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Sufficient conditions for quasiconformality of harmonic mappings of the upper halfplane onto itself

Abstract. In this paper we introduce a class of increasing homeomorphic self-mappings of $\mathbb{R}$. We define a harmonic extension of such functions to the upper halfplane by means of the Poisson integral. Our main results give some sufficient conditions for quasiconformality of the extension.

1. Introduction. Let F be a complex-valued sense-preserving diffeomorphism of the upper halfplane $\mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ onto itself, where $\mathbb{C}$ stands for the complex plane. Then the Jacobian

$$\mathrm{J}_F := |\partial F|^2 - |\bar{\partial} F|^2 \tag{1.1}$$

is positive on $\mathbb{C}^+$ and so the function

$$\mathbb{C}^+ \ni z \mapsto \mathrm{D}_F(z) := \frac{|\partial F(z)| + |\bar{\partial} F(z)|}{|\partial F(z)| - |\bar{\partial} F(z)|} \tag{1.2}$$

is well defined. We recall that $\mathrm{D}_F(z)$ is called the maximal dilatation of F at $z \in \mathbb{C}^+$. Here and in the sequel $\partial := (\partial_x - i\partial_y)/2$ and $\bar{\partial} := (\partial_x + i\partial_y)/2$ stands for the formal derivatives operators. From the analytical characterization of quasiconformal mappings (see [3]) it follows that for any $K \geq 1$, F is

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K-quasiconformal if and only if

$$\mathrm{D}_F(z) \leq K, \quad z \in \mathbb{C}^+. \tag{1.3}$$

Assume now that F is quasiconformal, i.e. F satisfies (1.3) for some $K \geq 1$. Then F has a unique homeomorphic extension F^* to the closure $\overline{\mathbb{C}^+} := \mathbb{C}^+ \cup \hat{\mathbb{R}}$, $\hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ (see [3]). The famous result of Beurling and Ahlfors (see [1]) says that a function f of $\mathbb{R}$ onto itself is the restriction of F^* if and only if f is quasisymmetric, i.e. f is a strictly increasing homeomorphism, such that

$$\frac{1}{M} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq M \tag{1.4}$$

for some constant $M \geq 1$ and for all $x \in \mathbb{R}$ and $t > 0$.

Assume additionally that F is a harmonic mapping, i.e. F satisfies the Laplace equation $\partial\bar{\partial}F = 0$ on $\mathbb{C}^+$. Kalaj and Pavlović proved in [2] that an increasing homeomorphism f of $\mathbb{R}$ onto itself is the restriction of F^* if and only if it is biLipschitz and the Hilbert transformation of f' is bounded.

Following the idea of Beurling and Ahlfors we are going to find an effective extension of f to F^*. For $f \in \mathcal{F}$, where $\mathcal{F}$ is considered in Section 2, we provide a construction of the harmonic extension $H[f]$ defined in Definition 3.1 by means of the Poisson integral. The main purpose of this paper is to give sufficient conditions on $f \in \mathcal{F}$, that guarantee quasiconformality of $H[f]$. In Section 3 we show that $H[f]$ is a homeomorphism of $\mathbb{C}^+$ onto itself provided $f \in \mathcal{F}$ has the biLipschitz property (3.2), cf. Proposition 3.2. In Section 4 we provide various auxiliary estimates dealing with partial derivatives of $H[f]$. Applying them we are able to estimate the maximal dilatation $\mathrm{D}_{H[f]}$ of $H[f]$ in case $f \in \mathcal{F}$ satisfies the biLipschitz property (3.2) and f' is a Dini-continuous function with respect to spherical distance (4.3). This is the main result of the paper and is stated in Theorem 5.2. In particular, if f' is Hölder-continuous with respect to spherical distance we obtain estimate of $\mathrm{D}_{H[f]}$ given in Theorem 5.3.

2. Preliminary notes. Let $\mathrm{Hom}^+(\mathbb{R})$ be the set of all increasing real line homeomorphisms onto itself. For $a \in \mathbb{R}$ we define

$$\mathcal{F}_a := \left\{ f \in \mathrm{Hom}^+(\mathbb{R}) : \mathrm{I}(f,a) < +\infty \right\},$$

where

$$\mathrm{I}(f,a) := \int_{-\infty}^{+\infty} \frac{|f(t) - at|}{1+t^2} \,\mathrm{d}\, t.$$

We define also

$$\mathcal{F} := \bigcup_{a>0} \mathcal{F}_a.$$

The following properties hold.

Proposition 2.1. *If $a < 0$, then $\mathcal{F}_a = \emptyset$.*

Proof. Let $f \in \mathrm{Hom}^+(\mathbb{R})$. There exists $T > 0$ such that $f(t) \geq 0$ for $t \geq T$. Hence, if $a < 0$, then $|f(t) - at| \geq f(t) + |a|t$ for $t \geq T$, which implies that

$$\mathrm{I}(f, a) \geq \int_T^{+\infty} \frac{f(t) + |a|t}{1 + t^2}\,\mathrm{d}\,t.$$

Since the last integral is divergent, $f \notin \mathcal{F}_a$ and we have a contradiction which completes the proof. □

Proposition 2.2. *If $a \neq b$, then $\mathcal{F}_a \cap \mathcal{F}_b = \emptyset$.*

Proof. Let $f \in \mathcal{F}_a \cap \mathcal{F}_b$, $a \neq b$. Observe, that

$$\int_{-\infty}^{+\infty} \frac{|(a-b)t|}{1+t^2}\,\mathrm{d}\,t \leq \int_{-\infty}^{+\infty} \frac{|f(t) - at|}{1+t^2}\,\mathrm{d}\,t + \int_{-\infty}^{+\infty} \frac{|f(t) - bt|}{1+t^2}\,\mathrm{d}\,t < +\infty.$$

But the first integral is divergent, thus we have a contradiction, which completes the proof. □

Remark 2.3. By Proposition 2.2, for every fixed $f \in \mathcal{F}$ there exists exactly one constant $a > 0$, such that $\mathrm{I}(f, a) < +\infty$.

Proposition 2.4. *If $f \in \mathcal{F}_a$, then $\tilde{f} \in \mathcal{F}_a$, where $\tilde{f}(t) := -f(-t)$, $t \in \mathbb{R}$.*

Proof. Consider $\mathrm{I}(\tilde{f}, a)$. Substituting $s := -t$ we have

$$\mathrm{I}(\tilde{f}, a) = -\int_{+\infty}^{-\infty} \frac{|\tilde{f}(-s) + as|}{1+s^2}\,\mathrm{d}\,s = \int_{-\infty}^{+\infty} \frac{|-\tilde{f}(-s) - as|}{1+s^2}\,\mathrm{d}\,s = \mathrm{I}(f, a). \quad \square$$

Proposition 2.5. *If $f \in \mathcal{F}$, then $\liminf_{t\to+\infty} f(t)/t \geq 0$.*

Proof. Assume that $\liminf_{t\to+\infty} f(t)/t < 0$, then there exists a sequence $\{t_n\}$ and $T \in \mathbb{R}$ such that $t_n \to +\infty$ and $f(t_n) < 0$ for $n \geq T$. But $f \in \mathrm{Hom}^+(\mathbb{R})$, i.e. f is an increasing homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$, thus we have a contradiction and the proof is completed. □

Proposition 2.6. *If $f \in \mathcal{F}$, then $\liminf_{t\to-\infty} f(t)/t \geq 0$.*

Proof. Consider $\tilde{f}(t) := -f(-t)$. By Proposition 2.4 we have $\tilde{f} \in \mathcal{F}_a$ and then by Proposition 2.5 we have

$$\liminf_{t\to+\infty} \tilde{f}(t)/t \geq 0.$$

This is equivalent to $\liminf_{t\to-\infty} f(t)/t \geq 0$, which completes the proof. □

Proposition 2.7. *If $f \in \mathcal{F}_a$, then a is an accumulation point of $f(t)/t$ in $+\infty$.*

Proof. Consider $f \in \mathcal{F}_a$ satisfying the condition

$$\forall_{T>0}\forall_{\delta>0}\exists_{t\geq T}\ \left|\frac{f(t)}{t} - a\right| < \delta.$$

If we put $T \coloneqq n$ and $\delta \coloneqq 1/n$, then we have

$$\forall_{n>0}\exists_{t\geq n}\ \left|\frac{f(t)}{t}-a\right|<\frac{1}{n}.$$

This means that a is an accumulation point of $f(t)/t$ in $+\infty$.

Assume that a is not an accumulation point of $f(t)/t$ in $+\infty$. This implies that

$$\exists_{T>0}\exists_{\delta>0}\forall_{t\geq T}\ \left|\frac{f(t)}{t}-a\right|\geq\delta.$$

Hence

$$\mathrm{I}(f,a)\geq\int_T^{+\infty}\frac{|f(t)+at|}{1+t^2}\,\mathrm{d}\,t\geq\int_T^{+\infty}\frac{\delta t}{1+t^2}\,\mathrm{d}\,t.$$

Since the last integral is divergent, this contradicts the assumption $f\in\mathcal{F}_a$, which completes the proof. □

Proposition 2.8. *If $f\in\mathcal{F}_a$, then a is an accumulation point of $f(t)/t$ in $-\infty$.*

Proof. Consider $\tilde{f}(t)\coloneqq -f(-t)$. By Proposition 2.4 we have $\tilde{f}\in\mathcal{F}_a$ and by Proposition 2.7 we obtain that a is an accumulation point of $\tilde{f}(t)/t$ in $+\infty$. This is equivalent to that a is an accumulation point of $f(t)/t$ in $-\infty$ and completes the proof. □

Theorem 2.9. *If $f\in\mathcal{F}_a$, then $\lim_{t\to+\infty}f(t)/t=a$.*

Proof. Note, that by Proposition 2.7 a is the accumulation point of $f(t)/t$ in $+\infty$. Assume that there exists $b\in\mathbb{R}$, $b\neq a$ which is an accumulation point of $f(t)/t$ in $+\infty$, i.e. there exists a sequence $\{t_n\}$, $t_n>0$, $t_n\to+\infty$, such that

$$\forall_{\varepsilon>0}\exists_{\tilde{n}}\forall_{n\geq\tilde{n}}\ \left|\frac{f(t_n)}{t_n}-b\right|<\varepsilon.$$

Set $\varepsilon\coloneqq|a-b|/3$ and denote

$$s_n\coloneqq\frac{2b+a}{2a+b}t_n.$$

In view of Proposition 2.5 we may restrict our consideration to $a\geq 0$ and $b\geq 0$.

If $b>a\geq 0$, then $s_n>t_n$ and for $t\in[t_n,s_n]$ we have the following estimate

$$f(t)-at\geq f(t_n)-as_n>\left(b-\epsilon-a\frac{2b+a}{2a+b}\right)t_n=\frac{(b-a)(2b+a)}{3(2a+b)}t_n>0.$$

We chose from $\{t_n\}$ a subsequence $\{t_{n_k}\}$, $k=1,2,3,\dots$ such that $t_{n_1}=t_{\tilde{n}}$ and for all k holds

$$t_{n_{k+1}}>s_{n_k}.$$

Hence, for $t \in [t_n, s_n]$ we have

$$
\begin{aligned}
\mathrm{I}(f,a) &\geq \int_0^{+\infty} \frac{|f(t)-at|}{1+t^2}\,\mathrm{d}\,t = \int_0^{t_{n_1}} \frac{|f(t)-at|}{1+t^2}\,\mathrm{d}\,t + \int_{t_{n_1}}^{+\infty} \frac{|f(t)-at|}{1+t^2}\,\mathrm{d}\,t \\
&\geq \sum_{n=1}^{+\infty} \int_{t_{n_k}}^{t_{n_{k+1}}} \frac{|f(t)-at|}{1+t^2}\,\mathrm{d}\,t \geq \sum_{n=1}^{+\infty} \int_{t_{n_k}}^{s_{n_k}} \frac{|f(t)-at|}{1+t^2}\,\mathrm{d}\,t \\
&\geq \sum_{n=1}^{+\infty} \int_{t_{n_k}}^{s_{n_k}} \frac{(b-a)(2b+a)t_{n_k}}{3(2a+b)(1+t_{n_k}^2)}\,\mathrm{d}\,t \\
&= \sum_{n=1}^{+\infty} \frac{(b-a)(2b+a)(s_{n_k}-t_{n_k})t_{n_k}}{3(2a+b)(1+s_{n_k}^2)} \\
&= \sum_{n=1}^{+\infty} \frac{(b-a)^2(2b+a)t_{n_k}^2}{3[(2a+b)^2+(2b+a)^2 t_{n_k}^2]}.
\end{aligned}
$$

Observe, that

$$
\lim_{n\to+\infty} \frac{(b-a)^2(2b+a)t_{n_k}^2}{3[(2a+b)^2+(2b+a)^2 t_{n_k}^2]} = \frac{(b-a)^2}{3(2b+a)} \neq 0. \tag{2.1}
$$

If $a > b \geq 0$, then $s_n < t_n$ and for $t \in [s_n, t_n]$ we have the following estimate

$$
f(t) - at \leq f(t_n) - as_n < \left(b + \epsilon - a\frac{2b+a}{2a+b}\right) t_n = \frac{(b-a)(2b+a)}{3(2a+b)} t_n < 0.
$$

We chose from $\{s_n\}$ a subsequence $\{s_{n_k}\}$, $k = 1, 2, 3, \dots$ such that $s_{n_1} = s_{\tilde{n}}$ and for all k holds

$$
s_{n_{k+1}} > t_{n_k}.
$$

Hence, for $t \in [s_n, t_n]$ we have

$$
\begin{aligned}
\mathrm{I}(f,a) &\geq \int_0^{+\infty} \frac{|f(t)-at|}{1+t^2}\,\mathrm{d}\,t = \int_0^{s_{n_1}} \frac{|f(t)-at|}{1+t^2}\,\mathrm{d}\,t + \int_{s_{n_1}}^{+\infty} \frac{|f(t)-at|}{1+t^2}\,\mathrm{d}\,t \\
&\geq \sum_{n=1}^{+\infty} \int_{s_{n_k}}^{s_{n_{k+1}}} \frac{|f(t)-at|}{1+t^2}\,\mathrm{d}\,t \geq \sum_{n=1}^{+\infty} \int_{s_{n_k}}^{t_{n_k}} \frac{|f(t)-at|}{1+t^2}\,\mathrm{d}\,t \\
&\geq \sum_{n=1}^{+\infty} \int_{s_{n_k}}^{t_{n_k}} \frac{(a-b)(2b+a)t_{n_k}}{3(2a+b)(1+t_{n_k}^2)}\,\mathrm{d}\,t = \sum_{n=1}^{+\infty} \frac{(a-b)(2b+a)(t_{n_k}-s_{n_k})t_{n_k}}{3(2a+b)(1+t_{n_k}^2)} \\
&= \sum_{n=1}^{+\infty} \frac{(a-b)^2(2b+a)t_{n_k}^2}{3(2a+b)^2(1+t_{n_k}^2)}.
\end{aligned}
$$

Observe, that

$$\lim_{n\to+\infty}\frac{(a-b)^2(2b+a)t_{n_k}^2}{3(2a+b)^2(1+t_{n_k}^2)}=\frac{(a-b)^2(2b+a)}{3(2a+b)^2}\neq 0. \tag{2.2}$$

Finally, (2.1) and (2.2), together, imply that $\mathrm{I}(f,a)=+\infty$, which contradicts the assumption $f\in\mathcal{F}$. Hence

$$\lim_{t\to+\infty} f(t)/t = a,$$

which completes the proof. □

Theorem 2.10. *If $f\in\mathcal{F}_a$, then $\lim_{t\to-\infty} f(t)/t=a$.*

Proof. Consider $\tilde{f}(t):=-f(-t)$. By Proposition 2.4 we have $\tilde{f}\in\mathcal{F}_a$ and by Theorem 2.9 we obtain

$$\lim_{t\to+\infty}\tilde{f}(t)/t=a.$$

This is equivalent to

$$\lim_{t\to-\infty} f(t)/t=a$$

and completes the proof. □

Remark 2.11. Every function $f\in\mathcal{F}_a$ has the form

$$\mathbb{R}\ni t\mapsto f(t)=at+g(t), \tag{2.3}$$

where $g(t)/t\to 0$ as $|t|\to+\infty$.

3. The harmonic extension $H[f]$. We introduce a harmonic extension of $f\in\mathcal{F}$ from $\mathbb{R}$ to $\mathbb{C}^+$. By the definition of the class $\mathcal{F}$ the following definition makes sense.

Definition 3.1. For $f\in\mathcal{F}_a$ we define $H[f]:\mathbb{C}^+\to\mathbb{C}^+$ as follows

$$H[f](z):=az+P[g](z),$$

where g is related to f by (2.3) and

$$P[g](z):=\int_{-\infty}^{+\infty}P_z(t)g(t)\,\mathrm{d}\,t \tag{3.1}$$

is the Poisson integral for $\mathbb{C}^+$ and

$$P_z(t):=\frac{1}{\pi}\frac{\mathrm{Im}\{z\}}{|z-t|^2}$$

is the Poisson kernel for $\mathbb{C}^+$.

Note, that $P[g](z) \in \mathbb{R}$ for every $z \in \mathbb{C}^+$ and let us denote

$$U(z) := \mathrm{Re}\{H[f](z)\} = a\,\mathrm{Re}\{z\} + P[g](z)$$

and

$$V(z) := \mathrm{Im}\{H[f](z)\} = a\,\mathrm{Im}\{z\}.$$

Throughout this paper U and V will always mean $\mathrm{Re}\{H[f]\}$ and $\mathrm{Im}\{H[f]\}$, respectively.

Recall that the biLipschitz condition on f, i.e.

$$\exists_{L_1,L_2>0}\forall_{t_1,t_2\in\mathbb{R}}\ L_2|t_2-t_1| \le |f(t_2)-f(t_1)| \le L_1|t_2-t_1| \tag{3.2}$$

is the necessary condition for $H[f]$ to be quasiconformal (see [2]).

Proposition 3.2. *If $f \in \mathcal{F}_a$ satisfies the biLipschitz condition* (3.2)*, then $H[f]$ is a homeomorphism of $\mathbb{C}^+$ onto itself.*

Proof. Fix $y > 0$ and let $z_1 = x_1 + iy$, $z_2 = x_2 + iy$, where $x_1, x_2 \in \mathbb{R}$. Since $P_z(t) > 0$, $t \in \mathbb{R}$ and

$$\int_{-\infty}^{+\infty} P_z(t)\,\mathrm{d}\,t = 1, \quad z \in \mathbb{C}^+,$$

we can write

$$\begin{aligned}
U(z_1) - U(z_2) &= ax_1 + P[g](z_1) - ax_2 - P[g](z_2)\\
&= \int_{-\infty}^{+\infty} \frac{1}{\pi}\frac{y}{(x_1-t)^2+y^2}[ax_1 + g(t)]\,\mathrm{d}\,t\\
&\quad - \int_{-\infty}^{+\infty} \frac{1}{\pi}\frac{y}{(x_2-t)^2+y^2}[ax_2 + g(t)]\,\mathrm{d}\,t\\
&= \int_{-\infty}^{+\infty} \frac{1}{\pi}\frac{y}{s^2+y^2}[a(x_1-s) + g(x_1-s) - a(x_2-s) - g(x_2-s)]\,\mathrm{d}\,s\\
&= \int_{-\infty}^{+\infty} \frac{1}{\pi}\frac{y}{s^2+y^2}[f(x_1-s) - f(x_2-s)]\,\mathrm{d}\,s.
\end{aligned}$$

Because f increases, then $U(z_1) > U(z_2)$ for $x_1 > x_2$. Hence U is univalent on every horizontal line. Since $V(z) = a\,\mathrm{Im}\{z\}$, $H[f]$ is univalent.

To show that U maps every horizontal line in the upper halfplane onto $\mathbb{R}$, we fix $y > 0$ and observe that

$$U(x+iy) - U(iy) = \int_{-\infty}^{+\infty} \frac{1}{\pi}\frac{y}{s^2+y^2}[f(x-s) - f(-s)]\,\mathrm{d}\,s.$$

Let $x > 0$. Since f increases and by applying the biLipschitz condition (3.2), we have

$$U(x+iy) - U(iy) \ge \int_{-\infty}^{+\infty} \frac{1}{\pi}\frac{y}{s^2+y^2}L_2|x|\,\mathrm{d}\,s = L_2 x.$$

Let $x < 0$. Analogically we obtain

$$U(x+iy) - U(iy) \le -\int_{-\infty}^{+\infty} \frac{1}{\pi} \frac{y}{s^2+y^2} L_2 |x| \,\mathrm{d}\, s = L_2 x.$$

Since $V(z) = a \operatorname{Im}\{z\}$, $H[f](\mathbb{C}^+) = \mathbb{C}^+$. □

The following example shows that not every function $f \in \mathcal{F}$ has the extension $H[f]$ which is quasiconformal.

Example 3.3. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(t) = t + |t|^{1/2} \operatorname{sgn} t$. Obviously, $f \in \mathcal{F}_1 \subset \mathcal{F}$ since

$$\int_{-\infty}^{+\infty} \frac{|t|^{1/2}}{1+t^2} \,\mathrm{d}\, t < +\infty.$$

On the other hand, we have

$$|f(t_1) - f(t_2)| = |t_1 - t_2| \left(1 + \frac{1}{\sqrt{t_1} + \sqrt{t_2}} \right),$$

where $t_1, t_2 > 0$. Hence, we see that

$$\forall_{L>0} \exists_{t_1, t_2 > 0} \; 1 + \frac{1}{\sqrt{t_1} + \sqrt{t_2}} > L,$$

e.g. putting $t_2 := t_1/4 := 1/(9L^2)$. This means that f is not biLipschitz and so it cannot have quasiconformal extension to the upper halfplane.

4. Estimates of partial derivatives of $H[f]$. Let $f \in \mathcal{F}_a$ and $z = x+iy$. We compute partial derivatives of U and V.

$$\begin{aligned}
\frac{\partial U}{\partial x}(z) &= a + \frac{\partial}{\partial x}(P[g](z)) = a + \int_{-\infty}^{+\infty} \frac{1}{\pi} \frac{-2y(x-t)}{[(x-t)^2+y^2]^2} g(t) \,\mathrm{d}\, t \\
&= a + \int_0^{+\infty} \frac{1}{\pi} \frac{2ys}{(s^2+y^2)^2} [g(x+s) - g(x-s)] \,\mathrm{d}\, s, \\
\frac{\partial U}{\partial y}(z) &= \frac{\partial}{\partial y}(P[g](z)) = \int_{-\infty}^{+\infty} \frac{1}{\pi} \frac{(x-t)^2 - y^2}{[(x-t)^2+y^2]^2} g(t) \,\mathrm{d}\, t \\
&= \int_0^{+\infty} \frac{1}{\pi} \frac{s^2-y^2}{(s^2+y^2)^2} [g(x+s) + g(x-s)] \,\mathrm{d}\, s, \\
\frac{\partial V}{\partial x}(z) &= 0, \\
\frac{\partial V}{\partial y}(z) &= a.
\end{aligned} \tag{4.1}$$

First, we give the estimates on $\partial U / \partial x$ under assumption, that $f \in \mathcal{F}$ is biLipschitz only.

Theorem 4.1. *If $f \in \mathcal{F}_a$ satisfies the biLipschitz condition* (3.2)*, then*

$$L_2 \le \frac{\partial U}{\partial x}(z) \le L_1, \quad z \in \mathbb{C}^+. \tag{4.2}$$

Proof. Observe, that (3.2) implies

$$2(L_2 - a)s \le g(x+s) - g(x-s) \le 2(L_1 - a)s$$

for every $s > 0$. Let $z = x + iy$. Then

$$\begin{aligned}
\frac{\partial U}{\partial x}(z) &= a + \int_0^{+\infty} \frac{1}{\pi} \frac{2ys}{(s^2+y^2)^2} [g(x+s) - g(x-s)] \,\mathrm{d}\, s \\
&\le a + \int_0^{+\infty} \frac{1}{\pi} \frac{4ys^2}{(s^2+y^2)^2} (L_1 - a) \,\mathrm{d}\, s = L_1, \\
\frac{\partial U}{\partial x}(z) &= a + \int_0^{+\infty} \frac{1}{\pi} \frac{2ys}{(s^2+y^2)^2} [g(x+s) - g(x-s)] \,\mathrm{d}\, s \\
&\ge a + \int_0^{+\infty} \frac{1}{\pi} \frac{4ys^2}{(s^2+y^2)^2} (L_2 - a) \,\mathrm{d}\, s = L_2.
\end{aligned}$$

□

As a corollary from the estimates of $\partial U/\partial x$ we obtain the estimates of the Jacobian $\mathrm{J}_{H[f]}$ of $H[f]$ defined in (1.1).

Corollary 4.2. *If $f \in \mathcal{F}_a$ satisfies the biLipschitz condition* (3.2)*, then*

$$aL_2 \le \mathrm{J}_{H[f]}(z) \le aL_1, \quad z \in \mathbb{C}^+.$$

Proof. We can rewrite the Jacobian of $H[f]$ in the form

$$J_{H[f]} = \frac{\partial U}{\partial x}\frac{\partial V}{\partial y} - \frac{\partial U}{\partial y}\frac{\partial V}{\partial x}.$$

Since $\partial V/\partial x = 0$ and $\partial V/\partial y = a$, by applying the inequalities (4.2) the proof is completed. □

Now, we give the estimate of $\partial U/\partial y$ under an additional assumption on f, but first we formulate the following lemma.

Lemma 4.3. *If $f \in \mathcal{F}$ is absolutely continuous function, then*

$$\frac{\partial U}{\partial y}(z) = \int_0^{+\infty} \frac{1}{\pi} \frac{s}{s^2+y^2} [f'(x+s) - f'(x-s)] \,\mathrm{d}\, s.$$

Proof. Recall that

$$\frac{\partial U}{\partial y}(z) = \int_0^{+\infty} \frac{1}{\pi} \frac{s^2-y^2}{(s^2+y^2)^2} [g(x+s) + g(x-s)] \,\mathrm{d}\, s,$$

where $z = x + iy$. Since f is absolutely continuous, f' exists almost everywhere and for almost all $t_1, t_2 \in \mathbb{R}$

$$f'(t_1) - f'(t_2) = g'(t_1) - g'(t_2).$$

Hence, integrating by parts we have

$$\frac{\partial U}{\partial y}(z) = -\frac{1}{\pi}\frac{s}{s^2+y^2}[g(x+s)+g(x-s)]\Big|_0^{+\infty} + \int_0^{+\infty} \frac{1}{\pi}\frac{s}{s^2+y^2}[g'(x+s)-g'(x-s)]\,\mathrm{d}\,s.$$

Since, by Theorem 2.9,

$$\lim_{t\to+\infty}\frac{g(t)}{t} = 0,$$

the proof is completed. □

Recall, that a continuous function φ is said to be Dini-continuous with respect to spherical distance if it satisfies the following condition

$$\int_0^{\varsigma} \frac{\omega(t)}{t}\,\mathrm{d}\,t = M_\varsigma < +\infty \tag{4.3}$$

for some $\varsigma \in (0,1]$, where $\omega : [0,1] \to [0,1]$,

$$\omega(t) := \sup\{d_s(\varphi(t_1),\varphi(t_2)) : d_s(t_1,t_2) < t\}$$

is the modulus of continuity of φ with respect to spherical distance d_s,

$$d_s(t_1,t_2) := \frac{|t_1-t_2|}{\sqrt{1+t_1^2}\sqrt{1+t_2^2}}.$$

Obviously, ω is non-decreasing function and

$$d_s(\varphi(t_1),\varphi(t_2)) \le \omega(d_s(t_1,t_2)) \tag{4.4}$$

holds for all $t_1, t_2 \in \mathbb{R}$.

Remark 4.4. If f satisfies the biLipschitz condition (3.2) and f' is Dini-continuous with respect to spherical distance a.e. in $\mathbb{R}$, then f' exists everywhere in $\hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ and $L_2 \le |f'(t)| \le L_1$, $t \in \hat{\mathbb{R}}$. In particular, there exists finite value of f' at the point ∞. If, additionally, $f \in \mathcal{F}_a$, then by Remark 2.11 f is of the form (2.3) and so we have

$$\lim_{t\to+\infty} f'(t) = \lim_{t\to-\infty} f'(t) = a.$$

Theorem 4.5. *If $f \in \mathcal{F}$ satisfies the biLipschitz condition* (3.2) *and if f' is Dini-continuous with respect to spherical distance* (4.3)*, then*

$$\left|\frac{\partial U}{\partial y}(z)\right| \le \frac{2(1+L_1^2)}{\pi}\left[\frac{M_\varsigma}{\sqrt{1-\delta^2}} + \log\left(\frac{1+\sqrt{1-\delta^2}}{\delta}\right)\right], \tag{4.5}$$

where $\delta := \min\{\varsigma, 1/\sqrt{1+M_\varsigma}\}$ and ς, M_ς satisfy (4.3).

Proof. Since f is biLipschitz, f is absolutely continuous and by Lemma 4.3 we have

$$\begin{aligned}\left|\frac{\partial U}{\partial y}(z)\right| &= \left|\frac{1}{\pi}\int_0^{+\infty}\frac{s}{s^2+y^2}[g'(x+s)-g'(x-s)]\,\mathrm{d}\,s\right|\\ &\le \frac{1}{\pi}\int_0^{+\infty}\frac{|g'(x+s)-g'(x-s)|}{s}\,\mathrm{d}\,s.\end{aligned}$$

From the Dini-continuity condition with respect to spherical distance (4.3) we have that (4.4) holds for f' and so we obtain

$$\left|\frac{\partial U}{\partial y}(z)\right| \le \frac{1}{\pi}\int_0^{+\infty}\left[\frac{\sqrt{1+[f'(x+s)]^2}\sqrt{1+[f'(x-s)]^2}}{s}\times\omega\left(\frac{2s}{\sqrt{1+(x+s)^2}\sqrt{1+(x-s)^2}}\right)\right]\mathrm{d}\,s.$$

Again, the biLipschitz condition for f gives

$$\left|\frac{\partial U}{\partial y}(z)\right| \le \frac{(1+L_1^2)}{\pi}\int_0^{+\infty}\frac{1}{s}\omega\left(\frac{2s}{\sqrt{1+(x+s)^2}\sqrt{1+(x-s)^2}}\right)\mathrm{d}\,s.$$

Setting

$$t := \frac{2s}{\sqrt{1+(x+s)^2}\sqrt{1+(x-s)^2}}, \tag{4.6}$$

we have

$$t' = \frac{-2s^4+2(1+x^2)^2}{(\sqrt{1+(x+s)^2}\sqrt{1+(x-s)^2})^3} = \frac{t^3[-2s^4+2(1+x^2)^2]}{4s^3}.$$

Let

$$\begin{aligned}&A := t^2, \quad B := [2t^2(1-x^2)-4], \quad C := t^2(1+x^2)^2,\\ &\Delta := B^2-4AC = 16(1-t^2)(1+x^2t^2).\end{aligned}$$

To apply the substitution (4.6) to the last integral we need to divide it into two integrals from 0 to $\sqrt{1+x^2}$ and from $\sqrt{1+x^2}$ to $+\infty$. Then we obtain

$$\begin{aligned}&\left|\frac{\partial U}{\partial y}(z)\right| \le \frac{4(1+L_1^2)}{\pi}\int_0^{+\infty}\frac{s^2}{(Bs^2+2C)}\frac{\omega(t)}{t^3}t'\,\mathrm{d}\,s\\ &= \frac{4(1+L_1^2)}{\pi}\left[\int_0^{\sqrt{1+x^2}}\frac{1}{(B+\frac{2C}{s^2})}\frac{\omega(t)}{t^3}t'\,\mathrm{d}\,s + \int_{\sqrt{1+x^2}}^{+\infty}\frac{1}{(B+\frac{2C}{s^2})}\frac{\omega(t)}{t^3}t'\,\mathrm{d}\,s\right].\end{aligned}$$

From (4.6) we compute two solutions

$$s^2 = \frac{-B-\sqrt{\Delta}}{2A} \quad \text{and} \quad s^2 = \frac{-B+\sqrt{\Delta}}{2A}$$

for $t \in (0,1)$. Hence, we have

$$\begin{aligned}\left|\frac{\partial U}{\partial y}(z)\right| &\le \frac{4(1+L_1^2)}{\pi}\int_0^1 \frac{1}{\sqrt{\Delta}}\frac{\omega(t)}{t}\,\mathrm{d}\,t + \frac{4(1+L_1^2)}{\pi}\int_1^0 \frac{-1}{\sqrt{\Delta}}\frac{\omega(t)}{t}\,\mathrm{d}\,t\\ &= \frac{8(1+L_1^2)}{\pi}\int_0^1 \frac{1}{\sqrt{\Delta}}\frac{\omega(t)}{t}\,\mathrm{d}\,t \le \frac{2(1+L_1^2)}{\pi}\int_0^1 \frac{1}{\sqrt{1-t^2}}\frac{\omega(t)}{t}\,\mathrm{d}\,t.\end{aligned}$$

Since, by definition, $\omega(t) \le 1$ and ω satisfies (4.3),

$$\begin{aligned}\left|\frac{\partial U}{\partial y}(z)\right| &\le \frac{2(1+L_1^2)}{\pi}\left[\int_0^\delta \frac{1}{\sqrt{1-t^2}}\frac{\omega(t)}{t}\,\mathrm{d}\,t + \int_\delta^1 \frac{\omega}{t\sqrt{1-t^2}}\,\mathrm{d}\,t\right]\\ &\le \frac{2(1+L_1^2)}{\pi}\begin{cases}\frac{1}{\sqrt{1-\varsigma^2}}\int_0^\varsigma \frac{\omega(t)}{t}\,\mathrm{d}\,t + \int_\varsigma^1 \frac{1}{t\sqrt{1-t^2}}\,\mathrm{d}\,t, & \delta \ge \varsigma,\\ \frac{1}{\sqrt{1-\delta^2}}\int_0^\delta \frac{\omega(t)}{t}\,\mathrm{d}\,t + \int_\delta^1 \frac{1}{t\sqrt{1-t^2}}\,\mathrm{d}\,t, & \delta < \varsigma\end{cases}\\ &\le \frac{2(1+L_1^2)}{\pi}\begin{cases}\frac{M_\varsigma}{\sqrt{1-\varsigma^2}} + \log\left(\frac{1+\sqrt{1-\varsigma^2}}{\varsigma}\right), & \delta \ge \varsigma,\\ \frac{M_\varsigma}{\sqrt{1-\delta^2}} + \log\left(\frac{1+\sqrt{1-\delta^2}}{\delta}\right), & \delta < \varsigma.\end{cases}\end{aligned}$$

Simple calculation shows that the above estimate is the best when $\delta = \min\{\varsigma, 1/\sqrt{1+M_\varsigma}\}$ and the proof is completed. □

In particular, if φ is Hölder-continuous with respect to spherical distance d_s, i.e.

$$d_s(\varphi(t_1),\varphi(t_2)) \le \lambda d_s(t_1,t_2)^\alpha \tag{4.7}$$

for all $t_1, t_2 \in \mathbb{R}$ and some constants $\lambda > 0$ and $\alpha \in (0,1]$, then φ is also Dini-continuous with respect to spherical distance.

We have the following corollary from the proof of Theorem 4.5.

Corollary 4.6. *If $f \in \mathcal{F}$ satisfies the biLipschitz condition* (3.2) *and f' is Hölder-continuous with respect to spherical distance* (4.7)*, then*

$$\left|\frac{\partial U}{\partial y}(z)\right| \le \frac{\lambda(1+L_1^2)}{\pi}\begin{cases}\mathrm{B}\left(\frac{\alpha}{2},\frac{1}{2};1\right), & \lambda \le 1,\\ \mathrm{B}\left(\frac{\alpha}{2},\frac{1}{2};\lambda^{-1/\alpha}\right) + \frac{2}{\lambda}\log\left(\lambda^{1/\alpha}+\sqrt{\lambda^{2/\alpha}-1}\right), & \lambda > 1.\end{cases} \tag{4.8}$$

where B *denotes the incomplete beta function and λ, α satisfy* (4.7)*.*

Proof. From the proof of Theorem 4.5 we have

$$\left|\frac{\partial U}{\partial y}(z)\right| \le \frac{2(1+L_1^2)}{\pi}\int_0^1 \frac{\omega(t)}{t\sqrt{1-t^2}}\,\mathrm{d}\,t,$$

where ω is the modulus of continuity of f' with respect to spherical distance. Since f' satisfies (4.7) and $\omega(t) \le 1$, we have

$$\omega(t) \le \min\{1, \lambda t^\alpha\}.$$

Hence

$$\left|\frac{\partial U}{\partial y}(z)\right| \leq \frac{2(1+L_1^2)}{\pi} \begin{cases} \int_0^1 \frac{\lambda t^\alpha}{t\sqrt{1-t^2}}\,\mathrm{d}\,t, & \lambda \leq 1, \\ \int_0^{\lambda^{-1/\alpha}} \frac{\lambda t^\alpha}{t\sqrt{1-t^2}}\,\mathrm{d}\,t + \int_{\lambda^{-1/\alpha}}^1 \frac{1}{t\sqrt{1-t^2}}\,\mathrm{d}\,t, & \lambda > 1. \end{cases}$$

Finally, recall that for $a > 0$, $b > 0$ and $c \in [0,1]$ the incomplete beta function is defined by the formula (see [4])

$$\mathrm{B}(a,b;c) := \int_0^c t^{a-1}(1-t)^{b-1}\,\mathrm{d}\,t.$$

Hence, the proof is completed. □

5. Quasiconformality of $\boldsymbol{H[f]}$. Using estimates on partial derivatives of the extension $H[f]$ we are able to estimate its maximal dilatation $\mathrm{D}_{H[f]}$, which is the main tool in studying quasiconformality of $H[f]$.

Theorem 5.1. *If $f \in \mathcal{F}_a$ satisfies the biLipschitz condition* (3.2) *and $|\partial U/\partial y| \leq A$ for some $A > 0$, then*

$$\mathrm{D}_{H[f]}(z) \leq \frac{L_1}{a} + \frac{A^2 + a^2}{aL_2}, \quad z \in \mathbb{C}^+.$$

Proof. We have

$$\begin{aligned}\mathrm{D}_{H[f]}(z) &\leq 2\frac{|\partial H(z)|^2 + |\overline{\partial} H(z)|^2}{\mathrm{J}_{H[f]}(z)} \\ &= \frac{\left(\frac{\partial U}{\partial x}(z)\right)^2 + \left(\frac{\partial U}{\partial y}(z)\right)^2 + \left(\frac{\partial V}{\partial x}(z)\right)^2 + \left(\frac{\partial V}{\partial y}(z)\right)^2}{\frac{\partial U}{\partial x}(z)\frac{\partial V}{\partial y}(z) - \frac{\partial U}{\partial y}(z)\frac{\partial V}{\partial x}(z)}.\end{aligned}$$

Combining this with (4.1) we obtain

$$\mathrm{D}_{H[f]}(z) \leq \frac{\frac{\partial U}{\partial x}(z)}{a} + \frac{\left(\frac{\partial U}{\partial y}(z)\right)^2 + a^2}{a\frac{\partial U}{\partial x}(z)}.$$

Applying (4.2) and the assumption $|\partial U/\partial y| \leq A$ the theorem follows. □

Theorem 5.2. *If $f \in \mathcal{F}_a$ satisfies the biLipschitz condition* (3.2) *and if f' is Dini-continuous with respect to spherical distance* (4.3)*, then*

$$\mathrm{D}_{H[f]}(z) \leq \frac{L_1}{a} + \frac{\frac{4}{\pi^2}\left(1+L_1^2\right)^2 \left[\frac{M_\varsigma}{\sqrt{1-\delta^2}} + \log\left(\frac{1+\sqrt{1-\delta^2}}{\delta}\right)\right]^2 + a^2}{aL_2}, \quad z \in \mathbb{C}^+,$$

where $\delta := \min\{\varsigma, 1/\sqrt{1+M_\varsigma}\}$ and ς, M_ς satisfy (4.3).

Proof. Theorem 4.5 gives the estimate (4.5) on $|\partial U/\partial y|$. Hence, the theorem follows from Theorem 5.1. □

Theorem 5.3. *If $f \in \mathcal{F}_a$ satisfies the biLipschitz condition* (3.2) *and f' is Hölder-continuous with respect to spherical distance* (4.7)*, then*

$$\mathrm{D}_{H[f]}(z) \leq \frac{L_1}{a} + \frac{A^2 + a^2}{aL_2}, \quad z \in \mathbb{C}^+,$$

where

$$A = \frac{\lambda(1+L_1^2)}{\pi} \begin{cases} \mathrm{B}\left(\frac{\alpha}{2}, \frac{1}{2}; 1\right), & \lambda \leq 1, \\ \mathrm{B}\left(\frac{\alpha}{2}, \frac{1}{2}; \lambda^{-1/\alpha}\right) + \frac{2}{\lambda} \log\left(\lambda^{1/\alpha} + \sqrt{\lambda^{2/\alpha} - 1}\right), & \lambda > 1 \end{cases}$$

and B *denotes the incomplete beta function and λ, α satisfy* (4.7)*.*

Proof. Corollary 4.6 gives the estimate (4.8) on $|\partial U/\partial y|$. Hence, the theorem follows from Theorem 5.1. □

References

[1] Ahlfors, L. V., *Lectures on Quasiconformal Mappings*, Van Nostrand Mathematical Studies, D. Van Nostrand, Princeton, 1966.

[2] Kalaj, D., Pavlović, M., *Boundary correspondence under quasiconformal harmonic diffeomorphisms of a half-plane*, Ann. Acad. Sci. Fenn. Ser. A. I. Math. **30** (2005), 159–165.

[3] Lehto, O., Virtanen, K. I., *Quasiconformal Mappings in the Plane*, 2nd ed., Grundlehren der matematischen Wissenschaften 126, Springer-Verlag, Berlin, 1973.

[4] Pearson, K., *Tables of the Incomplete Beta-Function*, Cambridge Univ. Press, Cambridge, 1934.

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