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UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

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SI DUC QUANG and TRAN VAN TAN

Uniqueness problem of meromorphic mappings with few targets

Abstract. In this paper, using techniques of value distribution theory, we give some uniqueness theorems for meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{C}P^n$.

1. Introduction. Using the Second Main Theorem of Value Distribution Theory and Borel's lemma, R. Nevanlinna [11] proved that for two nonconstant meromorphic functions f and g on the complex plane $\mathbf{C}$, if they have the same inverse images for five distinct values, then $f \equiv g$, and that g is a special type of linear fractional transformation of f if they have the same inverse images, counted with multiplicities, for four distinct values.

In 1975, H. Fujimoto [5] generalized Nevanlinna's result to the case of meromorphic mappings of $\mathbf{C}$ into $\mathbf{C}P^n$. He showed that for two linearly nondegenerate meromorphic mappings f and g of $\mathbf{C}$ into $\mathbf{C}P^n$, if they have the same inverse images, counted with multiplicities for $(3n+2)$ hyperplanes in $\mathbf{C}P^n$ located in general position, then $f \equiv g$, and there exists a projective linear transformation L of $\mathbf{C}P^n$ to itself such that $g = L \cdot f$ if they have the same inverse images counted with multiplicities for $(3n+1)$ hyperplanes in $\mathbf{C}P^n$ located in general position. Since that time, this problem has been studied intensively for the case of hyperplanes by H. Fujimoto ([7], [8]),

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W. Stoll [17], L. Smiley [14], S. Ji [9], M. Ru [13], Z. Ye [20], G. Dethloff–T. V. Tan ([2], [3], [4]), D. D. Thai–S. D. Quang [15] and others.

Let f be a linearly nondegenerate meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{C}P^n$. For each hyperplane H, we denote by $\nu_{(f,H)}$ the map of $\mathbf{C}^m$ into $\mathbf{N}_0$ whose value $\nu_{(f,H)}(a)$ $(a \in \mathbf{C}^m)$ is the intersection multiplicity of the image of f and H at $f(a)$.

Take q hyperplanes $H_1, \dots, H_q$ in $\mathbf{C}P^n$ located in general position with

a) $\dim\big(f^{-1}(H_i) \cap f^{-1}(H_j)\big) \leq m-2$ for all $1 \leq i < j \leq q$.

For each positive integer (or $+\infty$) M, denote by $\mathcal{G}\big(\{H_j\}_{j=1}^q, f, M\big)$ the set of all linearly nondegenerate meromorphic mappings g of $\mathbf{C}^m$ into $\mathbf{C}P^n$ such that

b) $\min\{\nu_{(g,H_j)}, M\} = \min\{\nu_{(f,H_j)}, M\}$, $j \in \{1, \dots, q\}$ and

c) $g = f$ on $\bigcup_{j=1}^q f^{-1}(H_j)$.

In 1983, L. Smiley [14] showed that:

Theorem A. *If $q \geq 3n+2$ then $g_1 = g_2$ for any $g_1, g_2 \in \mathcal{G}\big(\{H_j\}_{j=1}^q, f, 1\big)$.*

In 1998, H. Fujimoto [7] obtained the following theorem:

Theorem B. *If $q \geq 3n+1$ then $\mathcal{G}\big(\{H_j\}_{j=1}^q, f, 2\big)$ contains at most two mappings.*

He also gave the open question: Does his result remain valid if the number of hyperplanes is replaced by a smaller one? In 2006, G. Dethloff and T. V. Tan [4] showed that the above result of Fujimoto remains valid if $q \geq 3n-1$, $n \geq 7$. In this paper, by a different approach, we extend Theorem B to the case of

$$q > \max\bigg\{\frac{7(n+1)}{4}, \frac{\sqrt{17n^2+16n}+3n+4}{4}\bigg\}.$$

In 1980, W. Stoll [19] obtained the following theorem:

Theorem C. *Let $f_1, \dots, f_k$ $(k \geq 2)$ be linearly nondegenerate holomorphic mappings of $\mathbf{C}$ into $\mathbf{C}P^n$. Let $H_1, \dots, H_q$ $(q \geq (k+1)n+2)$ be hyperplanes in $\mathbf{C}P^n$ located in general position. Assume that*

i) *$f_1^{-1}(H_j) = \cdots = f_k^{-1}(H_j)$ for all $j \in \{1, \dots, q\}$,*

ii) *$f_1^{-1}(H_i) \cap f_1^{-1}(H_j) = \emptyset$ for all $1 \leq i < j \leq q$ and*

iii) *$f_1 \wedge \cdots \wedge f_k = 0$ on $\bigcup_{j=1}^q f_1^{-1}(H_j)$.*

Then $f_1 \wedge \cdots \wedge f_k \equiv 0$.

In 2001, M. Ru [13] generalized the above result to the case of moving hyperplanes. In the last part of this paper, we extend Theorem C to the case of moving hypersurfaces.

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2. Preliminaries. For $z = (z_1, \dots, z_m) \in \mathbf{C}^m$, we set

$$\|z\| = \Big(\sum_{j=1}^{m} |z_j|^2\Big)^{1/2}$$

and define

$$B(r) = \{z \in \mathbf{C}^m : \|z\| < r\}, \quad S(r) = \{z \in \mathbf{C}^m : \|z\| = r\},$$

$$d^c = \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial),\ \mathcal{V} = \big(dd^c\|z\|^2\big)^{m-1},\ \sigma = d^c \log \|z\|^2 \wedge \big(dd^c \log \|z\|\big)^{m-1}.$$

Let F be a nonzero holomorphic function on $\mathbf{C}^m$. For a set $\alpha = (\alpha_1, \dots, \alpha_m)$ of nonnegative integers, we set $|\alpha| = \alpha_1 + \dots + \alpha_m$ and $\mathcal{D}^\alpha F = \frac{\mathcal{D}^{|\alpha|}F}{\partial^{\alpha_1} z_1 \dots \partial^{\alpha_m} z_m}$. We define the map $\nu_F : \mathbf{C}^m \to \mathbf{N}_0$ by

$$\nu_F(a) = \max\{p : \mathcal{D}^\alpha F(a) = 0 \text{ for all } \alpha \text{ with } |\alpha| < p\}.$$

Let φ be a nonzero meromorphic function on $\mathbf{C}^m$. For each $a \in \mathbf{C}^m$, we choose nonzero holomorphic functions F and G on a neighborhood U of a such that $\varphi = \frac{F}{G}$ on U and $\dim\big(F^{-1}(0) \cap G^{-1}(0)\big) \le m - 2$ and we define the map $\nu_\varphi : \mathbf{C}^m \longrightarrow \mathbf{N}_0$ by $\nu_\varphi(a) = \nu_F(a)$. Set

$$|\nu_\varphi| = \overline{\{z : \nu_\varphi(z) \neq 0\}}.$$

Let k be positive integer or $+\infty$. Set $\nu_\varphi^{(k)}(z) = \min\{\nu_\varphi(z), k\}$, and

$$N_\varphi^{(k)}(r) := \int_1^r \frac{n^{(k)}(t)}{t^{2m-1}} dt \quad (1 < r < +\infty)$$

where

$$n^{(k)}(t) = \int_{|\nu_\varphi| \cap B(t)} \nu_\varphi^{(k)} \cdot \mathcal{V} \ \text{ for } m \ge 2$$

and

$$n^{(k)}(t) = \sum_{|z| \le t} \nu_\varphi^{(k)}(z) \ \text{ for } m = 1.$$

We simply write $N_\varphi(r)$ for $N_\varphi^{(+\infty)}(r)$. We have the following Jensen's formula:

$$N_\varphi(r) - N_{\frac{1}{\varphi}}(r) = \int_{S(r)} \log|\varphi|\sigma - \int_{S(1)} \log|\varphi|\sigma.$$

Let f be a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{C}P^n$. For arbitrary fixed homogeneous coordinates $(w_0 : \cdots : w_n)$ of $\mathbf{C}P^n$, we take a reduced representation $f = (f_0 : \cdots : f_n)$ which means that each f_i is holomorphic function on $\mathbf{C}^m$ and $f(z) = (f_0(z) : \cdots : f_n(z))$ outside the analytic $I(f) := \{z : f_0(z) = \cdots = f_n(z) = 0\}$ of codimension ≥ 2. Set $\|f\| = \max\{|f_0|, \ldots, |f_n|\}$.

The characteristic function of f is defined by

$$T_f(r) := \int\limits_{S(r)} \log \|f\| \sigma - \int\limits_{S(1)} \log \|f\| \sigma, \quad 1 < r < +\infty.$$

For a meromorphic function φ on $\mathbf{C}^m$, the characteristic function $T_\varphi(r)$ of φ is defined as φ is a meromorphic map of $\mathbf{C}^m$ into $\mathbf{C}P^1$. The proximity function $m(r, \varphi)$ is defined by

$$m(r, \varphi) = \int\limits_{S(r)} \log^+ |\varphi| \sigma,$$

where $\log^+ x = \max\{\log x, 0\}$ for $x \geq 0$.

Then

$$T_\varphi(r) = N_{\frac{1}{\varphi}}(r) + m(r, \varphi) + O(1).$$

We state the First and the Second Main Theorems of Value Distribution Theory:

Let f be a nonconstant meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{C}P^n$. We say that a meromorphic function φ on $\mathbf{C}^m$ is "small" with respect to f if $T_\varphi(r) = o(T_f(r))$ as $r \to \infty$ (outside a set of finite Lebesgues measure). Denote by $\mathcal{R}_f$ the field of all "small" (with respect to f) meromorphic functions on $\mathbf{C}^m$.

Theorem D (First Main Theorem). *Let f be a nonconstant meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{C}P^n$ and Q be a homogeneous polynomial of degree d in $\mathcal{R}_f[x_0, \ldots, x_n]$ such that $Q(f) \not\equiv 0$ then*

$$N_{Q(f)}(r) \leq d \cdot T_f(r) + o(T_f(r)) \quad \textit{for all } r > 1.$$

For a hyperplane $H : a_0 w_0 + \cdots + a_n w_n = 0$ in $\mathbf{C}P^n$ with $im f \not\subseteq H$, we denote $(f, H) := a_0 f_0 + \cdots + a_n f_n$, where $(f_0 : \cdots : f_n)$ again is a reduced representation of f.

As usual, by the notation "|| P" we mean the assertion P holds for all $r \in (1, +\infty)$ excluding a subset E of $(1, +\infty)$ of finite Lebesgue measure.

Theorem E (Second Main Theorem). *Let f be a linearly nondegenerate meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{C}P^n$ and $H_1, \ldots, H_q$ $(q \geq n+1)$ hyperplanes in $\mathbf{C}P^n$ located in general position, then*

$$|| \ (q - n - 1) T_f(r) \leq \sum_{j=1}^{q} N^{(n)}_{(f, H_j)}(r) + o\big(T_f(r)\big).$$

3. Uniqueness problem for hyperplanes. First of all, we give the following lemma, which is an extension of uniqueness theorem to the case of few hyperplanes.

Lemma 1. *Let* $f, g : \mathbf{C}^m \to \mathbf{C}P^n$ *be two linearly nondegenerate meromorphic mappings with reduced representations* $f = (f_0 : \cdots : f_n)$, $g = (g_0 : \cdots : g_n)$. *Let* $\{H_i\}_{i=1}^q$ *be q hyperplanes located in general position with* $\dim\big(f^{-1}(H_i) \cap f^{-1}(H_j)\big) \leq m-2$ *for all* $1 \leq i < j \leq q$. *Assume that*

$$q > \frac{\sqrt{17n^2+16n}+3n+4}{4}$$

and

(i) $\min\{\nu_{(f,H_i)}(z), n\} = \min\{\nu_{(g,H_i)}(z), n\}$, *for all* $i \in \{1, \dots, q\}$,

(ii) *Zero* $(f_j) \cap f^{-1}(H_i) =$ *Zero* $(g_j) \cap f^{-1}(H_i)$, *for all* $1 \leq i \leq q$, $0 \leq j \leq n$,

(iii) $\mathcal{D}^\alpha\big(\frac{f_k}{f_s}\big) = \mathcal{D}^\alpha\big(\frac{g_k}{g_s}\big)$ *on* $\big(\bigcup_{i=1}^q f^{-1}(H_i)\big) \backslash \big(\mathit{Zero}\,(f_s)\big)$, *for all* $|\alpha| \leq 1$, $0 \leq k \neq s \leq n$.

Then $f \equiv g$.

Proof. Assume that $f \not\equiv g$. We write $H_i : \sum_{j=0}^n a_{ij}\omega_j = 0$.

For any fixed index i, $(1 \leq i \leq q)$, it is easy to see that there exists $j \in \{1, \dots, q\}\backslash\{i\}$ (depending on i) such that

$$P_{ij} := \frac{(f, H_i)}{(f, H_j)} - \frac{(g, H_i)}{(g, H_j)} \not\equiv 0.$$

Set

$$I := I(f) \cup I(g) \cup \bigcup_{1\leq k<s\leq q} \{z \in \mathbf{C}^m : \nu_{(f,H_k)}(z) > 0 \text{ and } \nu_{(f,H_s)}(z) > 0\}.$$

Then I is an analytic subset of codimension ≥ 2.

Case 1. $n \geq 2$.

Let t be an arbitrary index in $\{1, \dots, q\}\backslash\{i, j\}$. For any fixed point $z_0 \notin I$ satisfying $\nu_{(f,H_t)}(z_0) > 0$, there exists $l \in \{0, \dots, n\}$ such that $f_l(z_0)g_l(z_0) \neq 0$. It follows that

$$\begin{aligned}\mathcal{D}^\alpha P_{ij}(z_0) &= \mathcal{D}^\alpha\left(\frac{(f, H_i)}{(f, H_j)} - \frac{(g, H_i)}{(g, H_j)}\right)(z_0)\\ &= \mathcal{D}^\alpha\left(\frac{\sum_{v=0}^n \frac{f_v}{f_l}a_{iv}}{\sum_{v=0}^n \frac{f_v}{f_l}a_{jv}} - \frac{\sum_{v=0}^n \frac{g_v}{g_l}a_{iv}}{\sum_{v=0}^n \frac{g_v}{g_l}a_{jv}}\right)(z_0) = 0,\end{aligned}$$

for all α with $|\alpha| < 2$. So

$$\nu_{P_{ij}}(z_0) \geq 2. \tag{3.1}$$

For any fixed point $z_1 \notin I$ satisfying $\nu_{(f,H_i)}(z_1) > 0$, we have

$$\nu_{P_{ij}}(z_1) \geq \min\{\nu_{(f,H_i)}(z_1), \nu_{(g,H_i)}(z_1)\} \geq \min\{\nu_{(f,H_i)}(z_1), n\}. \tag{3.2}$$

From (3.1) and (3.2), we have

$$\nu_{P_{ij}} \geq \min\{n, \nu_{(f,H_i)}\} + \sum_{t\in\{1,\dots,q\}\setminus\{i,j\}} 2\min\{1, \nu_{(f,H_t)}\},$$

(outside an analytic subset of codimension two).

It yields that

$$N_{P_{ij}}(r) \geq N^{(n)}_{(f,H_i)}(r) + \sum_{t\in\{1,\dots,q\}\setminus\{i,j\}} 2N^{(1)}_{(f,H_t)}(r) \tag{3.3}$$

It is clear that

$$N_{\frac{1}{P_{ij}}}(r) \leq N(r, \nu_j), \tag{3.4}$$

where $\nu_j(z) := \max\{\nu_{(f,H_j)}(z), \nu_{(g,H_j)}(z)\}$.

We have

$$\begin{aligned} m\left(r, \frac{(f,H_i)}{(f,H_j)}\right) &= T_{\frac{(f,H_i)}{(f,H_j)}}(r) - N_{(f,H_j)}(r) + O(1) \\ &\leq T_f(r) - N_{(f,H_j)}(r) + O(1), \end{aligned}$$

and

$$m\left(r, \frac{(g,H_i)}{(g,H_j)}\right) \leq T_g(r) - N_{(g,H_j)}(r) + O(1),$$

This implies that

$$\begin{aligned} m(r, P_{ij}) &\leq m\left(r, \frac{(f,H_i)}{(f,H_j)}\right) + m\left(r, \frac{(g,H_i)}{(g,H_j)}\right) + O(1) \\ &= T_f(r) + T_g(r) - N_{(f,H_j)}(r) - N_{(g,H_j)}(r) + O(1). \end{aligned}$$

Combining with (3.3) and (3.4) we get

$$\begin{aligned} N^{(n)}_{(f,H_i)}(r) + \sum_{t\in\{1,\dots,q\}\setminus\{i,j\}} 2N^{(1)}_{(f,H_t)}(r) &\leq N_{P_{ij}}(r) \leq T_{P_{ij}}(r) + O(1) \\ &= N_{\frac{1}{P_{ij}}}(r) + m(r, P_{ij}) + O(1) \\ &\leq T_f(r) + T_g(r) + N(r, \nu_j) - N_{(f,H_j)}(r) \\ &\quad - N_{(g,H_j)}(r) + o(T_f(r) + T_g(r)). \end{aligned}$$

This gives

$$\begin{aligned} N_{(f,H_j)}(r) + N_{(g,H_j)}(r) - N(r, \nu_j) + N^{(n)}_{(f,H_i)}(r) + \sum_{t\in\{1,\dots,q\}\setminus\{i,j\}} 2N^{(1)}_{(f,H_t)}(r) \\ \leq T_f(r) + T_g(r) + o(T_f(r) + T_g(r)). \end{aligned}$$

On the other hand, since

$$\nu_j(z) - \nu_{(f,H_j)} - \nu_{(g,H_j)} + \min\{n, \nu_{(f,H_j)}\} \leq 0$$

(outside an analytic subset of codimension two), we have

$$N(r,\nu_j) - N_{(f,H_j)}(r) - N_{(g,H_j)}(r) + N^{(n)}_{(f,H_j)}(r) \le 0.$$

Hence

$$N^{(n)}_{(f,H_i)}(r) + N^{(n)}_{(f,H_j)}(r) + \sum_{t\in\{1,\dots,q\}\setminus\{i,j\}} 2N^{(1)}_{(f,H_t)}(r) \\ \le T_f(r) + T_g(r) + o(T_f(r)+T_g(r)).$$

It implies that

$$(3.5)\qquad N^{(n)}_{(f,H_i)}(r) + \frac{2}{n}\sum_{t\in\{1,\dots,q\}\setminus\{i\}} N^{(n)}_{(f,H_t)}(r) \\ \le T_f(r) + T_g(r) + o(T_f(r)+T_g(r)),$$

(note that $n \ge 2$).

Taking summing-up of both sides of (3.5) over all $i \in \{1,\dots,q\}$, we obtain

$$(3.6)\qquad \left(1+\frac{2(q-1)}{n}\right)\sum_{i=1}^{q} N^{(n)}_{(f,H_i)}(r) \\ \le q(T_f(r)+T_g(r)) + o(T_f f(r)+T_g(r)).$$

On the other hand, by Theorem E we have

$$(3.7)\quad \|\ (q-n-1)(T_f(r)+T_g(r)) \le 2\sum_{i=1}^{q} N^{(n)}_{(f,H_i)}(r) + o(T_f(r)+T_g(r)).$$

From (3.6) and (3.7), letting $r \longrightarrow \infty$ we have

$$1+\frac{2(q-1)}{n} \le \frac{2q}{q-n-1}.$$

This contradicts to

$$q > \frac{\sqrt{17n^2+16n}+3n+4}{4}.$$

Thus $f \equiv g$.

Case 2. $n=1$. We have $q \ge 4$. If $\frac{(f,H_1)}{(f,H_4)} \equiv \frac{(g,H_1)}{(g,H_4)}$, then $f \equiv g$.

We now assume that

$$P_{14} := \frac{(f,H_1)}{(f,H_4)} - \frac{(g,H_1)}{(g,H_4)} \not\equiv 0.$$

Let t be an arbitrary index in $\{1,2,3\}$. For any fixed point $z_0 \notin I$ satisfying $\nu_{(f,H_t)}(z_0) > 0$, there exists $l \in \{0,1\}$ such that $f_l(z_0)g_l(z_0) \ne 0$. It follows that

$$\mathcal{D}^\alpha P_{14}(z_0) = \mathcal{D}^\alpha\left(\frac{(f,H_1)}{(f,H_4)} - \frac{(g,H_1)}{(g,H_4)}\right)(z_0)$$
$$= \mathcal{D}^\alpha\left(\frac{a_{10}\frac{f_0}{f_l} + a_{11}\frac{f_1}{f_l}}{a_{40}\frac{f_0}{f_l} + a_{41}\frac{f_1}{f_l}} - \frac{a_{10}\frac{g_0}{g_l} + a_{11}\frac{g_1}{g_l}}{a_{40}\frac{g_0}{g_l} + a_{41}\frac{g_1}{g_l}}\right)(z_0) = 0,$$

for all α with $|\alpha| < 2$. It implies that $\nu_{P_{14}}(z_0) \geq 2$. Hence, we have

$$\nu_{P_{14}} \geq 2\big(\min\{1, \nu_{(f,H_1)}\} + \min\{1, \nu_{(f,H_2)}\} + \min\{1, \nu_{(f,H_3)}\}\big),$$

(outside an analytic subset of codimension two). It implies that

$$N_{P_{14}}(r) \geq 2\Big(N^{(1)}_{(f,H_1)}(r) + N^{(1)}_{(f,H_2)}(r) + N^{(1)}_{(f,H_3)}(r)\Big). \tag{3.8}$$

Let z_1 be an arbitrary *pole* of P_{14} such that $z_1 \notin I$. Then z_1 is a *zero* of (f,H_4) and there exists $l \in \{0,1\}$ such that $f_l(z_1)g_l(z_1) \neq 0$. Then

$$\mathcal{D}^\alpha\Bigg(\left(a_{10}\frac{f_0}{f_l} + a_{11}\frac{f_1}{f_l}\right)\left(a_{40}\frac{g_0}{g_l} + a_{41}\frac{g_1}{g_l}\right)$$
$$- \left(a_{40}\frac{f_0}{f_l} + a_{41}\frac{f_1}{f_l}\right)\left(a_{10}\frac{g_0}{g_l} + a_{11}\frac{g_1}{g_l}\right)\Bigg)(z_1) = 0,$$

for all α with $|\alpha| < 2$. This implies that

$$\nu_{((f,H_1)(g,H_4)-(f,H_4)(g,H_1))}(z_1) \geq 2.$$

Then, we have

$$\nu_{\frac{1}{P_{14}}}(z_1) \leq \nu_{(f,H_4)}(z_1) + \nu_{(g,H_4)}(z_1) - 2.$$

Hence we see

$$\nu_{\frac{1}{P_{14}}} \leq \nu_{(f,H_4)} + \nu_{(g,H_4)} - 2\min\{1, \nu_{(f,H_4)}\},$$

(outside an analytic subset of codimension two). This implies that

$$N_{\frac{1}{P_{14}}}(r) \leq N_{(f,H_4)}(r) + N_{(g,H_4)}(r) - 2N^{(1)}_{(f,H_4)}(r).$$

Combining with (3.8) we have

$$\begin{aligned}2\Big(N^{(1)}_{(f,H_1)}(r)+N^{(1)}_{(f,H_2)}(r)+N^{(1)}_{(f,H_3)}(r)\Big) &\le N_{P_{14}}(r)\le T_{P_{14}}(r)+O(1)\\ &= m(r,P_{14})+N_{\frac{1}{P_{14}}}(r)+O(1)\\ &\le m\Big(r,\frac{(f,H_1)}{(f,H_4)}\Big)+m\Big(r,\frac{(g,H_1)}{(g,H_4)}\Big)\\ &\quad + N_{(f,H_4)}(r)+N_{(g,H_4)}(r)-2N^{(1)}_{(f,H_4)}(r)+O(1)\\ &= T_{\frac{(f,H_1)}{(f,H_4)}}(r)+T_{\frac{(g,H_1)}{(g,H_4)}}(r)-2N^{(1)}_{(f,H_4)}(r)+O(1)\\ &\le T_f(r)+T_g(r)-2N^{(1)}_{(f,H_4)}(r)+o(T_f(r)+T_g(r)).\end{aligned}$$

It implies that

$$\begin{aligned}2\Big(N^{(1)}_{(f,H_1)}(r)+N^{(1)}_{(f,H_2)}(r)+N^{(1)}_{(f,H_3)}(r)+N^{(1)}_{(f,H_4)}(r)\Big)&\\ \le T_f(r)+T_g(r)+o(T_f(r)+T_g(r)).&\end{aligned}$$

On the other hand, by Theorem E, we also have

$$||\ 2T_f(r)\le N^{(1)}_{(f,H_1)}(r)+N^{(1)}_{(f,H_2)}(r)+N^{(1)}_{(f,H_3)}(r)+N^{(1)}_{(f,H_4)}(r)+o(T_f(r))$$

and

$$\begin{aligned}||\ 2T_g(r) &\le N^{(1)}_{(g,H_1)}(r)+N^{(1)}_{(g,H_2)}(r)+N^{(1)}_{(g,H_3)}(r)+N^{(1)}_{(g,H_4)}(r)+o(T_g(r))\\ &= N^{(1)}_{(f,H_1)}(r)+N^{(1)}_{(f,H_2)}(r)+N^{(1)}_{(f,H_3)}(r)+N^{(1)}_{(f,H_4)}(r)+o(T_g(r))\end{aligned}$$

Hence, we have

$$||\ 2(T_f(r)+T_g(r))\le T_f(r)+T_g(r)+o(T_f(r)+T_g(r)).$$

Letting $r\longrightarrow\infty$, we have $2\le 1$. This is a contradiction, hence $f\equiv g$. We have completed the proof of Lemma 1. □

Let f be a linearly nondegenerate meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{C}P^n$ with reduced representation $f=(f_0:\cdots:f_n)$. Let d be a positive integer and let $H_1,\ldots,H_q$ be q hyperplanes in $\mathbf{C}P^n$ located in general position with

$$\dim\big\{z\in\mathbf{C}^m:\nu_{(f,H_i)}(z)>0 \text{ and } \nu_{(f,H_j)}(z)>0\big\}\le m-2$$

$(1\le i<j\le q)$.

Consider the set $\mathcal{F}(f,\{H_j\}_{j=1}^q,d)$ of all linearly nondegenerate meromorphic mappings $g:\mathbf{C}^m\to\mathbf{C}P^n$ with reduced representation $g=(g_0:\cdots:g_n)$ satisfying the conditions:

(a) $\min(\nu_{(f,H_i)},d)=\min(\nu_{(g,H_i)},d)$ $(1\le i\le q)$,

(b) $\mathit{Zero}\,(f_j)\cap f^{-1}(H_i)=\mathit{Zero}\,(g_j)\cap f^{-1}(H_i)$, for all $1\le i\le q$, $0\le j\le n$,

(c) $\mathcal{D}^{\alpha}\big(\frac{f_k}{f_s}\big) = \mathcal{D}^{\alpha}\big(\frac{g_k}{g_s}\big)$ on $\big(\bigcup_{i=1}^{q} f^{-1}(H_i)\big)\setminus\big(Zero\,(f_s)\big)$, for all $|\alpha| < d$, $0 \le k \ne s \le n$.

Take $M+1$ maps $f^0, \ldots, f^M \in \mathcal{F}(f, \{H_j\}_{j=1}^q, d)$ with reduced representations

$$f^k := \big(f_0^k : \cdots : f_n^k\big)$$

and set $T(r) := \sum_{k=0}^{M} T_{f^k}(r)$. For each $c = (c_0, \ldots, c_n) \in \mathbf{C}^{n+1} \setminus \{0\}$ we put

$$(f^k, c) := \sum_{i=0}^{n} c_i f_i^k \quad (0 \le k \le M).$$

Denote by $\mathcal{C}$ the set of all $c \in \mathbf{C}^{n+1} \setminus \{0\}$ such that

$$\dim\{z \in \mathbf{C}^m : (f^k, H_j)(z) = (f^k, c)(z) = 0\} \le m-2$$

$(1 \le j \le q,\ 0 \le k \le M)$.

Lemma A ([9], Lemma 5.1). *$\mathcal{C}$ is dense in $\mathbf{C}^{n+1}$.*

Lemma B ([7]). *For each $c \in \mathcal{C}$, we put $F_c^{jk} = \frac{(f^k, H_j)}{(f^k, c)}$. Then $T_{F_c^{jk}}(r) \le T_{f^k}(r) + o(T(r))$.*

Definition 1. Let $F_0, \ldots, F_M$ be meromorphic functions on $\mathbf{C}^m$, where $M \ge 1$. Take a set $\alpha := (\alpha^0, \ldots, \alpha^{M-1})$ whose components α^k are composed of n nonnegative integers, and set $|\alpha| = |\alpha^0| + \cdots + |\alpha^{M-1}|$. We define Cartan's auxiliary function by

$$\Phi^{\alpha}(F_0, \ldots, F_M) := F_0 \cdot F_1 \cdots F_M \times \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \mathcal{D}^{\alpha^0}(\frac{1}{F_0}) & \mathcal{D}^{\alpha^0}(\frac{1}{F_1}) & \cdots & \mathcal{D}^{\alpha^0}(\frac{1}{F_M}) \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha^{M-1}}(\frac{1}{F_0}) & \mathcal{D}^{\alpha^{M-1}}(\frac{1}{F_1}) & \cdots & \mathcal{D}^{\alpha^{M-1}}(\frac{1}{F_M}) \end{vmatrix}.$$

Lemma C ([7], Proposition 3.4). *If $\Phi^{\alpha}(F, G, H) = 0$ and $\Phi^{\alpha}(\frac{1}{F}, \frac{1}{G}, \frac{1}{H}) = 0$ for all α with $|\alpha| \le 1$, then one of the following conditions holds:*

i) *$F = G$ or $G = H$ or $H = F$.*

ii) *$\frac{F}{G}, \frac{G}{H}$ and $\frac{H}{F}$ are all constant.*

Lemma 2. *Assume that there exists $\Phi^{\alpha} := \Phi^{\alpha}\big(F_c^{j_0 0}, \ldots, F_c^{j_0 M}\big) \not\equiv 0$ for some $c \in \mathcal{C}$, $|\alpha| \le \frac{M(M-1)}{2}$, $d \ge |\alpha|$. Then, for each $0 \le i \le M$, the following holds:*

$$||\ N^{(d-|\alpha|)}_{(f^i, H_{j_0})}(r) + Md \sum_{j \ne j_0} N^{(1)}_{(f^i, H_j)}(r) \le N_{\Phi^{\alpha}}(r) \le T(r) + o(T(r)).$$

Proof. Denote by **P** the set of all β with $|\beta| \leq \frac{M(M-1)}{2}$, $d \geq |\beta|$ such that $\Phi^\beta = \Phi^\beta\big(F_c^{j_0 0}, \dots, F_c^{j_0 M}\big) \not\equiv 0$ for some $c \in \mathcal{C}$. Let α be the *minimal* multi-index in **P** (in the lexicographic order). Set

$$I := \bigcup_{t=0}^{M} I(f^t) \cup \bigcup_{1\leq t<j\leq q} \big((f,H_t)^{-1}\{0\} \cap (f,H_j)^{-1}\{0\}\big) \\ \cup \bigcup_{t=1}^{q} \big((f,H_t)^{-1}\{0\} \cap (f,c)^{-1}\{0\}\big).$$

Then I is an analytic subset of codimension ≥ 2.

Assume that a is a *zero* of some (f^i, H_j), $j \neq j_0$ such that $a \notin I$. Let Γ be an irreducible component of the *zero*-divisor of the function (f^i, H_j) which contains a. We take a holomorphic function h on C^m satisfying: $\nu_{h|_\Gamma} = 1$ and $\nu_{h|_{(C^n\setminus\Gamma)}} = 0$.

By the condition (c), we have that $\varphi_i := \big(\frac{1}{h^d F^{j_0 i}} - \frac{1}{h^d F^{j_0 M}}\big)$ is a holomorphic function on a neighborhood U of a for all $i \in \{0, \dots, M-1\}$. Since $\alpha := \min \mathbf{P}$, we have

$$\Phi^\alpha := h^{Md} F^{j_0 0} \cdots F^{j_0 M} \times \begin{vmatrix} \mathcal{D}^{\alpha^0}\varphi_0 & \cdots & \mathcal{D}^{\alpha^0}\varphi_{M-1} \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha^{M-1}}\varphi_0 & \cdots & \mathcal{D}^{\alpha^{M-1}}\varphi_{M-1} \end{vmatrix}.$$

It implies that

$$\nu_{\Phi^\alpha}(a) \geq Md. \tag{3.9}$$

Assume that b is a *zero* of (f^i, H_{j_0}) such that $b \notin I$. If $\nu_{(f^i,H_{j_0})}(b) \geq d$, we write

$$\Phi^\alpha = \sum_{\sigma\in S_{M+1}} \operatorname{sign}(\sigma) F^{j_0 0} \cdots F^{j_0 M} \\ \times \mathcal{D}^{\alpha^0}\Big(\frac{1}{F^{j_0(\sigma(2)-1)}}\Big) \cdots D^{\alpha^{M-1}}\Big(\frac{1}{F^{j_0(\sigma(M+1)-1)}}\Big).$$

Then

$$\nu_{\Phi^\alpha}(b) \geq d - |\alpha|. \tag{3.10}$$

If $\nu_{(f^i,H_{j_0})}(b) < d$, then $\nu_{(f^0,H_{j_0})}(b) = \cdots = \nu_{(f^M,H_{j_0})}(b) < d$. There exists a holomorphic function h on an open neighborhood U of b such that $\nu_h = \nu_{(f^i,H_{j_0})|_U}$.

We write

$$\Phi^\alpha = h^{-M}F_c^{j_0 0}\cdots F_c^{j_0 M}$$

$$\times\begin{vmatrix} \big(\mathcal{D}^{\alpha^0}\big(\frac{h}{F_c^{j_00}}\big)-D^{\alpha^0}\big(\frac{h}{F_c^{j_0M}}\big)\big) & \cdots & \big(\mathcal{D}^{\alpha^0}\big(\frac{h}{F_c^{j_0(M-1)}}\big)-D^{\alpha^0}\big(\frac{h}{F_c^{j_0M}}\big)\big) \\ \vdots & \vdots & \vdots \\ \big(\mathcal{D}^{\alpha^{M-1}}\big(\frac{h}{F_c^{j_00}}\big)-D^{\alpha^{M-1}}\big(\frac{h}{F_c^{j_0M}}\big)\big) & \cdots & \big(\mathcal{D}^{\alpha^{M-1}}\big(\frac{h}{F_c^{j_0(M-1)}}\big)-D^{\alpha^{M-1}}\big(\frac{h}{F_c^{j_0M}}\big)\big) \end{vmatrix}.$$

Then

$$\nu_{\Phi^\alpha}(b)\geq \nu_{(f^i,H_{j_0})}(b). \tag{3.11}$$

From (3.9), (3.10) and (3.11), we have

$$\min\big\{d-|\alpha|,\nu_{(f^i,H_{j_0})}\big\}+Md\sum_{j\in\{1,\ldots,q\}\setminus\{j_0\}}\min\big\{1,\nu_{(f^i,H_j)}\big\}\leq \nu_{\Phi^\alpha},$$

(outside an analytic subset of codimension two). It immediately follows the first inequality in the lemma.

It is easy to see that a *pole* of Φ^α is a *zero* or a *pole* of some $F_c^{j_0k}$. By (3.9), (3.10) and (3.11) we have that Φ^α is holomorphic at all *zeros* of $F_c^{j_0i}$, $(0\leq i\leq M)$. Then

$$N_{\frac{1}{\Phi^\alpha}}(r)\leq \sum_{i=0}^{M}N_{\frac{1}{F_c^{j_0i}}}(r).$$

On the other hand, it is easy to see that

$$\begin{aligned} m(r,\Phi^\alpha) &\leq \sum_{i=0}^{M}m(r,F_c^{j_0i})+O\bigg(\sum m\bigg(r,\frac{\mathcal{D}^{\alpha^i}(\varphi_c^{j_0k})}{\varphi_c^{j_0k}}\bigg)\bigg)+O(1)\\ &\leq \sum_{i=0}^{M}m(r,F_c^{j_0i})+o(T(r)), \end{aligned}$$

where $\varphi_c^{j_0k}=1/F_c^{j_0k}$. Hence, we have

$$\begin{aligned} N_{\Phi^\alpha}(r)\leq T_{\Phi^\alpha}(r)+O(1)&\leq m(r,\Phi^\alpha)+N_{\frac{1}{\Phi^\alpha}}(r)+O(1)\\ &\leq \sum_{i=0}^{M}\big(N_{\frac{1}{F_c^{j_0i}}}(r)+m(r,F_c^{j_0i})\big)+o(T(r))\\ &=\sum_{i=0}^{M}T_{F_c^{j_0i}}(r)+o(T(r))\leq T(r)+o(T(r)). \end{aligned}$$

□

Theorem 1. *If*

$$q>\max\bigg\{\frac{7(n+1)}{4},\frac{\sqrt{17n^2+16n}+3n+4}{4}\bigg\}$$

then $\mathcal{F}(f, \{H_i\}_{i=1}^q, 2)$ *contains at most two mappings.*

Proof. If $n = 1$, by Lemma 1 we have $\sharp\mathcal{F}(f, \{H_i\}_{i=1}^q, 1) = 1$.

We prove the theorem for the case of $n \geq 2$. Assume that there exist three distinct mappings $f^0, f^1, f^2 \in \mathcal{F}(f, \{H_i\}_{i=1}^q, 2)$.

Denote by $\mathcal{Q}$ the set of all indices $j \in \{1, 2, \ldots, q\}$ satisfying the following: There exist $c \in \mathcal{C}$ and $\alpha \in \mathbf{Z}_+^n$ with $|\alpha| \leq 1$ such that $\Phi^\alpha\big(F_c^{j0}, F_c^{j1}, F_c^{j2}\big) \not\equiv 0$.

Set $T(r) = T_{f^0}(r) + T_{f^1}(r) + T_{f^2}(r)$.

We now prove that $\mathcal{Q} = \emptyset$. Suppose that there exists $j_0 \in \mathcal{Q}$. By Lemma 2, we have

$$(3.12)\qquad \begin{aligned} || \ N^{(1)}_{(f^i, H_{j_0})}(r) + 4 \sum_{j \in \{1, \ldots, q\} \setminus \{j_0\}} N^{(1)}_{(f^i, H_j)}(r) \\ \leq N(r, \nu_{\Phi^\alpha}) \leq T(r) + o(T(r)). \end{aligned}$$

$(0 \leq i \leq 2)$.

By Theorem E, we have

$$|| \ \sum_{j \neq j_0} N^{(1)}_{(f^i, H_j)}(r) \geq \frac{q - n - 2}{3n} T(r) + o(T(r))$$

and

$$\sum_{j=0}^{q} N^{(1)}_{(f^i, H_j)}(r) \geq \frac{q - n - 1}{3n} T(r) + o(T(r)).$$

This implies that

$$(3.13)\qquad \begin{aligned} || \ N^{(1)}_{(f^i, H_{j_0})}(r) + 4 \sum_{j \in \{1, \ldots, q\} \setminus \{j_0\}} N^{(1)}_{(f^i, H_j)}(r) \\ \geq \frac{4(q - n - 2) + 1}{3n} T(r) + o(T(r)). \end{aligned}$$

From (3.12) and (3.13), letting $r \to \infty$ we get

$$4(q - n - 2) + 1 \leq 3n \Leftrightarrow q \leq \frac{7(n + 1)}{4}.$$

This is a contradiction. Hence $\mathcal{Q} = \emptyset$. Then for each $1 \leq j \leq q$, $c \in \mathcal{C}$, $\alpha \in \mathbf{Z}_+^n$, $|\alpha| < 2$ we have $\Phi^\alpha\big(F_c^{j0}, F_c^{j1}, F_c^{j2}\big) \equiv 0$. Since $\mathcal{C}$ is dense in $\mathbf{C}^{n+1}$, we have that

$$\Phi^\alpha\big(F_i^{j0}, F_i^{j1}, F_i^{j2}\big) \equiv 0 \ (1 \leq i, j \leq q), \text{ for all } |\alpha| < 2,$$

where $F_i^{jt} := \frac{(f^t, H_j)}{(f^t, H_i)}$, $0 \leq t \leq 2$. By Lemma C, for each $1 \leq i, j \leq q$, there exists a nonzero constant χ_{ij} such that $F_i^{j0} = \chi_{ij} F_i^{j1}$, $F_i^{j1} = \chi_{ij} F_i^{j2}$ or $F_i^{j2} =$

$\chi_{ij}F_i^{j0}$. We now show that $\chi_{ij}=1$. Indeed, if $\chi_{ij}\neq 1$, without loss of generality we may assume that $F_i^{j0}=\chi_{ij}F_i^{j1}$. Then $\bigcup_{t\in\{1,\dots,q\}\setminus\{i,j\}} f^{-1}(H_t)=\emptyset$. Thus, by Theorem E, we have

$$|| \ (q-n-3)T_f(r)\leq \sum_{t\in\{1,\dots,q\}\setminus\{i,j\}} N^{(n)}_{(f,H_t)}(r)+o(T_f(r))=o(T_f(r)).$$

Letting $r\longrightarrow +\infty$, we obtain $q-n-3\leq 0$. This contradicts to $n\geq 2$. Thus,

$$\chi_{ij}=1 \quad (1\leq i,j\leq q).$$

We take an arbitrary element $k\in\{0,1,2\}$ and an index $i\in\{1,\dots,q\}$. We will show that $\nu_{(f^k,H_i)}=\nu_{(f^l,H_i)}$ or $\nu_{(f^k,H_i)}=\nu_{(f^t,H_i)}$, where $\{l,t\}:=\{0,1,2\}\setminus\{k\}$. In fact, if there is no index $j\neq i$ such that $F_i^{jk}=F_i^{jl}$ or $F_i^{jk}=F_i^{jt}$, then since $\chi_{ij}=1$ we have $F_i^{jl}=F_i^{jt}$ for all $j\neq i$. This implies that $f^k\equiv f^l$. This is a contradiction. Hence there exists $j\neq i$ such that $F_i^{jk}=F_i^{jl}$ or $F_i^{jk}=F_i^{jt}$. This yields that

$$\nu_{(f^k,H_i)}=\nu_{(f^l,H_i)} \text{ or } \nu_{(f^k,H_i)}=\nu_{(f^t,H_i)} \tag{3.14}$$

for all $k\in\{0,1,2\}$, $i\in\{1,\dots,q\}$. For any fixed index $i\in\{1,\dots,q\}$, by (3.14) (with $k=0$) we may assume that $\nu_{(f^0,H_i)}=\nu_{(f^1,H_i)}$. By (3.14) (with $k=2$) we obtain $\nu_{(f^2,H_i)}=\nu_{(f^0,H_i)}$ or $\nu_{(f^2,H_i)}=\nu_{(f^1,H_i)}$. This implies that $\nu_{(f^0,H_i)}=\nu_{(f^1,H_i)}=\nu_{(f^2,H_i)}$ for all $i\in\{1,\dots,q\}$. By Lemma 1, we have $f^0\equiv f^1\equiv f^2$. This is a contradiction.

Thus, $\sharp\mathcal{F}(f,\{H_i\}_{i=1}^q,2)\leq 2$ if

$$q>\max\Big\{\frac{7(N+1)}{4},\frac{\sqrt{17N^2+16N}+3N+4}{4}\Big\}. \qquad \square$$

4. Uniqueness problem for hypersurfaces. Let f be a nonconstant meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{C}P^n$. We say that a meromorphic function φ on $\mathbf{C}^m$ is "small" with respect to f if $T_\varphi(r)=o(T_f(r))$ as $r\to\infty$ (outside a set of finite Lebesgues measure). Denote by $\mathcal{R}_f$ the field of all "small" (with respect to f) meromorphic functions on $\mathbf{C}^m$.

Take a reduced representation $(f_0:\cdots:f_n)$ of f. We say that f is algebraically nondegenerate over $\mathcal{R}_f$ if there is no nonzero homogeneous polynomial $Q\in\mathcal{R}_f[x_0,\dots,x_n]$ such that $Q(f):=Q(f_0,\dots,f_n)\equiv 0$.

For a homogeneous polynomial $Q\in\mathcal{R}_f[x_0,\dots,x_n]$, denote by $Q(z)$ the homogeneous polynomial over $\mathbf{C}$ obtained by substituting a specific point $z\in\mathbf{C}^m$ into the coefficients of Q.

We say that a set $\{Q_j\}_{j=0}^n$ of homogeneous polynomials of the same degree in $\mathcal{R}_f[x_0,\dots,x_n]$ is admissible if there exists $z\in\mathbf{C}^m$ such that the system of equations

$$\begin{cases} Q_j(z)(w_0, \dots, w_n) = 0 \\ 0 \le j \le n \end{cases}$$

has only the trivial solution $w = (0, \dots, 0)$ in $\mathbf{C}^{n+1}$.

First of all, we give the following lemma:

Lemma 3. *Let f be a nonconstant meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{C}P^n$ and $\{Q_j\}_{j=0}^n$ be an admissible set of homogeneous polynomials of degree d in $\mathcal{R}_f[x_0, \dots, x_n]$. Let $\gamma_0, \dots, \gamma_n$ be $(n+1)$ nonzero meromorphic functions in $\mathcal{R}_f$.*

Put $P = \gamma_0 Q_0^p + \cdots + \gamma_n Q_n^p$, where p is a positive integer, $p > n(n+1)$.

Assume that f is algebraically nondegenerate over $\mathcal{R}_f$. Then

$$\|\ d(p - n(n+1))T_f(r) \le N^{(n)}_{P(f)}(r) + o(T_f(r)).$$

Proof. Set $\mathcal{T}_d := \{I := (i_0, \dots, i_n) \in \mathbf{N}_0^{n+1} : i_0 + \cdots + i_n = d\}$.

Assume that

$$Q_j = \sum_{I \in \mathcal{T}_d} a_{jI} x^I \quad (j = 0, \dots, n).$$

where $a_{jI} \in \mathcal{R}_f$, $x^I = x_0^{i_0} \cdots x_n^{i_n}$.

Set

$$F = \big(\gamma_0 Q_0^p(f) : \cdots : \gamma_n Q_n^p(f)\big) : \mathbf{C}^m \longrightarrow \mathbf{C}P^n.$$

Since f is algebraically nondegenerate over $\mathcal{R}_f$ we have that F is linearly nondegenerate (over $\mathbf{C}$).

Assume that $\left(\frac{\gamma_0 Q_0^p(f)}{h} : \cdots : \frac{\gamma_n Q_n^p(f)}{h}\right)$ is a reduced representation of F, where h is a meromorphic function on $\mathbf{C}^m$. Put $F_i = \frac{\gamma_i Q_i^p(f)}{h}$, $i \in \{0, \dots, n\}$.

We have

$$\max_{0 \le j \le n} |Q_j^p(f)| \le |h| \cdot \left(\sum_{i=0}^n \left|\frac{1}{\gamma_i}\right|\right) \cdot \max_{1 \le i \le n+1} |F_i|. \tag{4.1}$$

Let $t = (\dots, t_{kI}, \dots)$ be a family of variables, $(k \in \{0, \dots, n\}, I \in \mathcal{T}_d)$.

Set

$$\widetilde{Q}_j = \sum_{I \in \mathcal{T}_d} t_{jI} x^I \in \mathbf{Z}[t, x], \quad j = 0, \dots, n.$$

Let $\widetilde{R} \in \mathbf{Z}[t]$ be the resultant of $\widetilde{Q}_0, \dots, \widetilde{Q}_n$.

Since $\{Q_j\}_{j=0}^n$ is an admissible set, $R := \widetilde{R}(\dots, a_{kI}, \dots) \not\equiv 0$. It is clear that $R \in \mathcal{R}_f$ since $a_{kI} \in \mathcal{R}_f$.

By Theorems 3.4 and 3.5 in [10], there exists a positive integer $s > d$ and polynomials $\{\widetilde{R}_{ij}\}_{0 \le i,j \le n}$ in $\mathbf{Z}[t, x]$ which are zero or homogeneous in x of degree $s-d$ such that

$$x_i^s \cdot \widetilde{R} = \sum_{j=0}^{n} \widetilde{R}_{ij} \cdot \widetilde{Q}_j \quad \text{for all } i \in \{0, \ldots, n\}.$$

Set

$$R_{ij} = \widetilde{R}_{ij}\big((\ldots, a_{kI}, \ldots), (f_0, \ldots, f_n)\big), \quad 0 \le i, j \le n.$$

Then,

$$f_i^s \cdot R = \sum_{j=0}^{n} R_{ij} \cdot Q_j(f_0, \ldots, f_n) \quad \text{for all } i \in \{0, \ldots, n\}. \tag{4.2}$$

So,

$$\begin{aligned} |f_i^s \cdot R| &= \Big|\sum_{j=0}^{n} R_{ij} \cdot Q_j(f_0, \ldots, f_n)\Big| \\ &\le \sum_{j=0}^{n} |R_{ij}| \cdot \max_{k \in \{0,\ldots,n\}} |Q_k(f_0, \ldots, f_n)| \end{aligned} \tag{4.3}$$

for all $i \in \{0, \ldots, n\}$.

We write,

$$R_{ij} = \sum_{I \in \mathcal{T}_{s-d}} \beta_I^{ij} f^I, \quad \beta_I^{ij} \in \mathcal{R}_f.$$

By (4.3), we have

$$|f_i^s \cdot R| \le \Bigg(\sum_{\substack{0 \le j \le n \\ I \in \mathcal{T}_{s-d}}} |\beta_I^{ij}| \cdot \|f\|^{s-d}\Bigg) \cdot \max_{k \in \{0,\ldots,n\}} |Q_k(f_0, \ldots, f_n)|,$$

$i \in \{0, \ldots, n\}$. So,

$$\frac{|f_i|^s}{\|f\|^{s-d}} \le \Bigg(\sum_{\substack{0 \le j \le n \\ I \in \mathcal{T}_{s-d}}} \Big|\frac{\beta_I^{ij}}{R}\Big|\Bigg) \cdot \max_{k \in \{0,\ldots,n\}} |Q_k(f_0, \ldots, f_n)|$$

for all $i \in \{0, \ldots, n\}$.

Thus

$$\|f\|^d \le \Bigg(\sum_{\substack{0 \le i, j \le n \\ I \in \mathcal{T}_{s-d}}} \Big|\frac{\beta_I^{ij}}{R}\Big|\Bigg) \max_{k \in \{0,\ldots,n\}} |Q_k(f_0, \ldots, f_n)|. \tag{4.4}$$

By (4.1) and (4.4) we have

$$\|f\|^{dp} \le \Big(\sum_{\substack{0\le i,j\le n\\ I\in\mathcal{T}_{s-d}}}\Big|\frac{\beta_I^{ij}}{R}\Big|\Big)^p\cdot|h|\cdot\Big(\sum_{i=0}^n\Big|\frac{1}{\gamma_i}\Big|\Big)\cdot\|F\|. \tag{4.5}$$

By (4.2) and since $\Big(\frac{\gamma_0Q_0^p(f)}{h}:\cdots:\frac{\gamma_nQ_n^p(f)}{h}\Big)$ is a reduced representation of F, we have

$$N_h(r)\le pN_R(r)+\sum_{i=0}^n N_{\gamma_i}(r)=o(T_f(r))$$

and

$$N_{\frac{1}{h}}(r)\le \sum_{\substack{0\le j\le n\\ I\in\mathcal{T}_d}} N_{\frac{1}{a_{jI}}}(r)+\sum_{i=0}^n N_{\frac{1}{\gamma_i}}=o(T_f(r)).$$

By (4.5), we have

$$\begin{aligned}
dp\cdot T_f(r)&=pd\int\limits_{S(r)}\log\|f\|\sigma+O(1)\\
&\le\int\limits_{S(r)}\log\Big(\sum_{\substack{0\le i,j\le n\\ I\in\mathcal{T}_{s-d}}}\Big|\frac{\beta_I^{ij}}{R}\Big|\Big)^p|h|\Big(\sum_{i=0}^n\Big|\frac{1}{\gamma_i}\Big|\Big)\sigma+T_F(r)+O(1)\\
&\le p\int\limits_{S(r)}\log^+\Big(\sum_{\substack{0\le i,j\le n\\ I\in\mathcal{T}_{s-d}}}\Big|\frac{\beta_I^{ij}}{R}\Big|\Big)\sigma+\int\limits_{S(r)}\log^+\Big(\sum_{i=0}^n\Big|\frac{1}{\gamma_i}\Big|\Big)\sigma\\
&\quad+\int\limits_{S(r)}\log|h|\sigma+T_F(r)+O(1)\\
&\le p\sum_{\substack{0\le i,j\le n\\ I\in\mathcal{T}_{s-d}}}m\Big(r,\frac{\beta_I^{ij}}{R}\Big)+\sum_{i=0}^n m\Big(r,\frac{1}{\gamma_i}\Big)\\
&\quad+N_h(r)-N_{\frac{1}{h}}(r)+T_F(r)+O(1)\\
&=T_F(r)+o(T_f(r)).
\end{aligned}\tag{4.6}$$

By (4.6) and Theorem E, we have

$$
\begin{aligned}
\| \; dp\cdot T_f(r) &\le T_F(r)+o(T_f(r))\\
&\le \sum_{i=0}^{n} N^{(n)}_{\frac{\gamma_i Q_i^p(f)}{h}}(r)+N^{(n)}_{\sum_{i=0}^{n}\frac{\gamma_i Q_i^p(f)}{h}}(r)+o(T_f(r))\\
&\le \sum_{i=0}^{n} N^{(n)}_{\frac{\gamma_i Q_i^p(f)}{h}}(r)+N^{(n)}_{\frac{P(f)}{h}}(r)+o(T_f(r))\\
&\le \sum_{i=0}^{n} N^{(n)}_{Q_i^p(f)}(r)+\sum_{i=0}^{n} N^{(n)}_{\gamma_i}(r)+(n+2)N_{\frac{1}{h}}(r)+N^{(n)}_{P(f)}(r)+o(T_f(r))\\
&\le \sum_{i=0}^{n} nN_{Q_i(f)}(r)+N^{(n)}_{P(f)}(r)+o(T_f(r))\\
&\le d(n+1)nT_f(r)+N^{(n)}_{P(f)}(r)+o(T_f(r)).
\end{aligned}
$$

This implies that

$$\| \; d(p-(n+1)n)T_f(r)\le N^{(n)}_{P(f)}(r)+o(T_f(r)).$$

This has completed the proof of the lemma. □

Theorem 2. *Let $f_1,\dots,f_k$ $(k\ge 2)$ be nonconstant meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{C}P^n$ and $\{Q_j\}_{j=0}^n$ be an admissible set of homogeneous polynomials of degree d in $\mathcal{R}_{f_1}[x_0,\dots,x_n]$. Let $\gamma_0,\dots,\gamma_n$ be $(n+1)$ nonzero meromorphic functions in $\mathcal{R}_{f_1}$.*

Put $P=\gamma_0 Q_0^p+\cdots+\gamma_n Q_n^p$, where p is a positive integer, $p>\frac{n(d(n+1)+k)}{d}$. Assume that f_i is algebraically nondegenerate over $\mathcal{R}_{f_i}$ for all $i\in\{1,\dots,k\}$, and

i) *Zero$\big(P(f_i)\big)=$ Zero$\big(P(f_1)\big)$, for all $i\in\{2,\dots,k\}$, and*
ii) *$f_1\wedge\cdots\wedge f_k=0$ on Zero$\big(P(f_1)\big)$.*

Then $f_1\wedge\cdots\wedge f_k\equiv 0$.

Proof. Assume that $f_1\wedge\cdots\wedge f_k\not\equiv 0$. We denote by $\mu_{f_1\wedge\cdots\wedge f_k}$ the divisor associated with $f_1\wedge\cdots\wedge f_k$. Denote $N_{\mu_{f_1\wedge\cdots\wedge f_k}}(r)$ the counting function associated with the divisor $\mu_{f_1\wedge\cdots\wedge f_k}$. It is easy to see that

$$N_{\mu_{f_1\wedge\cdots\wedge f_k}}(r)\le \sum_{i=1}^{k} T_{f_i}(r)+O(1).$$

Since *Zero*$\big(P(f_i)\big)=$ *Zero*$\big(P(f_1)\big)$, for all $i\in\{2,\dots,k\}$, we have,

$$N^{(1)}_{P(f_1)}(r)\le N_{\mu_{f_1\wedge\cdots\wedge f_k}}(r)\le \sum_{i=1}^{k} T_{f_i}(r)+O(1)\le \sum_{i=1}^{k} T_{f_i}(r)+O(1).$$

Thus, since $Zero\big(P(f_i)\big) = Zero\big(P(f_1)\big)$, for all $i \in \{2, \dots, k\}$, we have

$$\sum_{i=1}^{k} N^{(n)}_{P(f_i)}(r) \le nkN^{(1)}_{P(f_1)}(r) \le nk\sum_{i=1}^{k} T_{f_i}(r) + O(1). \tag{4.7}$$

By Lemma 3 we have

$$\begin{aligned} d(p-n(n+1))T_{f_1}(r) &\le N^{(n)}_{P(f_1)}(r) + o(T_{f_1}(r)) \\ &\le nN^{(1)}_{P(f_i)}(r) + o(T_{f_1}(r)) \\ &\le ndpT_{f_i}(r) + o(T_{f_1}(r)) \quad (1 \le i \le k). \end{aligned}$$

This implies that $\mathcal{R}_{f_1} \subset \mathcal{R}_{f_i}$ for all $2 \le i \le k$. Thus, by Lemma 3 we have

$$d(p-n(n+1))T_{f_i}(r) \le N^{(n)}_{P(f_i)}(r) + o(T_{f_i}(r)) \quad (1 \le i \le k).$$

Combining with (4.7) we have

$$d(p-n(n+1))\sum_{i=1}^{k} T_{f_i}(r) \le nk\sum_{i=1}^{k} T_{f_i}(r) + o\Big(\sum_{i=1}^{k} T_{f_i}(r)\Big).$$

This contradicts to $p > \frac{n(d(n+1)+k)}{d}$. Thus, $f_1 \wedge \cdots \wedge f_k \equiv 0$. □

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Si Duc Quang
Department of Mathematics
Hanoi National University of Education
136-Xuan Thuy street, Cau Giay, Hanoi
Vietnam
e-mail: quangdhsp@yahoo.com

Tran Van Tan
Department of Mathematics
Hanoi National University of Education
136-Xuan Thuy street, Cau Giay, Hanoi
Vietnam
e-mail: tranvantanhn@yahoo.com

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