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Growth of polynomials whose zeros are outside a circle

Abstract. If $p(z)$ be a polynomial of degree n, which does not vanish in $|z| < k$, $k < 1$, then it was conjectured by Aziz [Bull. Austral. Math. Soc. **35** (1987), 245–256] that

$$\max_{|z|=r} |p(z)| \geq \frac{r^n + k^n}{1 + k^n} \max_{|z|=1} |p(z)| \text{ for } k^2 < r < 1.$$

In this paper, we consider the case $k < r < 1$ and present a generalization as well as improvement of the above inequality.

1. Introduction and statement of results. Let $p(z)$ be a polynomial of degree n and let $M(p,R) = \max_{|z|=R} |p(z)|$. Then it is a simple consequence of maximum modulus principle (for reference see [4, vol. I, p. 137, prob. III, 269] that

$$M(p,R) \leq R^n M(p,1) \text{ for } R \geq 1. \tag{1.1}$$

The result is best possible and equality holds for $p(z) = \alpha z^n$, where $|\alpha| = 1$.

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It was shown by Ankeny and Rivlin [1] that if $p(z) \neq 0$ in $|z| < 1$, then inequality (1.1) can be replaced by

$$M(p, R) \leq \frac{R^n + 1}{2} M(p, 1) \text{ for } R \geq 1. \tag{1.2}$$

The above inequality is best possible and equality holds for $p(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$.

As a generalization of inequality (1.2), Aziz [2] conjectured the following results.

Conjectured results. *If $p(z)$ is a polynomial of degree n, which does not vanish in $|z| < k$, then*

$$M(p, r) \geq \frac{r^n + k^n}{1 + k^n} M(p, 1) \text{ for } k^2 < r < 1,\ k < 1 \tag{1.3}$$

and

$$M(p, R) \leq \frac{R^n + k^n}{1 + k^n} M(p, 1) \text{ for } R > k^2,\ k > 1. \tag{1.4}$$

In the same paper, Aziz [2] was able to prove inequality (1.4).

In an attempt to answer the inequality (1.3) conjectured by Aziz, we have been able to prove the following result.

Theorem 1. *If $p(z)$ is a polynomial of degree n, which does not vanish in $|z| < k$, $k < 1$, then for $0 < k < r < \lambda \leq 1$*

$$M(p, r) \geq \frac{r^n + k^n}{\lambda^n + k^n} M(p, \lambda), \tag{1.5}$$

provided $|p'(z)|$ and $|q'(z)|$ attain the maximum at the same point on $|z| = 1$, where $q(z) = z^n \overline{p(1/\overline{z})}$. The result is best possible and equality holds for $p(z) = z^n + k^n$.

If we take $\lambda = 1$ in Theorem 1, then inequality (1.5) reduces to the following result, which is similar to inequality (1.3).

Corollary 1. *If $p(z)$ is a polynomial of degree n, which does not vanish in $|z| < k$, $k < 1$, then for $0 < k < r < 1$*

$$M(p, r) \geq \frac{r^n + k^n}{1 + k^n} M(p, 1), \tag{1.6}$$

provided $|p'(z)|$ and $|q'(z)|$ attain the maximum at the same point on $|z| = 1$, where $q(z) = z^n \overline{p(1/\overline{z})}$. The result is best possible and equality holds for $p(z) = z^n + k^n$.

Our next result further improves upon inequality (1.6).

Theorem 2. *If $p(z)$ is a polynomial of degree n, which does not vanish in $|z| < k$, $k < 1$, then for $0 < k < r < 1$*

$$M(p,r) \geq \left(\frac{r^n + k^n}{1+k^n}\right) M(p,1) + \left(\frac{1-r^n}{1+k^n}\right) m(p,k) \tag{1.7}$$

provided $|p'(z)|$ and $|q'(z)|$ attain the maximum at the same point on $|z| = 1$, where $q(z) = z^n\overline{p(1/\overline{z})}$ and $m(p,k) = \min_{|z|=k} |p(z)|$. The result is best possible and equality holds for $p(z) = z^n + k^n$.

2. Lemma. For the proofs of the theorems we need the following lemma due to Govil [3].

Lemma. *If $p(z)$ is a polynomial of degree n, which does not vanish in $|z| < k$, $k \leq 1$, then*

$$M(p',1) \leq \frac{n}{1+k^n} M(p,1), \tag{2.1}$$

provided $|p'(z)|$ and $|q'(z)|$ attain the maximum at the same point on $|z| = 1$, where $q(z) = z^n\overline{p(1/\overline{z})}$.

3. Proofs of the theorems.

Proof of Theorem 1. If $p(z) \neq 0$ in $|z| < k$, $k < 1$ and $0 < t < 1$, $k < t$, then $P(z) = p(tz)$ has no zero in $|z| < k/t$, $k/t < 1$. Hence applying above Lemma to the polynomial $P(z)$, we get

$$M(P',1) \leq \frac{n}{1 + k^n/t^n} M(P,1),$$

which is equivalent to

$$M(p',t) \leq \frac{nt^{n-1}}{t^n + k^n} M(p,t). \tag{3.1}$$

For $0 < r < \lambda \leq 1$ and $0 < \theta \leq 2\pi$, we have

$$p(\lambda e^{i\theta}) - p(re^{i\theta}) = \int_r^\lambda e^{i\theta} p'(te^{i\theta})dt.$$

This implies

$$\left|p(\lambda e^{i\theta}) - p(re^{i\theta})\right| \leq \int_r^\lambda \left|p'(te^{i\theta})\right| dt,$$

which gives

$$\left|p(\lambda e^{i\theta})\right| \leq \left|p(re^{i\theta})\right| + \int_r^\lambda \left|p'(te^{i\theta})\right| dt,$$

which further implies

$$M(p,\lambda) \leq M(p,r) + \int_r^\lambda M(p',t)dt.$$

Combining the above inequality with (3.1), we get

$$M(p,\lambda) \leq M(p,r) + \int_r^\lambda \frac{nt^{n-1}}{t^n+k^n} M(p,t)dt. \tag{3.2}$$

If we choose

$$\phi(\lambda) = M(p,r) + \int_r^\lambda \frac{nt^{n-1}}{t^n+k^n} M(p,t)dt,$$

then

$$\phi'(\lambda) = \frac{n\lambda^{n-1}}{\lambda^n+k^n} M(p,\lambda),$$

and inequality (3.2), gives

$$\phi'(\lambda) - \frac{n\lambda^{n-1}}{\lambda^n+k^n}\phi(\lambda) \leq 0.$$

Multiplying the above inequality by $(\lambda^n + k^n)^{-1}$, we get

$$\frac{d}{d\lambda}\left\{(\lambda^n+k^n)^{-1}\phi(\lambda)\right\} \leq 0,$$

which implies that $(\lambda^n+k^n)^{-1}\phi(\lambda)$ is a non-increasing function of λ in (0,1). Therefore for $0 < k < r < \lambda \leq 1$, we have

$$\phi(r) \geq \left(\frac{r^n+k^n}{\lambda^n+k^n}\right)\phi(\lambda).$$

Now since $\phi(r) = M(p,r)$ and $\phi(\lambda) \geq M(p,\lambda)$, we get

$$M(p,r) \geq \left(\frac{r^n+k^n}{\lambda^n+k^n}\right) M(p,\lambda).$$

Which completes the proof of Theorem 1. □

Proof of Theorem 2. If $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k < 1$ and if $m(p,k) = \min_{|z|=k}|p(z)|$, then for every α with $|\alpha| < 1$, the polynomial $p(z) - \alpha m(p,k)$ has no zero in $|z| < k$, $k < 1$. This result is clear if $p(z)$ has a zero on $|z| = k$, for then $m(p,k) = 0$ and therefore $p(z) - \alpha m(p,k) = p(z)$. In case $p(z)$ has no zero on $|z| = k$, then for every α with $|\alpha| < 1$, we have $|p(z)| > |\alpha|\, m(p,k)$ on $|z| = k$ and on applying Rouche's theorem the result follows. Thus $p(z) - \alpha m(p,k)$ has no zero in $|z| < k$, $k < 1$ and hence, applying inequality (1.6) to $p(z) - \alpha m(p,k)$, we get

$$M(p - \alpha m(p,k), r) \geq \left(\frac{r^n+k^n}{1+k^n}\right) M(p-\alpha m(p,k), 1),$$

which implies

$$M(p - \alpha m(p,k), r) \geq \left(\frac{r^n+k^n}{1+k^n}\right)\left\{M(p,1) - |\alpha|\, m(p,k)\right\}. \tag{3.3}$$

Now choosing argument of α on left hand side of (3.3), we get

$$M(p,r) - |\alpha|\, m(p,k) \geq \left(\frac{r^n + k^n}{1 + k^n}\right) \left\{M(p,1) - |\alpha|\, m(p,k)\right\},$$

which is equivalent to

$$M(p,r) \geq \left(\frac{r^n + k^n}{1 + k^n}\right) M(p,1) + \left(\frac{1 - r^n}{1 + k^n}\right) |\alpha|\, m(p,k)$$

and letting $|\alpha| \to 1$, we get

$$M(p,r) \geq \left(\frac{r^n + k^n}{1 + k^n}\right) M(p,1) + \left(\frac{1 - r^n}{1 + k^n}\right) |\alpha|\, m(p,k).$$

This completes the proof of Theorem 2. $\square$

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