) - L(u-)) \in A\}$ is a Poisson random measure, independent of the Brownian motion, with a time-inhomogeneous, deterministic intensity measure $\nu_t(dz)dt$ (a compensator). Let us recall that $\tilde{M}((s,t] \times A) = M((s,t] \times A) - \int_s^t \nu_u(A)du$ is a martingale-valued measure,

that is $\tilde{M}((s,t] \times A)$ is a $(\mathbb{P}, \mathbb{F}^F)$-martingale for all $t \in (s, T]$ and all Borel sets $A \in \mathcal{B}(\mathbb{R} - \{0\})$. For more information concerning additive processes, Lévy processes and Poisson random measures we refer the reader to Applebaum [1], Cont and Tankov [9] and Sato [17].

We make the following assumptions concerning the coefficients and the intensity measure:

A1: $r : [0,T] \to [0,\infty)$, $\mu : [0,T] \to [0,\infty)$, $\sigma : [0,T] \to [0,\infty)$ are Lipschitz continuous functions,

A2: we set $\xi(t) = 1$ for all $t \in [0,T]$, this is no loss of generality as the process $\int_{(0,t]} \xi(s) dL(s)$ is also additive and satisfies the Lévy–Itô decomposition,

A3: $(\nu_t, 0 \leq t \leq T)$ is a family of Lévy measures on $(-1, \infty)$, such that $\inf_{t \in [0,T]} \triangle L(t) > -1$, $\int_0^T \int_{z > -1} z^2 \nu_s(dz) ds < \infty$ and $\nu_t(\{0\}) = 0$ for all $t \in [0,T]$,

A4: $|\nu_t(A) - \nu_s(A)| \leq p(A)|t-s|$ for all Borel sets, where $p(\cdot)$ is a Lévy measure on $(-1, \infty)$, such that $\int_{z > -1} z^2 p(dz) < \infty$ and $p(\{0\}) = 0$.

The stochastic differential equation (2.2) has the unique, positive and almost surely finite solution, given explicitly by the Doléans–Dade exponential, see Applebaum [1].

We refer the interested reader to Chan [7] for more properties of such financial model. Alternatively, one can start with an exponential additive process as a model of a stock price, see Cont and Tankov [9]. Doléans–Dade exponential shows that these two approaches to price modelling are equivalent.

2.2. The stochastic default intensity process and default time. We apply a reduced-form model of default time, see for example Jeanblanc and Rutkowski [13] or Bielecki and Rutkowski [3] for more information. We assume that τ, which is the moment of default in the financial market, is a positive random variable, a stopping time with respect to filtration $\mathcal{F}_t^\tau$, exponentially distributed with survival function

$$(2.4) \qquad \mathbb{P}(\tau > t | \mathcal{F}_t^D) = e^{-\int_0^t \lambda(s) ds},$$

where $(\lambda(t), 0 \leq t \leq T)$ is a default intensity process. The default intensity is a stochastic process of diffusion type, which dynamics is given by a stochastic differential equation

$$(2.5) \qquad d\lambda(t) = a(t, \lambda(t)) dt + b(t, \lambda(t)) d\bar{W}(t), \quad \lambda(0) = \lambda_0 > 0,$$

where $(\bar{W}(t), 0 \leq t \leq T)$ is an $\mathcal{F}_t^D$-adapted $\mathbb{P}$-Brownian motion, independent of the Brownian motion $(W(t), 0 \leq t \leq T)$ and Poisson random measure $M((0,t] \times A)$. One can associate a one-jump point-process $(N(t), 0 \leq t \leq T)$ with the random variable τ, and the process defined as $N(t) = \mathbf{1}\{t \geq \tau\}$ is called *doubly stochastic with intensity λ* or *Cox process*.

We make the following assumptions concerning the stochastic default intensity process:

B1: $a : [0,T] \times (0,\infty) \to \mathbb{R}$, $b : [0,T] \times (0,\infty) \to (0,\infty)$ are continuous functions, locally Lipschitz continuous in λ, uniformly in t,

B2: there exists a sequence $(A_n)_{n \in \mathbb{N}}$ of bounded domains with $\bar{A}_n \subseteq (0,\infty)$ and $\bigcup_{n \geq 1} A_n = (0,\infty)$, each with a $\mathcal{C}^2$-boundary, such that the functions $a(t,\lambda)$ and $b^2(t,\lambda)$ are uniformly Lipschitz continuous on $[0,T] \times \bar{A}_n$,

B3: $\mathbb{P}(\forall_{s \in [t,T]} \lambda(s) > 0 | \lambda(t) = \lambda) = 1$ and

$$\sup_{s \in [t,T]} \mathbb{E}^{\mathbb{P}}[|\lambda(s)|^2 | \lambda(t) = \lambda] < \infty$$

for all starting points $(t,\lambda) \in [0,T] \times (0,\infty)$.

Under the assumptions B1 and B3 the default intensity process is nonexplosive on $[t,T]$ and there exists the unique strong solution to the stochastic differential equation (2.5) for each starting point $(t,\lambda) \in [0,T] \times (0,\infty)$, such that the mapping $(t,\lambda,s) \to \lambda^{t,\lambda}(s)$ is $\mathbb{P}$-a.s continuous, see Kunita [14]. The assumption B2 is required in the verification results.

3. Optimal investment and consumption problem. We assume that an agent makes decisions based on his/her utility preferences and let $u : \mathbb{R} \to \mathbb{R}$ denote the agent's utility function. This utility function should be increasing and concave as economic theory indicates. Let $X(t)$, for $0 \leq t \leq T$, denote the value of the agent's wealth at time t arising from trading in the financial market. The agent adopts the consumption strategy $(c(t), 0 < t \leq T)$ and the investment strategy $(\pi(t), 0 < t \leq T)$, where $c(t)$ denotes the rate of consumption at time t and $\pi(t)$ denotes the fraction of the wealth invested in the risky asset at time t. The remaining fraction of the available wealth, $1 - \pi(t)$, is invested in the risk-free asset. The dynamics of the agent's wealth process $(X^{c,\pi}(t), 0 \leq t \leq T)$ is given by a stochastic differential equation

$$
\begin{aligned}
(3.1) \quad dX^{c,\pi}(t) &= \pi(t)X^{c,\pi}(t-)\left(\mu(t)dt + \sigma(t)dW(t) + \int_{z>-1} z\tilde{M}(dt \times dz)\right) \\
&\quad + X^{c,\pi}(t-)(1 - \pi(t))r(t)dt - c(t)dt, \quad X(0) = x_0,
\end{aligned}
$$

where x_0 denotes the initial available wealth of the agent.

Let us introduce the set of admissible strategies for our problem.

Definition 3.1. The control $(c(t), \pi(t), s < t \leq T)$ is an admissible on the time interval $(s,T]$, $(c,\pi) \in \mathcal{A}(s,T]$, if it satisfies the following assumptions:

(1) $\pi : (s,T] \times \Omega \to [0,1]$ and $c : (s,T] \times \Omega \to [0,\infty)$ are predictable mappings with respect to filtration $\mathbb{F}$,

(2) the stochastic differential equation (3.1) has the unique, strong and positive solution on $[s,T]$ given the initial condition $X(s) = x$, for all $x \in (0,\infty)$.

We deal with the following optimization problem:

$$\sup_{(c,\pi)\in\mathcal{A}(0,\tau\wedge T]} \mathbb{E}\bigg[\int_0^T \mathbf{1}\{\tau \geq s\}e^{-\rho s}u(c(s))ds$$

$$(3.2) \qquad\qquad + \alpha e^{-\rho\tau}u(X^{c,\pi}(\tau))\mathbf{1}\{\tau \leq T\}$$

$$+ \beta e^{-\rho T}u(X^{c,\pi}(T))\mathbf{1}\{\tau > T\}\bigg],$$

where $\rho \geq 0$ is a intertemporal agent's discount factor and parameters $\alpha, \beta > 0$ attach weights to utility at the default time τ and at the terminal time T. The possibility of liquidating the assets even if the default occurs before terminal time T is called in life-cycles models a bequest motive. In the view of optimization criterium (3.2) the agent is trying to maximize the expected total discounted utility of consumption rates up to the terminal time T or up to the default time τ, whichever occurs first, and the discounted utility of the wealth at the terminal time T or at the default time τ, whichever occurs first. This problem is an extension of the well-known Merton problem.

4. The stochastic control problem. The investment/consumption problem stated in the previous section can be solved by applying stochastic control theory. The solution of the optimization problem (3.2), together with the verification theorem is, to the best of our knowledge, the new one in the financial literature.

4.1. Hamilton–Jacobi–Bellman equation. In this subsection we derive the Hamilton–Jacobi–Bellman equation and prove the verification theorem. In the next subsections, the classical solutions are found in the case of HARA utility functions.

Let $\mathcal{L}_F$ denote the integro-differential operator given by

$$\mathcal{L}_F^{c(t),\pi(t)}\phi(t,x) = \big(\pi(t)x(\mu(t) - r(t)) + xr(t) - c(t)\big)\frac{\partial\phi}{\partial x}(t,x)$$

$$(4.1) \qquad + \frac{1}{2}\pi^2(t)x^2\sigma^2(t)\frac{\partial^2\phi}{\partial x^2}(t,x)$$

$$+ \int_{z>-1}\bigg(\phi(t,x+\pi(t)xz) - \phi(t,x) - \pi(t)xz\frac{\partial\phi}{\partial x}(t,x)\bigg)\nu_t(dz),$$

and let $\mathcal{L}_D$ denote the differential operator given by

$$(4.2) \qquad \mathcal{L}_D\phi(t,\lambda) = a(t,\lambda)\frac{\partial\phi}{\partial\lambda}(t,\lambda) + \frac{1}{2}b^2(t,\lambda)\frac{\partial^2\phi}{\partial\lambda^2}(t,\lambda).$$

These two operators are defined for functions ϕ such that $\mathcal{L}\phi$ are well-defined pointwise and all derivatives appearing in $\mathcal{L}\phi$ exist and are continuous functions.

Let us define the optimal value function for the optimization problem:

$$V(t, x, \lambda) = \sup_{(c,\pi) \in \mathcal{A}(t,\tau \wedge T]} \mathbb{E}\left[\int_t^T \mathbf{1}\{\tau \geq s\}e^{-\rho s}u(c(s))ds\right.$$

(4.3)
$$+ \alpha e^{-\rho\tau}u(X^{c,\pi}(\tau))\mathbf{1}\{\tau \leq T\} + \beta e^{-\rho T}u(X^{c,\pi}(T))\mathbf{1}\{\tau > T\}|$$

$$\left. X(t) = x, \lambda(t) = \lambda, N(t) = 0\right], \quad 0 \leq t \leq T.$$

Let us denote the expectations $\mathbb{E}[\cdot|X(t) = x, \lambda(t) = \lambda, N(t) = 0]$ in the form $\mathbb{E}^{t,x,\lambda}[\cdot]$. Below we prove the classical stochastic verification theorem.

Theorem 4.1. *Let $v \in \mathcal{C}^{1,2,2}([0,T) \times (0,\infty) \times (0,\infty)) \cap \mathcal{C}([0,T] \times (0,\infty) \times (0,\infty))$ satisfies, for all $(c,\pi) \in \mathcal{A}(t,\tau \wedge T]$,*

(4.4)
$$0 \geq \frac{\partial v}{\partial t}(t,x,\lambda) + e^{-\rho t}u(c(t)) + \mathcal{L}_F^{c(t),\pi(t)}v(t,x,\lambda) + \mathcal{L}_D v(t,x,\lambda)$$
$$+ \lambda\big(\alpha e^{-\rho t}u(x) - v(t,x,\lambda)\big),$$

(4.5)
$$v(T,x,\lambda) = \beta e^{-\rho T}u(x),$$

(4.6)
$$\mathbb{E}^{t,x,\lambda}\left[\int_t^T \int_{z \geq 1} \mathbf{1}\{\tau \geq s\}\big(v(s, X^{c,\pi}(s-) + \pi(s)X^{c,\pi}(s-)z, \lambda(s))\right.$$
$$\left. - v(s, X^{c,\pi}(s-), \lambda(s))\big)^2 \nu_s(dz)ds\right] < \infty,$$

(4.7)
$$\sup_{s \in [t,T]} \mathbb{E}^{t,x,\lambda}\left[\big(|v(s, X^{c,\pi}(s), \lambda(s))|^2 + |u(X^{c,\pi}(s))|^2\big)\mathbf{1}\{\tau \geq s\}\right] < \infty,$$

and for all $(t,x,\lambda) \in [0,T] \times (0,\infty) \times (0,\infty)$. Then

(4.8)
$$v(t,x,\lambda) \geq V(t,x,\lambda), \quad \forall (t,x,\lambda) \in [0,T] \times (0,\infty) \times (0,\infty).$$

Moreover, if there exists an admissible feedback control $(\tilde{c}, \tilde{\pi}) \in \mathcal{A}(0, \tau \wedge T]$ such that

(4.9)
$$0 = \frac{\partial v}{\partial t}(t, X^{\tilde{c},\tilde{\pi}}(t-), \lambda(t)) + e^{-\rho t}u(\tilde{c}(t))$$
$$+ \mathcal{L}_F^{\tilde{c}(t),\tilde{\pi}(t)}v(t, X^{\tilde{c},\tilde{\pi}}(t-), \lambda(t)) + \mathcal{L}_D v(t, X^{\tilde{c},\tilde{\pi}}(t-), \lambda)$$
$$+ \lambda(t)\big(\alpha e^{-\rho t}u(X^{\tilde{c},\tilde{\pi}}(t-)) - v(t, X^{\tilde{c},\tilde{\pi}}(t-), \lambda)\big),$$

holds $\mathbb{P}$-a.s. for a.a. $0 < t \leq T$ with respect to Lebesque measure, and

(4.10)
the family $\{v(\mathcal{T}, X^{\tilde{c},\tilde{\pi}}(\mathcal{T}), \lambda(\mathcal{T}))\}_{t<\mathcal{T}\leq T}$ is uniformly integrable for all $\mathbb{F}$-stopping times $\mathcal{T}$,

then

(4.11)
$$v(t,x,\lambda) = V(t,x,\lambda), \quad \forall (t,x,\lambda) \in [0,T] \times (0,\infty) \times (0,\infty),$$

and $(\tilde{c}, \tilde{\pi})$ is the optimal strategy for the optimization problem (4.3).

Proof. Let us fix t, $0 \leq t \leq T$. Let us assume that an arbitrary admissible control $(c, \pi) \in \mathcal{A}(t, \tau \wedge T]$ is applied and let $v \in \mathcal{C}^{1,2,2}([0, T) \times (0, \infty) \times (0, \infty)) \cap \mathcal{C}([0, T] \times (0, \infty) \times (0, \infty))$ denote a function which satisfies (4.4)–(4.7). Notice that

$$
\begin{aligned}
\mathbb{E}^{t,x,\lambda}\big[&\alpha e^{-\rho\tau} u(X^{c,\pi}(\tau))\mathbf{1}\{\tau \leq T\} \\
&+ \beta e^{-\rho T} u(X^{c,\pi}(T))\mathbf{1}\{\tau > T\} - v(t, x, \lambda)\big] \\
= \mathbb{E}^{t,x,\lambda}\big[&\alpha e^{-\rho\tau} u(X^{c,\pi}(\tau))\mathbf{1}\{\tau \leq T\} \\
&+ \beta e^{-\rho T} u(X^{c,\pi}(T))\mathbf{1}\{\tau > T\} \\
&- v(\tau \wedge T, X^{c,\pi}(\tau \wedge T), \lambda(\tau \wedge T)) \\
&+ v(\tau \wedge T, X^{c,\pi}(\tau \wedge T), \lambda(\tau \wedge T)) - v(t, x, \lambda)\big] \\
= \mathbb{E}^{t,x,\lambda}\big[&\big(\alpha e^{-\rho\tau} u(X^{c,\pi}(\tau)) - v(\tau, X^{c,\pi}(\tau), \lambda(\tau))\big)\mathbf{1}\{\tau \leq T\}\big] \\
&+ \mathbb{E}^{t,x,\lambda}\big[v(\tau \wedge T, X^{c,\pi}(\tau \wedge T), \lambda(\tau \wedge T)) - v(t, x, \lambda)\big].
\end{aligned}
\tag{4.12}
$$

Let us deal with the first factor in (4.12). We have

$$
\begin{aligned}
\mathbb{E}^{t,x,\lambda}&\big[\big(\alpha e^{-\rho\tau} u(X^{c,\pi}(\tau)) - v(\tau, X^{c,\pi}(\tau), \lambda(\tau))\big)\mathbf{1}\{\tau \leq T\}\big] \\
&= \mathbb{E}^{t,x,\lambda}\big[\big(\alpha e^{-\rho\tau} u(X^{c,\pi}(\tau-)) \\
&\qquad - v(\tau, X^{c,\pi}(\tau-), \lambda(\tau))\big)\mathbf{1}\{\tau \leq T\}\big] \\
&= \mathbb{E}^{t,x,\lambda}\big[\mathbb{E}^{t,x,\lambda}\big[\big(\alpha e^{-\rho\tau} u(X^{c,\pi}(\tau-)) \\
&\qquad - v(\tau, X^{c,\pi}(\tau-), \lambda(\tau))\big)\mathbf{1}\{\tau \leq T\}|\mathcal{F}_T^D\big]\big] \\
&= \mathbb{E}^{t,x,\lambda}\bigg[\int_t^T \mathbb{E}^{t,x,\lambda}\big[\big(\alpha e^{-\rho s} u(X^{c,\pi}(s-)) \\
&\qquad - v(s, X^{c,\pi}(s-), \lambda(s))\big)|\mathcal{F}_T^D\big]\lambda(s)e^{-\int_t^s \lambda(w)dw}ds\bigg] \\
&= \mathbb{E}^{t,x,\lambda}\bigg[\int_t^T \mathbb{E}^{t,x,\lambda}\big[\big(\alpha e^{-\rho s} u(X^{c,\pi}(s-)) \\
&\qquad - v(s, X^{c,\pi}(s-), \lambda(s))\big)\lambda(s)\mathbf{1}\{\tau \geq s\}|\mathcal{F}_T^D\big]ds\bigg] \\
&= \mathbb{E}^{t,x,\lambda}\bigg[\mathbb{E}^{t,x,\lambda}\bigg[\int_t^T \mathbf{1}\{\tau \geq s\}\lambda(s)\big(\alpha e^{-\rho s} u(X^{c,\pi}(s-)) \\
&\qquad - v(s, X^{c,\pi}(s-), \lambda(s))\big)ds|\mathcal{F}_T^D\bigg]\bigg] \\
&= \mathbb{E}^{t,x,\lambda}\bigg[\int_t^T \mathbf{1}\{\tau \geq s\}\lambda(s)\big(\alpha e^{-\rho s} u(X^{c,\pi}(s-)) \\
&\qquad - v(s, X^{c,\pi}(s-), \lambda(s))\big)ds\bigg],
\end{aligned}
\tag{4.13}
$$

where we have used:

1. the observation that $X^{c,\pi}(\tau)\mathbf{1}\{t < \tau \leq T\} = X^{c,\pi}(\tau-)\mathbf{1}\{t < \tau \leq T\}$ holds $\mathbb{P}$-a.s.,
2. the property of conditional expectations,
3. the independence of the wealth process $(X^{c,\pi}(t), 0 \leq t \leq T)$ and the random variable τ, conditioned on filtration $\mathcal{F}_T^D$,
4. the distribution of the random variable τ conditioned on filtration $\mathcal{F}_T^D$,
5. Fubini theorem for conditional expectation, integrability is justified by (4.7) and B3.

Let us now deal with the second factor in (4.12). Let us introduce the sequence of stopping times

$$t_n = \inf\{s \in (t,T] : |X(s) - x| + |\lambda(s) - \lambda| > \theta n\}, \quad \theta > 0.$$

Clearly, $t_1 \leq \cdots \leq t_n \to T$ holds $\mathbb{P}$-a.s. Let us choose an arbitrary $0 < \varepsilon < T - t$. Applying Itô's lemma we arrive at

$$\mathbb{E}^{t,x,\lambda}\big[v(\tau \wedge t_n \wedge (T - \varepsilon), X^{c,\pi}(\tau \wedge t_n \wedge (T - \varepsilon)), \lambda(\tau \wedge t_n \wedge (T - \varepsilon)))$$

$$- v(t,x,\lambda)\big]$$

$$(4.14) \quad = \mathbb{E}^{t,x,\lambda}\bigg[\int_t^{T-\varepsilon} \mathbf{1}\{\tau \geq s, t_n \geq s\}\bigg(\frac{\partial v}{\partial t}(s, X^{c,\pi}(s-), \lambda(s))$$

$$+ \mathcal{L}_F^{c(s),\pi(s)} v(s, X_0^\pi(s-), \lambda(s)) + \mathcal{L}_D v(s, X^{c,\pi}(s-), \lambda(s))\bigg)ds\bigg],$$

where we have used the martingale property of the stochastic integrals, resulting from the boundness of the integrands and the assumption (4.6). The next steps are rather standard and we refer the reader to Øksendal, Sulem [16] for details. Taking the limit $n \to \infty$, $\varepsilon \to 0$ in (4.14) we arrive at

$$\mathbb{E}^{t,x,\lambda}\big[v(\tau \wedge T, X^{c,\pi}(\tau \wedge T), \lambda(\tau \wedge T)) - v(t,x,\lambda)\big]$$

$$\leq -\mathbb{E}^{t,x,\lambda}\bigg[\int_t^T \mathbf{1}\{\tau \geq s\}e^{-\rho s}u(c(s))ds$$

$$(4.15)$$

$$+ \int_t^T \mathbf{1}\{\tau \geq s\}\lambda(s)\big(\alpha e^{-\rho s}u(X^{c,\pi}(s-))$$

$$- v(s, X^\pi(s-), \lambda(s))\big)ds\bigg],$$

which in combination with (4.13) proves (4.8). In order to prove (4.11) one should apply the control $(\tilde{c}, \tilde{\pi})$ on $(t, \tau \wedge T]$. $\qquad\square$

In the next two subsections, the solutions of the derived Hamilton–Jacobi–Bellman equation are found for power and logarithmic utility functions.

4.2. Power utility functions. In this subsection we assume that the agent applies a power utility function of the form $u(x) = \frac{x^\gamma}{\gamma}$, $\gamma \in (0,1)$. We postulate that the optimal value function for the problem (4.3) is given by

$$(4.16) \qquad V(t,x,\lambda) = \psi(t,\lambda)\frac{x^\gamma}{\gamma}, \quad \forall (t,x,\lambda) \in [0,T] \times (0,\infty) \times (0,\infty).$$

Substituting (4.16) into (4.4) we arrive at

$$0 = \frac{x^\gamma}{\gamma}\frac{\partial \psi}{\partial t}(t,\lambda)$$
$$+ \sup_{c(t) \in [0,\infty)} \left\{ e^{-\rho t}\frac{(c(t))^\gamma}{\gamma} - x^{\gamma-1}\psi(t,\lambda)c(t) \right\} + x^\gamma \psi(t,\lambda)r(t)$$
$$(4.17) \qquad + \frac{x^\gamma}{\gamma}\psi(t,\lambda) \sup_{\pi(t) \in [0,1]} \left\{ \gamma\pi(t)(\mu(t) - r(t)) + \frac{1}{2}\gamma(\gamma-1)\pi^2(t)\sigma^2(t) \right.$$
$$\left. + \int_{z>-1} ((1+\pi(t)z)^\gamma - 1 - \gamma\pi(t)z)\nu_t(dz) \right\}$$
$$+ \frac{x^\gamma}{\gamma}\mathcal{L}^D\psi(t,\lambda) + \frac{x^\gamma}{\gamma}\lambda(\alpha(t)e^{-\rho t} - \psi(t,\lambda)).$$

This yields that $\tilde{\pi}(t)$ the optimal investment strategy at time t is determined by the point π which maximizes the concave function

$$(4.18) \qquad \begin{aligned} F(\pi,t) &= \gamma\pi(\mu(t) - r(t)) + \frac{1}{2}\gamma(\gamma-1)\pi^2\sigma^2(t) \\ &\quad + \int_{z>-1} ((1+\pi z)^\gamma - 1 - \gamma\pi z)\nu_t(dz). \end{aligned}$$

This function has a unique maximum in the interval $[0,1]$. The derived optimal investment strategy is the same as in a classical investment problem with a fixed investment time-horizon, see Choulli and Hurd [8], and it is not affected by the uncertainty over time of exiting the market. However, the optimal consumption rate is affected. This phenomena has been explained by Blanchet-Scalliet et al. [4]. The optimal consumption rate is given by

$$(4.19) \qquad \tilde{c}(t) = e^{-\frac{\rho}{1-\gamma}t}(\psi(t,\lambda))^{-\frac{1}{1-\gamma}}x,$$

where the function $\psi(t,\lambda)$ solves the reaction-diffusion partial differential equation

$$(4.20) \qquad \begin{aligned} 0 &= \frac{\partial \psi}{\partial t}(t,\lambda) + \mathcal{L}^D\psi(t,\lambda) + ((r(t) + F(\tilde{\pi}(t),t))\gamma - \lambda)\psi(t,\lambda) \\ &\quad + (1-\gamma)e^{-\frac{\rho}{1-\gamma}t}(\psi(t,\lambda))^{-\frac{\gamma}{1-\gamma}} + \alpha e^{-\rho t}\lambda, \quad \psi(T,\lambda) = \beta e^{-\rho T}. \end{aligned}$$

We prove, following Becherer and Schweizer [2] that the equation (4.20) has the unique solution of the class $\mathcal{C}^{1,2}([0,T] \times (0,\infty)) \cap \mathcal{C}([0,T] \times (0,\infty))$.

Lemma 4.1. *The function $F(\tilde{\pi}(t), t) : [0, T] \rightarrow \mathbb{R}$, defined in (4.18), is Lipschitz continuous.*

Proof. Let $0 \leq s < t \leq T$. Notice that $F(\tilde{\pi}(t), t) \geq F(\tilde{\pi}(s), t)$ and $F(\tilde{\pi}(s), s) \geq F(\tilde{\pi}(t), s)$. Then

$$F(\tilde{\pi}(s), t) - F(\tilde{\pi}(s), s) \leq F(\tilde{\pi}(t), t) - F(\tilde{\pi}(s), s) \leq F(\tilde{\pi}(t), t) - F(\tilde{\pi}(t), s).$$

We have

$$
\begin{aligned}
|F(\tilde{\pi}(s), t) - F(\tilde{\pi}(s), s)| &\leq \gamma\tilde{\pi}(s)|\mu(t) - \mu(s)| + \gamma\tilde{\pi}(s)|r(t) - r(s)| \\
&\quad + \frac{1}{2}\gamma(1 - \gamma)\tilde{\pi}^2(s)|\sigma^2(t) - \sigma^2(s)| \\
&\quad + \left| \int_{z>-1} \left((1 + \tilde{\pi}(s)z)^\gamma - 1 - \gamma\tilde{\pi}(s)z \right)(\nu_t(dz) - \nu_s(dz)) \right| \\
&\leq K|t - s|,
\end{aligned}
$$

which follows from the assumed Lipschitz continuity of $\mu(t)$, $r(t)$, $\sigma(t)$ in A1 and the condition A4 concerning the measure ν_t. Lipschitz continuity of the right hand side is proved analogously. $\qquad \square$

Lemma 4.2. *Let us define the operator $\mathcal{A}$ on functions φ by*

$$
\begin{aligned}
(4.21) \qquad (\mathcal{A}\varphi)(t, \lambda) = \mathbb{E}\Bigg[&\beta e^{-\rho T} e^{\int_t^T \left((r(s) + F(\tilde{\pi}(s), s))\gamma - \lambda(s) \right)ds} \\
&+ \int_t^T \Bigg((1 - \gamma)e^{-\frac{\rho}{1-\gamma}s}(\varphi(s, \lambda(s))^{-\frac{\gamma}{1-\gamma}} \\
&+ \alpha e^{-\rho s}\lambda(s) \Bigg) e^{\int_t^s \left((r(u) + F(\tilde{\pi}(u), u))\gamma - \lambda(u) \right)du} ds \Big| \lambda(t) = \lambda \Bigg].
\end{aligned}
$$

The operator $\mathcal{A}$ defines the mapping of continuous functions φ which are bounded away from zero and bounded from above into itself and is a contraction with respect to the norm

$$
(4.22) \qquad \|\varphi\| = \sup_{(t,\lambda)\in[0,T]\times(0,\infty)} e^{-\kappa(T-t)}|\varphi(t, \lambda)|,
$$

for large $\kappa < \infty$.

Proof. Notice that $F(\tilde{\pi}(t), t) \geq 0$ and

$$
\begin{aligned}
(4.23) \qquad \mathcal{A}\varphi(t, \lambda) &> \mathbb{E}\Big[\beta e^{-\rho T} e^{-\int_t^T \lambda(s)ds} + \alpha e^{-\rho T}\Big(1 - e^{-\int_t^T \lambda(s)ds} \Big) \Big| \lambda(t) = \lambda \Big] \\
&\geq (\alpha \wedge \beta)e^{-\rho T} > 0,
\end{aligned}
$$

for an arbitrary positive function φ bounded away from zero. We have an uniform lower bound, as well as uniform upper bound, and we can find κ as in Proposition 2.1 of Becherer and Schweizer [2], to which we refer the reader for details. $\qquad \square$

Theorem 4.2. *The reaction-diffusion partial differential equation* (4.20) *has the unique solution of class* $\mathcal{C}^{1,2}([0,T] \times (0,\infty)) \cap \mathcal{C}([0,T] \times (0,\infty))$, *which is bounded away from zero and bounded from above, given by the fixed point of the operator* $\mathcal{A}$ *from Lemma 4.2.*

Proof. The proof of Proposition 2.3 from Becherer and Schweizer [2] can also be applied in our setting. The difference is that our solution ψ is approximated by the sequence of bounded away from zero and bounded from above functions $(\varphi)_n = (\mathcal{A}\varphi)_{n-1}$ if we only choose φ_0 bounded away from zero and locally Hölder continuous in (t, λ). $\square$

4.3. Logarithmic utility functions. In this subsection we assume that the agent applies a logarithmic utility function of the form $u(x) = \log x$. We postulate that the optimal value function for the problem (4.3) is given by

(4.24) $V(t,x,\lambda) = f(t,\lambda)\log x + g(t,\lambda), \forall (t,x,\lambda) \in [0,T] \times (0,\infty) \times (0,\infty).$

Substituting (4.24) into (4.4) we arrive at

$$
0 = \log x \frac{\partial f}{\partial t}(t,\lambda) + \frac{\partial g}{\partial t}(t,\lambda) + \sup_{c(t)\in[0,\infty)} \left\{ e^{-\rho t} \log c(t) - f(t,\lambda)\frac{c(t)}{x} \right\}
$$

$$
+ f(t,\lambda)r(t) + f(t,\lambda) \sup_{\pi(t)\in[0,1]} \left\{ \pi(t)(\mu(t) - r(t)) - \frac{1}{2}\pi^2(t)\sigma^2(t) \right.
$$

(4.25)

$$
\left. + \int_{z>-1} \big(\log(1 + \pi(t)z) - \pi(t)z \big)\nu_t(dz) \right\}
$$

$$
+ \log x \mathcal{L}^D f(t,\lambda) + \mathcal{L}^D g(t,\lambda)
$$

$$
+ \lambda(\alpha e^{-\rho t}\log x - f(t,\lambda)\log x - g(t,\lambda)).
$$

This yields that $\tilde{\pi}(t)$ the optimal investment strategy at time t is determined by the point π which maximizes the concave function

(4.26)
$$
G(\pi,t) = \pi(t)(\mu(t) - r(t)) - \frac{1}{2}\pi^2(t)\sigma^2(t)
$$
$$
+ \int_{z>-1} \big(\log(1 + \pi(t)z) - \pi(t)z \big)\nu_t(dz).
$$

Again, it is not affected by the uncertainty over exit time. The function $G(\pi,t)$ has the unique maximum in the interval $[0,1]$, see Choulli and Hurd [8]. The optimal consumption rate is given by

(4.27) $$\tilde{c}(t) = e^{-\rho t}\frac{x}{f(t,\lambda)},$$

where the function $f(t,\lambda)$ solves the partial differential equation

(4.28) $\dfrac{\partial f}{\partial t}(t,\lambda) + \mathcal{L}^D f(t,\lambda) - \lambda f(t,\lambda) + e^{-\rho t}(\alpha\lambda + 1) = 0, \; f(T,\lambda) = \beta e^{-\rho T}$

Applying the Theorem 1 from Heath and Schweizer [12] we conclude that this equation has the unique solution of the class $\mathcal{C}^{1,2}([0,T] \times (0,\infty)) \cap \mathcal{C}([0,T] \times (0,\infty))$ which has the probabilistic representation given by Feynman–Kac formula

$$
(4.29) \quad
\begin{aligned}
f(t,\lambda) = \mathbb{E}\bigg[& \beta e^{-\rho T} e^{-\int_t^T \lambda(s)ds} \\
& + \int_t^T e^{-\rho s}(\alpha\lambda(s)+1)e^{-\int_t^s \lambda(u)du}ds \Big| \lambda(t) = \lambda \bigg].
\end{aligned}
$$

Notice that in the case of a logarithmic utility function the rate of wealth consumed by an agent does not depend on the financial market, in contrast to a power utility function. For both HARA utility functions the relative consumption rate is bounded from above which means that even in the case of increasing default intensities the agent is not consuming all his wealth. This is due to the inclusion of the bequest motive. Clearly, for the logarithmic utility we have

$$
(4.30) \quad
\begin{aligned}
f(t,\lambda) &> \mathbb{E}\Big[\beta e^{-\rho T} e^{-\int_t^T \lambda(s)ds} + \alpha e^{-\rho T}\Big(1 - e^{-\int_t^T \lambda(s)ds}\Big) \Big| \lambda(t) = \lambda \Big] \\
&\geq (\alpha \wedge \beta)e^{-\rho T} > 0.
\end{aligned}
$$

Notice that for $\alpha = \beta$ we have the following representation

$$
(4.31) \quad f(t,\lambda) = \mathbb{E}\bigg[\alpha e^{-\rho(\tau \wedge T)} + \int_t^{\tau \wedge T} e^{-\rho s}ds \Big| \lambda(t) = \lambda, \tau > t \bigg],
$$

which coincides with the result from Øksendal [1].

The function $g(t,\lambda)$ satisfies the partial differential equation

$$
(4.32) \quad
\begin{aligned}
& \frac{\partial g}{\partial t}(t,\lambda) + \mathcal{L}^D g(t,\lambda) - \lambda g(t,\lambda) + f(t,\lambda)(r(t) + G(\tilde{\pi}(t),t)) \\
& - e^{-\rho t}(\log f(t,\lambda) + \rho t + 1) = 0, \quad g(T,\lambda) = 0.
\end{aligned}
$$

One can show that the function $G(\tilde{\pi}(t),t)$ is Lipschitz continuous, in the same way as in Lemma 4.1, and that the function $f(t,\lambda)$ is locally Hölder continuous in (t,λ), see Becherer and Schweizer [2]. Applying again the Theorem 1 from Heath and Schweizer we conclude that the equation (4.32) has the unique solution of the class $\mathcal{C}^{1,2}([0,T] \times (0,\infty)) \cap \mathcal{C}([0,T] \times (0,\infty))$.

Finally, let us consider the agent's wealth process under the optimal investment/consumption strategy $(\tilde{X}(t), 0 \leq t \leq T)$. Its dynamic is given by the stochastic differential equation

$$
(4.33) \quad
\begin{aligned}
\frac{d\tilde{X}(t)}{\tilde{X}(t-)} = {}& \big(\tilde{\pi}(t)(\mu(t) - r(t)) + r(t) - \varrho(t,\lambda(t))\big)dt + \tilde{\pi}(t)\sigma(t)dW(t) \\
& + \tilde{\pi}(t)\int_{z>-1} z\tilde{M}(dt \times dz), \quad X(0) = x_0,
\end{aligned}
$$

where $\varrho(t, \lambda(t))$ is the relative consumption rate corresponding to a power or a logarithmic utility. The stochastic differential equation (4.33) has the solution given by Doléans–Dade exponential, see Applebaum [1]

$$
\begin{aligned}
\tilde{X}(t) = x_0 \exp\Bigg\{ & \int_0^t \Big(\tilde{\pi}(s)(\mu(s) - r(s)) + r(s) \\
& - \varrho(s, \lambda(s)) - \frac{1}{2}\tilde{\pi}^2(s)\sigma^2(s) \Big)ds + \int_0^t \tilde{\pi}(s)\sigma(s)dW(s) \\
& + \int_0^t \int_{z>-1} \log(1 + \tilde{\pi}(s)z)\tilde{M}(ds \times dz) \\
& + \int_0^t \int_{z>-1} \big(\log(1 + \tilde{\pi}(s)z) - \tilde{\pi}(s)z \big)\nu_s(dz)ds \Bigg\}.
\end{aligned}
$$

(4.34)

We left it to the reader to check that the assumptions of Theorem 4.1 are satisfied for power and logarithmic utility functions.

5. Insurance applications. In our setting the default time is independent of the financial market. This implies that the default in the financial market is due to the appearance of some market-independent event. It might be too strong assumption in many financial applications. However, our main motivation for this paper comes from life and pension insurance. In this case the default intensity process is the mortality intensity process of an insured person and the assumption of the independence of person's life-time and the financial market clearly holds true.

We refer the reader to Delong [10], where indifference pricing of a life policy is concerned with the presence of systematic mortality risk. This involves maximization of the expected utility of the insurer's wealth at the terminal time or at the moment of the death of the insured person, whichever occurs first. The classical solutions are found for exponential and quadratic utility functions.

In the next paper, we are going to deal with some pension problem, in which a pensioner solves an investment/consumption problem, applying subjective mortality intensities reflecting his/her current health status, whereas an insurer, when calculating the cost of annuity, uses objective mortality intensities based on its portfolio of pensioner. This is based on the results from this paper.

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