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ANNA WALCZUK

Central limit theorem for an additive functional of a Markov process, stable in the Wesserstein metric

Abstract. We study the question of the law of large numbers and central limit theorem for an additive functional of a Markov processes taking values in a Polish space that has Feller property under the assumption that the process is asymptotically contractive in the Wasserstein metric.

1. Introduction. In the present note we are concerned with the problem of proving the law of large numbers (LLN) and central limit theorem (CLT) for Markov processes $\{X_t,\ t \geq 0\}$ that take values in a Polish metric space $\mathcal{X}$. Our principal assumption is that the considered process is asymptotically contractive in the Wasserstein metric, i.e. that there exist constants $c, \gamma > 0$ such that $d(\mu P^t, \nu P^t) \leq c e^{-\gamma t} d(\mu, \nu)$ for all $\mu, \nu \in \mathcal{M}_1(\mathcal{X})$ and $t \geq 0$. Here $\mathcal{M}_1(\mathcal{X})$ denotes the set of all Borel, probability measures on $\mathcal{X}$, P^t is the dual to the transition probability operator, acting on such measures, $d(\cdot, \cdot)$ is the Wasserstein metric, see (2.1) below. The LLN and CLT we have in mind concern the case when the process is *out of the equilibrium*, i.e. we do not assume that the initial state of the process is invariant. This of course implies that the process under consideration *needs not be stationary.* The question of establishing the LLN and CLT for an additive functional

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of a Markov process is one of the most fundamental in probability theory and there exists a rich literature on the subject, see e.g. the monograph of Meyn and Tweedie [7] and the citations therein. However, in most of the existing results, see e.g. [6, 9, 8], it is usually assumed that the process under consideration is stationary and its equilibrium state μ_* is stable in some sense, usually in the L^2, or total variation norm. Our stability condition is formulated without invoking any reference measure and in a weaker metric than the total variation distance.

The organization of this note is as follows. In Section 2 we present some preliminaries concerning the basic notions appearing in the article. We shall also formulate our main result, see Theorem 2.1 below. Its proof is given in Section 3. It is based on the martingale approach of Kipnis and Varadhan, see [6].

2. Preliminaries and the statement of the main theorem. Let $(\mathcal{X},\rho)$ be a Polish metric space, $\mathfrak{B}$ denote its Borel σ-algebra. Let $B_b(\mathcal{X})$ be the space of bounded, Borel measurable functions. For a real-valued function f on $\mathcal{X}$, its Lipschitz seminorm is defined by $\|f\|_L := \sup_{x\neq y}|f(x)-f(y)|/\rho(x,y)$. Note that $\|f\|_L = 0$ if and only if f is constant. Let also $\|f\|_\infty := \sup_x |f(x)|$ and $\|f\|_{Lip} := \|f\|_L + \|f\|_\infty$. By $Lip_b(\mathcal{X})$ (resp. $C_b(\mathcal{X})$) we denote the space of bounded, Lipschitz continuous (resp. continuous) functions. Below, we recall basic notation related to Markov processes theory. An interested reader should consult [4] for details. Consider a Markov process $\{X_t, t\geq 0\}$ taking values in $\mathcal{X}$. We say that the Markov process is stochastically continuous at point s if $\lim_{t\to s}\mathbb{P}[|X_t - X_s|\geq\varepsilon] = 0$ for every $\varepsilon>0$. Denote by $\{P^t, t\geq 0\}$ the transition probability semi-group defined on $B_b(\mathcal{X})$. It is a semi-group of contractions under the supremum norm. We have then

$$\mathbb{E}[f(X_t)|\mathfrak{F}_s] = \big(P^{t-s}f\big)(X_s),\ \text{for } t\geq s,\ f\in B_b(\mathcal{X}).$$

Here $\mathfrak{F}_s = \sigma(X_h, h\leq s)$. We can also write $P^t f(x) := \int P(t,x,dy)f(y)$, where $P(t,x,\cdot)$, $t\geq 0$, $x\in\mathcal{X}$, are transition probability functions corresponding to the process. Let L be the generator of the semi-group and $\mathcal{D}(L)$ be its domain. Assume that $\mathbb{P}_\mu$ is the law of the Markov process X_t, $t\geq 0$ with the initial distribution μ on the appropriate path space and $\mathbb{E}_\mu$ the expectation with respect to $\mathbb{P}_\mu$. In the case when $\mu=\delta_x$ we use the notation $\mathbb{P}_x$, $\mathbb{E}_x$. We say that processes have the Feller property if for $f\in C_b(\mathcal{X})$ we have $P^t f\in C_b(\mathcal{X})$, $t\geq 0$. We denote $\langle\mu,f\rangle = \int_{\mathcal{X}} f(x)\mu(dx)$. Let $\mu P^t(A) := \int\mu(dx)P(t,x,A)$, $A\in\mathfrak{B}$. Notice that $\langle\mu,P^t f\rangle = \langle\mu P^t, f\rangle$. For any laws μ and ν on $\mathcal{X}$ define their Wasserstein's distance

$$d(\mu,\nu) := \sup_{\|\psi\|_{Lip}\leq 1}\left|\int\psi\,d\mu - \int\psi\,d\nu\right|, \tag{2.1}$$

see e.g. [2], p. 310. For any Polish metric space $(\mathcal{X},\rho)$, d metrizes the weak* topology on $\mathcal{M}_1(\mathcal{X})$ and the space $(\mathcal{M}_1(\mathcal{X}),d)$ is complete. This can be seen from the fact that $(\mathcal{M}_1(\mathcal{X}),\rho_1)$ is complete, where ρ_1 is the Levy–Prokhorov's metric (that also metrizes the topology of weak convergence of measures), see e.g. [1] p. 73, and the fact that $\rho_1(\mu,\nu)\le 2\sqrt{d(\mu,\nu)}$, see p. 311 of [2].

We say that μ_* is invariant if $\mu_*=\mu_*P^t$ or equivalently $\langle\mu_*,P^tf\rangle=\langle\mu_*,f\rangle$, for every $t\ge 0$. Now we state our main result.

Theorem 2.1. *In addition to the above, suppose that the following conditions are satisfied:*

(i) *there exist constants $c,\gamma>0$ such that:*

$$d\big(\mu_1P^t,\mu_2P^t\big)\le c\,e^{-\gamma t}\,d(\mu_1,\mu_2)\quad \text{for every } t\ge 0,\ \mu_1,\mu_2\in\mathcal{M}_1(\mathcal{X}), \tag{2.2}$$

(ii) $\psi\in Lip_b(\mathcal{X})$.

Then,

(i) *there is a unique invariant measure μ_* for the process $\{X_t,t\ge 0\}$,*

(ii) *the weak law of large numbers holds*

$$\frac{1}{T}\int_0^T\psi(X_s)ds\xrightarrow{T\to\infty}v_* \tag{2.3}$$

in $\mathbb{P}_\mu$ probability for any initial distribution μ, where $v_*:=\int\psi d\mu_*$,

(iii) *central limit theorem: there exists $\sigma>0$*

$$\mathbb{P}_\mu\left(\frac{\int_0^T\psi(X_s)ds-v_*T}{\sqrt{T}}<\xi\right)\xrightarrow{T\to\infty}\Phi_\sigma(\xi),\quad \xi\in\mathbb{R} \tag{2.4}$$

where $\Phi_\sigma(\xi)=(2\pi\sigma)^{-\frac{1}{2}}\int_{-\infty}^{\xi}e^{-y^2/2\sigma^2}dy$,

(iv) *let $D=\sigma^2$. Then,*

$$\frac{1}{T}\mathbb{E}_\mu\left(\int_0^T\psi(X_s)ds-v_*T\right)^2\xrightarrow{T\to\infty}D\ge 0. \tag{2.5}$$

Remark 2.2. From condition (2.2) it follows that for any $\psi\in C_b(\mathcal{X})$

$$\frac{1}{T}\int_0^T P^t\psi(x)\,dt\xrightarrow{T\to\infty}v_*\quad \text{for every } x\in\mathcal{X}. \tag{2.6}$$

Indeed, suppose that μ_* is the unique invariant measure and $\psi\in C_b(\mathcal{X})$. We have

$$\sup_{\|\psi\|_{Lip}\le 1}\left|\int\psi(y)\delta_xP^t(dy)-\int\psi(y)\mu_*P^t(dy)\right|\le ce^{-\gamma t}d(\delta_x,\mu_*).$$

From weak convergence we have the weak convergence of ergodic averages, so

$$\sup_{\|\psi\|_{Lip}\le 1}\left|\frac{1}{T}\int_0^T P^t\psi(x)dt-v_*\right|\le ce^{-\gamma t}d(\delta_x,\mu_*).$$

If $T \to \infty$ we obtain (2.6).

Remark 2.3. From condition (2.6) it follows that the invariant measure, μ_*, if exists, is unique.

Indeed, assume that there exists another invariant measure ν_*. Then uniqueness can be concluded from (2.2) because

$$d(\nu_*, \mu_*) = d\big(\nu_* P^t, \mu_* P^t\big) \leq 2ce^{-\gamma t}.$$

Taking $t \to \infty$ we get the result.

3. The proof of Theorem 2.1. We take t_0, such that $ce^{-\gamma t_0} < 1$. Then from (2.2) P^{t_0} is contraction in metric d. The space $\mathcal{M}_1(\mathcal{X})$ with Wesserstein metric d is complete, so from the Banach contraction principle there exists μ_*^0 such that $\mu_*^0 P^{t_0} = \mu_*^0$. Let $\mu_* := t_0^{-1} \int_0^{t_0} \mu_*^0 P^s ds$. Notice that μ_* is invariant for every $t \geq 0$, so part (i) is proved.

Let $v(T) := \int_0^T \psi(X_s)ds$. In order to prove (ii) part of theorem it suffices to show that: $\mathbb{E}_\mu v(T)/T \xrightarrow{T\to\infty} v_*$ and $\mathbb{E}_\mu (v(T)/T)^2 \xrightarrow{T\to\infty} v_*^2$. Then by of Chebyshev's inequality we obtain the result. As X_s is a Markov process:

$$\begin{aligned}\mathbb{E}_\mu \frac{v(T)}{T} &= \mathbb{E}_\mu \frac{1}{T}\int_0^T \psi(X_s)ds = \frac{1}{T}\int_0^T \int P^s\psi(x)\mu(dx)ds \\ &= \int \frac{1}{T}\int_0^T P^s\psi(x)ds\mu(dx) \xrightarrow{T\to\infty} \int \psi d\mu_* = v_*.\end{aligned}$$

By symmetry we have

$$\begin{aligned}\mathbb{E}_\mu\left(\frac{v(T)}{T}\right)^2 &= \frac{1}{T^2}\,\mathbb{E}_\mu\left(\int_0^T \psi(X_t)dt \int_0^T \psi(X_s)ds\right) \\ &= \frac{2}{T^2}\,\mathbb{E}_\mu \int_0^T dt \int_0^t ds\, \psi(X_t)\psi(X_s).\end{aligned}$$

Using Markov property we obtain that the right hand side equals

$$\begin{aligned}&\frac{2}{T^2}\int_0^T dt \int_0^t ds\, \mathbb{E}_\mu\big(\psi(X_s)P^{t-s}\psi(X_s)\big) \\ &\qquad = \frac{2}{T^2}\int_0^T dt \int_0^t ds \int \mathbb{E}_x\big(\psi(X_s)P^{t-s}\psi(X_s)\big)\mu(dx) \\ &\qquad = \frac{2}{T^2}\int_0^T dt \int_0^t ds \int \big(P^s(\psi P^{t-s}\psi)\big)(x)\mu(dx).\end{aligned}$$

In order to finish the first part of the proof we need the following.

Lemma 3.1. *For every $\varepsilon > 0$ and compact $K \subset \mathcal{X}$ there exists t_0 such that for every $t \geq t_0$*

$$\sup_{x\in K}\left|\frac{1}{t}\int_0^t P^s\psi(x)ds - v_*\right| < \varepsilon. \tag{3.1}$$

Proof. Note that $\{P^s\psi, s \geq 0\}$ is uniformly continuous. Indeed, from equality $\langle \delta_x, P^s\psi\rangle = \langle \delta_x P^s, \psi\rangle$ and assumptions of Theorem 2.1 we have:

$$\begin{aligned}
\big|P^s\psi(x_1) - P^s\psi(x_2)\big| &= \left|\int P^s\psi(y)\delta_{x_1}(dy) - \int P^s\psi(y)\delta_{x_2}(dy)\right| \\
&= \left|\int \psi(y)\delta_{x_1}P^s(dy) - \int \psi(y)\delta_{x_2}P^s(dy)\right| \\
&\leq d\big(\delta_{x_1}P^s, \delta_{x_2}P^s\big)\|\psi\|_{Lip} \\
&\leq e^{-\gamma s} d(\delta_{x_1}, \delta_{x_2})\|\psi\|_{Lip} \\
&\leq e^{-\gamma s}\rho(x_1, x_2)\|\psi\|_{Lip}.
\end{aligned}$$

Suppose now that $t_n \to +\infty$. The above shows that $\{(1/t_n)\int_0^{t_n} P^s\psi(x)ds, n \geq 1\}$ is a sequence of functions uniformly continuous on K. It is also bounded. The result follows then from Arzela–Ascoli theorem, see [1], and assumption (i) Theorem 2.1. □

Because

$$\frac{1}{T}\int_0^T P^t\psi(x)\,dt = \frac{1}{T}\int_0^T \int \psi(y)\delta_x P^t(dy)\,dt \xrightarrow{T\to\infty} \int \psi d\mu_*$$

we get that $\{T^{-1}\int_0^T \mu P^t\,dt, T \geq 0\}$ converges weakly to μ_*, as $T \to \infty$. Then, the above family of measures is relatively compact and by Prokhorov theorem it is tight, see [1]. Using tightness of $\big\{T^{-1}\int_0^T \mu P^t\,dt, T \geq 0\big\}$, for every $\varepsilon > 0$ one can find K compact such that

$$\frac{1}{T}\int_0^T \mu P^t(K^c)\,dt < \varepsilon \quad \text{for every } T \geq 0. \tag{3.2}$$

Suppose that we know that

$$\left|\frac{2}{T^2}\int_0^T dt\int_0^t ds\int P^s(\psi(P^{t-s}\psi - \upsilon_*))d\mu\right| < \varepsilon \tag{3.3}$$

then

$$\begin{aligned}
\lim_{T\to\infty}\mathbb{E}_\mu\left(\frac{v(T)}{T}\right)^2 &= \lim_{T\to\infty}\frac{2}{T^2}\int_0^T dt\int_0^t ds\int P^s\big(\psi P^{t-s}\psi\big)d\mu \\
&= \lim_{T\to\infty}\frac{2}{T^2}\int_0^T dt\int_0^t ds\int P^s(\psi\upsilon_*)\,d\mu \\
&= \lim_{T\to\infty}\frac{2}{T^2}\upsilon_*\int_0^T t\,dt\int\frac{1}{t}\int_0^t P^s\psi\,ds\,d\mu \to \upsilon_*^2.
\end{aligned}$$

Now we prove inequality (3.3). Note that

$$
\begin{aligned}
&\left|\frac{2}{T^2}\int_0^T dt\int_0^t ds\int P^s\big(\psi\big(P^{t-s}\psi-\upsilon_*\big)\big)d\mu\right|\\
&\quad\le\left|\frac{2}{T^2}\int_0^T dt\int_0^t ds\int P^s\big(\psi\big(P^{t-s}\psi-\upsilon_*\big)\mathbf{1}_K\big)d\mu\right|\\
&\qquad+\left|\frac{2}{T^2}\int_0^T dt\int_0^t ds\int P^s\big(\psi\big(P^{t-s}\psi-\upsilon_*\big)\mathbf{1}_{K^c}\big)d\mu\right|.
\end{aligned}
$$

Denote the first and second terms on the right hand side above by I and II respectively. Note that from contractivity of operator P^s on $B_b(\mathfrak{X})$ and inequality (3.1) we have:

$$
\begin{aligned}
I&=\left|\frac{2}{T^2}\int_0^T dt\int_0^t ds\int P^s\big(\psi\big(P^{t-s}\psi-\upsilon_*\big)\mathbf{1}_K\big)d\mu\right|\\
&=\left|\frac{2}{T^2}\int_0^T (T-s)ds\int P^s\left(\psi\left(\frac{1}{T-s}\int_0^{T-s}P^t\psi dt-\upsilon_*\right)\mathbf{1}_K\right)d\mu\right|\\
&\le\frac{2}{T^2}\int_0^T (T-s)ds\|\psi\|_\infty\left\|\left(\frac{1}{T-s}\int_0^{T-s}P^t\psi dt-\upsilon_*\right)\mathbf{1}_K\right\|_\infty\\
&<\frac{2\varepsilon}{T^2}\|\psi\|_\infty\int_0^T s\,ds\\
&=\varepsilon\|\psi\|_\infty.
\end{aligned}
$$

Next from contractivity of P^{t-s} and inequality (3.2):

$$
\begin{aligned}
II&=\left|\frac{2}{T^2}\int_0^T dt\int_0^t ds\int \psi\big(P^{t-s}\psi-\upsilon_*\big)\mathbf{1}_{K^c}\mu P^s(dx)\right|\\
&\le\frac{2}{T^2}\int_0^T\int_0^t\int\left(2\|\psi\|_\infty^2\right)\mathbf{1}_{K^c}\,\mu P^s dx\,ds\,dt\\
&=\frac{4}{T^2}\|\psi\|_\infty^2\int_0^T t\left(\frac{1}{t}\int_0^t\mu P^s(K^c)ds\right)dt\\
&<2\varepsilon\|\psi\|_\infty^2.
\end{aligned}
$$

Hence, we obtain expression (3.3) which completes the part (ii) of the proof.

We will prove now parts (iii) and (iv) of the theorem. We need the following lemma.

Lemma 3.2. *Let* $\widetilde{\psi}:=\psi-\int\psi\,d\mu_*$*. The integral* $\chi:=\int_0^\infty P^s\widetilde{\psi}\,ds$ *converges in* $C_b(\mathfrak{X})$*.*

Proof. We show that the sequence $\left\{ \int_0^{t_n} P^s\widetilde{\psi}(x)\,ds,\ n \geq 1\right\}$ satisfies Cauchy condition, as $t_n \to \infty$. For $t_m > t_n$ we have

$$\begin{aligned}
&\left|\int_0^{t_m} P^s\widetilde{\psi}(x)\,ds - \int_0^{t_n} P^s\widetilde{\psi}(x)\,ds\right| = \left|\int_{t_n}^{t_m} P^s\widetilde{\psi}(x)\,ds\right| \\
&\quad = \left|\int_{t_n}^{t_m}\left(P^s\psi(x) - \int P^s\psi(y)\mu_*(dy)\right)ds\right| \\
&\quad \leq \int_{t_n}^{t_m}\left|\int P^s\psi(y)\delta_x(dy) - \int P^s\psi(y)\mu_*(dy)\right| ds \\
&\quad \leq \int_{t_n}^{t_m} \sup_{\|\psi\|_{Lip}\leq 1}\left|\int \psi(y)\delta_x P^s(dy) - \int \psi(y)\mu_* P^s(dy)\right| ds \\
&\quad = \int_{t_n}^{t_m} \|\psi\|_{Lip}\, d(\delta_x P^s, \mu_* P^s)\,ds \leq \|\psi\|_{Lip}\int_{t_n}^{t_m} c\,e^{-\gamma s} d(\delta_x,\mu_*)\,ds \\
&\quad \leq 2c\,\|\psi\|_{Lip}\frac{e^{-\gamma t_n}}{\gamma}.
\end{aligned}$$

So the sequence satisfies Cauchy condition, thus it converges in $C_b(\mathcal{X})$. □

Let $\chi_T = \int_0^T P^t\widetilde{\psi}\,dt$. Next, we note that $\chi \in D(L)$ for

$$L\chi_T = L\int_0^T P^t\widetilde{\psi}\,dt = \int_0^T LP^t\widetilde{\psi}\,dt = P^T\widetilde{\psi} - \widetilde{\psi}.$$

Indeed, because $\chi = \lim_{T\to\infty}\chi_T$ in $C_b(\mathcal{X})$ we have

$$\lim_{T\to\infty} L\chi_T = \lim_{T\to\infty} P^T\widetilde{\psi} - \widetilde{\psi} = -\widetilde{\psi}$$

in $C_b(\mathcal{X})$. We have

$$L\chi = L\lim_{T\to\infty}\chi_T = \lim_{T\to\infty} L\chi_T = -\widetilde{\psi}.$$

We show the central limit theorem for $\{t^{-1/2}\int_0^t(\psi(X_s) - v_*)ds\}$, as $t \to +\infty$. With no loss of generality we assume that $v_* := \int \psi d\mu_* = 0$, otherwise take $\widetilde{\psi} := \psi - v_*$.

$$\begin{aligned}
&\frac{\int_0^T \psi(X_s)\,ds - Tv_*}{\sqrt{T}} = \frac{\int_0^T (\psi(X_s) - v_*)\,ds}{\sqrt{T}} \\
&\quad = \frac{\int_0^T \widetilde{\psi}(X_s)\,ds}{\sqrt{T}} = -\frac{1}{\sqrt{T}}\int_0^T L\chi(X_s)\,ds \\
&\quad = -\frac{1}{\sqrt{T}}\left\{\int_0^T L\chi(X_s)\,ds - \chi(X_T) + \chi(X_0)\right\} + \frac{1}{\sqrt{T}}\big(\chi(X_0) - \chi(X_T)\big) \\
&\quad = \frac{M_T + \big(\chi(X_0) - \chi(X_T)\big)}{\sqrt{T}}
\end{aligned}$$

where

$$M_T = \chi(X_T) - \chi(x) - \int_0^T L\chi(X_s)\,ds.$$

So it suffices to verify the central limit theorem for $M_T/\sqrt{T}$. Note that $\{M_T, T \geq 0\}$ is a square integrable martingale and

$$\frac{1}{N}\sum_{n=0}^{N-1} \mathbb{E}_x\big((M_{n+1} - M_n)^2 | \mathfrak{F}_n\big) \xrightarrow[\mathbb{P}]{N\to\infty} -2\int L\chi\,\chi\,d\mu_*. \tag{3.4}$$

The central limit theorem is then a consequence of a version of Billingsley's central limit theorem for martingale increments, see e.g. [5]. Because we could not find the formulation of the result in the precise form we need, we provide its proof in the appendix. Note that

$$\mathbb{E}_x(M_{n+1} - M_n)^2 = \mathbb{E}_x\left[\chi(X_{n+1}) - \chi(X_n) - \int_n^{n+1} L\chi(X_s)\,ds\right]^2 = P^n F(x),$$

where

$$F(x) := P^0 F(x) = \mathbb{E}_x\left[\chi(X_1) - \chi(x) - \int_0^1 L\chi(X_s)\,ds\right]^2.$$

Hence, by the Markov property

$$\begin{aligned}
&\mathbb{E}_x\left[\frac{1}{N}\sum_{n=0}^{N-1} \mathbb{E}_x\big((M_{n+1} - M_n)^2 | \mathfrak{F}_n\big)\right] = \frac{1}{N}\sum_{n=0}^{N-1} \mathbb{E}_x\mathbb{E}_x(M_{n+1} - M_n)^2 \\
&= \frac{1}{N}\sum_{n=0}^{N-1} P^n F(x) = \frac{1}{N}\sum_{n=0}^{N-1} \int P^n F(y)\delta_x(dy) \qquad (3.5)\\
&= \frac{1}{N}\sum_{n=0}^{N-1} \int F(y)\delta_x P^n(dy) = \int F(y)\frac{1}{N}\sum_{n=0}^{N-1} \delta_x P^n(dy) \to \int F\,d\mu_*.
\end{aligned}$$

The final limit holds due to $\delta_x P^n \xrightarrow{n\to\infty} \mu_*$. Next, by the Markov property

$$\begin{aligned}
&\mathbb{E}_x\left[\frac{1}{N}\sum_{n=0}^{N-1} \mathbb{E}_x((M_{n+1} - M_n)^2 | \mathfrak{F}_n)\right]^2 \\
&= \mathbb{E}_x\left[\frac{1}{N^2}\sum_{n=0}^{N-1} F^2(X_n)\right] + \mathbb{E}_x\left\{\frac{2}{N^2}\sum_{n=1}^{N-1}\Big[\sum_{m=0}^{n-2} F(X_m)\Big]F(X_n)\right\} \\
&= \frac{2}{N^2}\sum_{n=1}^{N-1}\sum_{m<n} \mathbb{E}_x\big[F(X_m)P^{n-m}F(X_m)\big] + O\left(\frac{1}{N}\right) \\
&= \frac{2}{N^2}\sum_{m=0}^{N-2}\sum_{n=m+1}^{N-1} P^m(FP^{n-m}F)(x) + O\left(\frac{1}{N}\right) \xrightarrow{N\to\infty} \left(\int F\,d\mu_*\right)^2.
\end{aligned}$$

The last formulas and (3.5) together imply (3.4), which ends the proof of part (iii) of the theorem.

As for part (iv), we can write

$$\frac{1}{T}\mathbb{E}_\mu\Big(\int_0^T \widetilde{\psi}\, ds\Big)^2 = \frac{2}{T}\int_0^T dt \int_0^t ds \int P^s\Big(\widetilde{\psi} P^{t-s}\widetilde{\psi}\Big)\, d\mu$$
$$= \frac{2}{T}\int_0^T ds \int P^s\Big(\widetilde{\psi}\int_0^{T-s} P^t\widetilde{\psi}\, dt\Big)\, d\mu.$$

Note that:

$$\Big|\frac{2}{T}\int_0^T ds \int P^s\Big(\widetilde{\psi}\int_0^{T-s} P^t\widetilde{\psi}\, dt\Big)\, d\mu - \frac{2}{T}\int_0^T ds \int P^s\Big(\widetilde{\psi}\int_0^{\infty} P^t\widetilde{\psi}\, dt\Big)\, d\mu\Big|$$
$$= \frac{2}{T}\Big|\int_0^T ds \int P^s\Big(\widetilde{\psi}\int_{T-s}^{\infty} P^t\widetilde{\psi}\, dt\Big)\, d\mu\Big|$$
$$\le \frac{4\|\psi\|_{Lip}\|\widetilde{\psi}\|_\infty}{T\gamma}\int_0^T e^{-\gamma(T-s)} ds \xrightarrow{T\to\infty} 0.$$

Hence we obtain

$$\lim_{T\to\infty}\frac{1}{T}\mathbb{E}_\mu\Big(\int_0^T \widetilde{\psi}\, ds\Big)^2 = \lim_{T\to\infty}\frac{2}{T}\int_0^T \int P^s\Big(\widetilde{\psi}\chi\Big)\, d\mu\, ds$$
$$= \lim_{T\to\infty} 2\frac{1}{T}\int_0^T P^s\Big(\int\Big(\widetilde{\psi}\chi\Big)\, d\mu\Big)\, ds \xrightarrow{T\to\infty} \int 2\int\Big(\widetilde{\psi}\chi\Big)\, d\mu\, d\mu_*$$
$$= 2\int\Big(\widetilde{\psi}\chi\Big)\, d\mu_* = D$$

and part (iv) is proved. □

Appendix. Central limit theorem for martingales. We prove here a version of the central limit theorem for martingales. This is obtained by well known methods, see e.g. [5]. We present it for the convenience of a reader.

Theorem A.1. *Let $\{Z_j, j \ge 0\}$ be a sequence of bounded random variables adapted with respect to a filtration $\{\mathcal{F}_j, j \ge 0\}$. Assume that $\mathbb{E}_x[Z_j|\mathcal{F}_{j-1}] = 0$ for $j \ge 1$ and that*

$$\frac{1}{N}\sum_{1\le j\le N}\mathbb{E}_x\big[Z_j^2|\mathcal{F}_{j-1}\big] \to \sigma^2,$$

as $N \to +\infty$ in $\mathbb{P}_x$ probability. Then, for all x in $\mathcal{X}$, as $N \uparrow \infty$,

$$\frac{M_N}{\sqrt{N}} = \frac{1}{\sqrt{N}}\sum_{j=1}^N Z_j$$

converges in $\mathbb{P}_x$ distribution to a mean zero Gaussian random variable with variance σ^2.

Proof. For $|\theta|$ small enough, so $|\theta Z_j| < \pi$, we may define

$$A_j(\theta) \coloneqq \log \mathbb{E}_x[\exp\{i\theta Z_j\}|\mathcal{F}_{j-1}].$$

Fix $\theta \in \mathbb{R}$. An elementary computation shows that for all N large enough (to make sure that $\left|(\theta Z_i)/\sqrt{N}\right| < \pi$),

$$\mathbb{E}_x\Big[\exp\Big\{\big(i\theta/\sqrt{N}\big)M_N - \sum_{j=1}^{N} A_j\big(\theta/\sqrt{N}\big)\Big\}\Big] = 1.$$

It follows from the second order Taylor expansion (taking into account that $\mathbb{E}_x[Z_j|\mathcal{F}_{j-1}] = 0$) that

$$\sum_{j=1}^{N} A_j\big(\theta/\sqrt{N}\big) = -\frac{\theta^2}{2N}\sum_{j=1}^{N}\mathbb{E}_x\big[Z_j^2|\mathcal{F}_{j-1}\big] + \frac{1}{\sqrt{N}}R_N,$$

for some random variable R_N bounded from above by a constant. Since

$$\frac{1}{N}\sum_{1\le j\le N}\mathbb{E}_x\big[Z_j^2|\mathcal{F}_{j-1}\big] \xrightarrow[\mathbb{P}]{N\to\infty} \sigma^2,$$

$\sum_{1\le j\le N} A_j\big(\theta/\sqrt{N}\big)$ converges $\mathbb{P}_x$-a.s. to $\big(-\theta^2\sigma^2\big)/2$. In particular,

$$\lim_{N\to\infty}\mathbb{E}_x\big[\exp\big(i\theta/\sqrt{N}\big)M_N\big] = e^{-\theta^2\sigma^2/2}$$

and this ends the proof of the theorem. □

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Anna Walczuk
Institute of Mathematics
M. Curie-Skłodowska University
pl. Marii Curie-Skłodowskiej 1
20-031 Lublin, Poland
e-mail: walczuk.anna@gmail.com

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