DOI: 10.2478/v10062-008-0004-4

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXII, 2008 SECTIO A 37–48

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Continuity of the quenching time in a semilinear parabolic equation

Abstract. In this paper, we consider the following initial-boundary value problem

$$\begin{cases} u_t = \Delta u - u^{-p} \quad \text{in} \quad \Omega \times (0,T), \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \quad \text{in} \quad \overline{\Omega}, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p > 0$, Δ is the Laplacian, ν is the exterior normal unit vector on $\partial\Omega$. Under some assumptions, we show that the solution of the above problem quenches in a finite time and estimate its quenching time. We also prove the continuity of the quenching time as a function of the initial data u_0. Finally, we give some numerical results to illustrate our analysis.

1. Introduction.

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider the following initial-boundary value problem

$$u_t = \Delta u - u^{-p} \quad \text{in} \quad \Omega \times (0,T), \tag{1}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{2}$$

$$u(x,0) = u_0(x) > 0 \quad \text{in} \quad \overline{\Omega}, \tag{3}$$

2000 *Mathematics Subject Classification.* 35B40, 35B50, 35K60, 65M06.

Key words and phrases. Quenching, nonlinear parabolic equation, numerical quenching time.

where $p > 0$, Δ is the Laplacian, ν is the exterior normal unit vector on $\partial\Omega$. The initial data $u_0 \in C^1(\overline{\Omega})$ and satisfies the compatibility conditions. Here $(0,T)$ is the maximal time interval of existence of the solution u. The time T may be finite or infinite. When T is infinite, we say that the solution u exists globally. When T is finite, the solution u develops a singularity in a finite time, namely

$$\lim_{t\to T} u_{\min}(t) = 0,$$

where $u_{\min}(t) = \min_{x\in\overline{\Omega}} u(x,t)$. In this last case, we say that the solution u quenches in a finite time and the time T is called the quenching time of the solution u. Thus we have $u(x,t) > 0$ in $\overline{\Omega} \times [0,T)$. Solutions of semilinear parabolic equations which quench in a finite time have been the subject of investigation of many authors (see [2]–[4], [6], [7], [10], [11], [13], [14], [20], [24], [25], [27]–[29] and the references cited therein). In particular, in [7], the problem (1)–(3) has been studied. The local in time existence of a classical solution has been proved and this solution is unique (see [7]). It is also shown that the solution of (1)–(3) quenches in a finite time and its quenching time has been estimated (see [7]). In this paper, we are interested in the continuity of the quenching time as a function of the initial data u_0. More precisely, we consider the following initial-boundary value problem

$$v_t = \Delta v - v^{-p} \quad \text{in} \quad \Omega \times (0,T_h), \tag{4}$$

$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0,T_h), \tag{5}$$

$$v(x,0) = v_0(x) > 0 \quad \text{in} \quad \overline{\Omega}, \tag{6}$$

where $v_0(x) = u_0(x) + h(x)$, $h \in C^1(\overline{\Omega})$, $\frac{\partial h}{\partial \nu} = 0$ on $\partial\Omega$, $h(x) \geq 0$ in $\overline{\Omega}$. Here $(0,T_h)$ is the maximal time interval on which the solution v of (4)–(6) exists. When T_h is finite, we say that the solution v of (4)–(6) quenches in a finite time and the time T_h is called the quenching time of the solution v. From the maximum principle, we have $v \geq u$ as long as all of them are defined. We deduce that $T_h \geq T$. In the present paper, we prove that if $\|h\|_\infty$ is small enough, then the solution v of (4)–(6) quenches in a finite time and its quenching time T_h goes to T as $\|h\|_\infty$ goes to zero where T is the quenching time of the solution u of (1)–(3) and $\|h\|_\infty = \sup_{x\in\overline{\Omega}} |h(x)|$. Similar results have been obtained in [5], [8], [16], [19], [18], [21]–[23], [30] where the authors have considered the phenomenon of blow-up (we say that a solution blows up in a finite time if it reaches the value infinity in a finite time). The rest of the paper is organized as follows. In the next section, under some assumptions, we show that the solution v of (4)–(6) quenches in a finite time and estimate its quenching time. In the third section, we prove the continuity of the quenching time and finally in the last section, we give some numerical results to illustrate our analysis.

2. Quenching time. In this section, under some assumptions, we show that the solution v of (4)–(6) quenches in a finite time and estimate its quenching time.

Using an idea of Friedman and McLeod in [15], we may prove the following result.

Theorem 2.1. *Suppose that there exists a constant $A \in (0,1]$ such that the initial data at* (6) *satisfies*

$$\Delta v_0(x) - v_0^{-p}(x) \leq -Av_0^{-p}(x) \quad in \quad \Omega. \tag{7}$$

Then, the solution v of (4)–(6) *quenches in a finite time T_h which obeys the following estimate*

$$T_h \leq \frac{\|v_0\|_{\inf}^{p+1}}{A(p+1)},$$

where $\|v_0\|_{\inf} = \min_{x\in\overline{\Omega}} v_0(x)$.

Proof. Since $(0, T_h)$ is the maximal time interval of existence of the solution v, our aim is to show that T_h is finite and satisfies the above inequality. Introduce the function $J(x,t)$ defined as follows

$$J(x,t) = v_t(x,t) + Av^{-p}(x,t) \quad \text{in} \quad \overline{\Omega} \times (0, T_h).$$

A straightforward computation yields

$$J_t - \Delta J = (v_t - \Delta v)_t - Apv^{-p-1}v_t - A\Delta v^{-p} \quad \text{in} \quad \Omega \times [0, T_h). \tag{8}$$

By a direct calculation, we observe that

$$\Delta v^{-p} = p(p+1)v^{-p-2}|\nabla v|^2 - pv^{-p-1}\Delta v,$$

which implies that $\Delta v^{-p} \geq -pv^{-p-1}\Delta v$. Using this estimate and (8), we arrive at

$$J_t - \Delta J \leq (v_t - \Delta v)_t - Apv^{-p-1}(v_t - \Delta v) \quad \text{in} \quad \Omega \times (0, T_h). \tag{9}$$

It follows from (4) and (9) that

$$J_t - \Delta J \leq pv^{-p-1}v_t + Apv^{-2p-1} \quad \text{in} \quad \Omega \times (0, T_h).$$

Taking into account the expression of J, we find that

$$J_t - \Delta J \leq pv^{-p-1}J \quad \text{in} \quad \Omega \times (0, T_h).$$

We also have

$$\frac{\partial J}{\partial \nu} = \left(\frac{\partial v}{\partial \nu}\right)_t - Apv^{-p-1}\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0, T_h)$$

and due to (7), we discover that

$$J(x,0) = \Delta v_0(x) - v_0^{-p}(x) + Av_0^{-p}(x) \leq 0 \quad \text{in} \quad \Omega.$$

From the maximum principle, we have

$$J(x,t) \leq 0 \quad \text{in} \quad \Omega \times (0, T_h),$$

which implies that

$$v_t(x,t) + Av^{-p}(x,t) \leq 0 \quad \text{in} \quad \Omega \times (0, T_h).$$

This estimate may be rewritten in the following manner

$$v^p dv \leq -Adt \quad \text{in} \quad \Omega \times (0, T_h). \tag{10}$$

Integrate the above inequality over $(0, T_h)$ to obtain

$$T_h \leq \frac{(v(x,0))^{p+1}}{A(p+1)} \quad \text{for} \quad x \in \Omega. \tag{11}$$

We deduce that

$$T_h \leq \frac{\|v_0\|_{inf}^{p+1}}{A(p+1)}. \tag{12}$$

Consequently, v quenches at the time T_h because the quantity on the right hand side of the above inequality is finite and the proof is complete. □

Remark 2.1. Let $t_0 \in (0, T_h)$. Integrating the inequality in (10) from t_0 to T_h, we get

$$T_h - t_0 \leq \frac{(v(x,t_0))^{p+1}}{A(p+1)} \quad \text{for} \quad x \in \Omega.$$

We deduce that

$$T_h - t_0 \leq \frac{(v_{\min}(t_0))^{p+1}}{A(p+1)}.$$

3. Continuity of the quenching time. In this section, under some assumptions, we show that the solution v of (4)–(6) quenches in a finite time and its quenching time goes to that of the solution u of (1)–(3) when $\|h\|_\infty$ goes to zero.

Firstly, we show that the solution v approaches the solution u in $\overline{\Omega} \times [0, T-\tau]$ with $\tau \in (0,T)$ when $\|h\|_\infty$ tends to zero. This result is stated in the following theorem.

Theorem 3.1. *Let u be the solution of* (1)–(3)*. Suppose that $u \in C^{2,1}(\overline{\Omega} \times [0, T-\tau])$ and $\min_{t\in[0,T-\tau]} u_{\min}(t) = \alpha > 0$ with $\tau \in (0,T)$. Then, the problem* (4)–(6) *admits a unique solution $v \in C^{2,1}(\overline{\Omega} \times [0, T_h))$ and the following relation holds*

$$\sup_{t\in[0,T-\tau]} \|v(\cdot,t) - u(\cdot,t)\|_\infty = 0(\|h\|_\infty) \quad as \quad \|h\|_\infty \to 0.$$

Proof. The problem (4)–(6) has for each h, a unique solution $v \in C^{2,1}(\overline{\Omega} \times [0, T_h))$. In the introduction of the paper, we have seen that $T_h \geq T$. Let $t(h) \leq T$ the greatest value of $t > 0$ such that

$$\|v(\cdot,t) - u(\cdot,t)\|_\infty \leq \frac{\alpha}{2} \quad \text{for} \quad t \in (0, t(h)). \tag{13}$$

By a direct calculation, we see that $\|v(\cdot,0)-u(\cdot,0)\|_\infty = \|v_0-u_0\|_\infty = \|h\|_\infty$, which implies that $\|v(\cdot,0)-u(\cdot,0)\|_\infty$ tends to zero as $\|h\|_\infty$ goes to zero. Due to this fact, we deduce that $t(h) > 0$ for $\|h\|_\infty$ sufficiently small. By the triangle inequality, we obtain

$$v_{\min}(t) \geq u_{\min}(t) - \|v(\cdot,t)-u(\cdot,t)\|_\infty \quad \text{for} \quad t \in (0,t(h)),$$

which leads us to

$$v_{\min}(t) \geq \alpha - \frac{\alpha}{2} = \frac{\alpha}{2} \quad \text{for} \quad t \in (0,t(h)).$$

Introduce the function $e(x,t)$ defined as follows

$$e(x,t) = v(x,t) - u(x,t) \quad \text{in} \quad \overline{\Omega} \times [0,t(h)).$$

A routine computation reveals that

$$e_t - \Delta e = p\theta^{-p-1} e \quad \text{in} \quad \Omega \times (0,t(h)),$$

$$\frac{\partial e}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0,t(h)),$$

$$e(x,0) \geq h(x) \quad \text{in} \quad \Omega,$$

where θ is an intermediate value between u and v. Let

$$z(x,t) = e^{(L+1)t}\|h\|_\infty \quad \text{in} \quad \overline{\Omega} \times [0,T],$$

where $L = p(\frac{\alpha}{2})^{-p-1}$. It is not hard to see that $L = p(\frac{\alpha}{2})^{-p-1} \geq p\theta^{-p-1}$. Thanks to this observation, a straightforward calculation yields

$$z_t - \Delta z \geq p\theta^{-p-1} z \quad \text{in} \quad \Omega \times (0,t(h)),$$

$$\frac{\partial z}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0,t(h)),$$

$$z(x,0) \geq e(x,0) \quad \text{in} \quad \Omega.$$

It follows from the maximum principle that

$$z(x,t) \geq e(x,t) \quad \text{in} \quad \Omega \times (0,t(h)).$$

By the same way, we also prove that

$$z(x,t) \geq -e(x,t) \quad \text{in} \quad \Omega \times (0,t(h)),$$

which implies that

$$\|e(\cdot,t)\|_\infty \leq e^{(L+1)t}\|h\|_\infty \quad \text{for} \quad t \in (0,t(h)).$$

Let us show that $t(h) = T$. Suppose that $t(h) < T$. From (13), we obtain

$$\frac{\alpha}{2} = \|v(\cdot,t(h)) - u(\cdot,t(h))\|_\infty \leq e^{(L+1)T}\|h\|_\infty.$$

Since the term on the right hand side of the above inequality goes to zero as $\|h\|_\infty$ goes to zero, we deduce that $\frac{\alpha}{2} \leq 0$, which is impossible. Consequently, $t(h) = T$ and the proof is complete. □

Now, we are in a position to prove the main result of the paper.

Theorem 3.2. *Suppose that the problem* (1)–(3) *has a solution u which quenches at the time T and $u \in C^{2,1}(\overline{\Omega} \times [0,T))$. Under the assumption of Theorem 2.1, the problem* (4)–(6) *has a solution v which quenches in a finite time T_h and the following relation holds*

$$\lim_{\|h\|_\infty \to 0} T_h = T.$$

Proof. Let $\varepsilon > 0$. There exists $\rho > 0$ such that

$$\frac{y^{p+1}}{A(p+1)} \leq \frac{\varepsilon}{2}, \quad 0 \leq y \leq \rho. \tag{14}$$

Since u quenches in a finite time T, there exists $T_0 \in (T - \frac{\varepsilon}{2}, T)$ such that

$$0 < u_{\min}(t) < \frac{\rho}{2} \quad \text{for} \quad t \in [T_0, T).$$

Set $T_1 = \frac{T_0+T}{2}$. It is not hard to see that

$$u_{\min}(t) > 0 \quad \text{for} \quad t \in [0, T_1].$$

From Theorem 3.1, the problem (4)–(6) has a solution v and we get

$$\|v(\cdot,t) - u(\cdot,t)\|_\infty < \frac{\rho}{2} \quad \text{for} \quad t \in [0, T_1],$$

which implies that $\|v(\cdot,T_1) - u(\cdot,T_1)\|_\infty \leq \frac{\rho}{2}$. An application of the triangle inequality leads us to

$$v_{\min}(T_1) \leq \|v(\cdot,T_1) - u(\cdot,T_1)\|_\infty + u_{\min}(T_1) \leq \frac{\rho}{2} + \frac{\rho}{2} = \rho.$$

From Theorem 2.1, v quenches at the time T_h. We deduce from Remark 2.1 and (14) that

$$0 \leq T_h - T = T_h - T_1 + T_1 - T \leq \frac{(v_{\min}(T_1))^{p+1}}{A(p+1)} + \frac{\varepsilon}{2} \leq \varepsilon,$$

and the proof is complete. □

4. Numerical results. In this section, we give some computational experiments to confirm the theory given in the previous section. We consider the radial symmetric solution of the following initial-boundary value problem

$$u_t = \Delta u - u^{-p} \quad \text{in} \quad B \times (0,T),$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad S \times (0,T),$$

$$u(x,0) = u_0(x) \quad \text{in} \quad B,$$

where $B = \{x \in \mathbb{R}^N;\ \|x\| < 1\}$, $S = \{x \in \mathbb{R}^N; \|x\| = 1\}$. The above problem may be rewritten in the following form

$$u_t = u_{rr} + \frac{N-1}{r} u_r - u^{-p}, \quad r \in (0,1), \quad t \in (0,T), \tag{15}$$

$$u_r(0,t) = 0, \quad u_r(1,t) = 0, \quad t \in (0,T), \tag{16}$$

$$u(r,0) = \varphi(r), \quad r \in (0,1). \tag{17}$$

Here, we take $\varphi(r) = \frac{2+cos(\pi r)}{4} + \varepsilon(1+\cos(\pi r))$ where $\varepsilon \in [0,1]$. We start with the construction of an adaptive scheme as follows. Let I be a positive integer and let $h = 1/I$. Define the grid $x_i = ih$, $0 \le i \le I$ and approximate the solution u of (15)–(17) by the solution $U_h^{(n)} = (U_0^{(n)}, \dots, U_I^{(n)})^T$ of the following explicit scheme

$$\begin{aligned}
\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} &= N \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} - (U_0^{(n)})^{-p-1} U_0^{(n+1)}, \\
\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} &= \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \frac{(N-1)}{ih} \frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h} \\
&\quad - (U_i^{(n)})^{-p-1} U_i^{(n+1)}, \quad 1 \le i \le I-1, \\
\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} &= N \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2} - (U_I^{(n)})^{-p-1} U_I^{(n+1)}, \\
U_i^{(0)} &= \varphi_i, \quad 0 \le i \le I,
\end{aligned}$$

where $\Delta t_n = \min\left\{ \frac{(1-h^2)h^2}{2N}, h^2 \|U_h^{(n)}\|_{\text{inf}}^{p+1} \right\}$ with $\|U_h^{(n)}\|_{\text{inf}} = \min_{0 \le i \le I} U_i^{(n)}$. Let us notice that the restriction on the time step ensures the positivity of the discrete solution. We also approximate the solution u of (15)–(17) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\begin{aligned}
\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} &= N \frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2} - (U_0^{(n)})^{-p-1} U_0^{(n+1)} \\
\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} &= \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} + \frac{(N-1)}{ih} \frac{U_{i+1}^{(n+1)} - U_{i-1}^{(n+1)}}{2h} \\
&\quad - (U_i^{(n)})^{-p-1} U_i^{(n+1)}, \quad 1 \le i \le I-1, \\
\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} &= N \frac{2U_{I-1}^{(n+1)} - 2U_I^{(n+1)}}{h^2} - (U_I^{(n)})^{-p-1} U_I^{(n+1)}, \\
U_i^{(0)} &= \varphi_i, \quad 0 \le i \le I,
\end{aligned}$$

where $\Delta t_n = h^2 \|U_h^{(n)}\|_{\text{inf}}^{p+1}$.

Let us again remark that for the above implicit scheme, the existence and positivity of the discrete solution is also guaranteed using standard methods (see for instance [6]). It is not hard to see that $u_{xx}(1,t)=\lim_{r\to 1}\frac{u_r(r,t)}{r}$ and $u_{xx}(0,t)=\lim_{r\to 0}\frac{u_r(r,t)}{r}$. Hence, if $r=0$ and $r=1$, we see that

$$u_t(0,t)=Nu_{rr}(0,t)-u^{-p}(0,t),\quad t\in(0,T),$$

$$u_t(1,t)=Nu_{rr}(1,t)-u^{-p}(1,t),\quad t\in(0,T).$$

These observations have been taken into account in the construction of our schemes when $i=0$ and $i=I$. We need the following definition.

Definition 4.1. We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme quenches in a finite time if

$$\lim_{n\to\infty}\|U_h^{(n)}\|_{\inf}=0$$

and the series $\sum_{n=0}^{\infty}\Delta t_n$ converges. The quantity $\sum_{n=0}^{\infty}\Delta t_n$ is called the numerical quenching time of the discrete solution $U_h^{(n)}$.

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical quenching time $T^n=\sum_{j=0}^{n-1}\Delta t_j$ which is computed at the first time when

$$\Delta t_n=|T^{n+1}-T^n|\leq 10^{-16}.$$

The order(s) of the method is computed from

$$s=\frac{\log((T_{4h}-T_{2h})/(T_{2h}-T_h))}{\log(2)}.$$

Numerical experiments for $p=1$, $N=2$.

First case: $\varepsilon=0$.

Table 1: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

I	T^n	n	CPU_t	s
16	0.055728	1921	4	-
32	0.054213	7048	27	-
64	0.053488	25371	195	1.06
128	0.053127	82924	1836	1.00

Table 2: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method.

I	T^n	n	CPU_t	s
16	0.054900	1908	5	-
32	0.053847	7020	29	-
64	0.053319	28007	429	1.00
128	0.053111	87262	1965	0.99

Second case: $\varepsilon = 1/50$.

Table 3: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

I	T^n	n	CPU_t	s
16	0.058875	1946	5	-
32	0.057148	7152	28	-
64	0.056321	25803	215	0.88
128	0.056012	102818	20237	0.82

Table 4: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method.

I	T^n	n	CPU_t	s
16	0.057270	1934	5	-
32	0.055651	7100	30	-
64	0.054877	25590	204	1.05
128	0.054525	98037	1203	1.15

Third case: $\varepsilon = 1/100$.

Table 5: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

I	T^n	n	CPU_t	s
16	0.057270	1934	5	-
32	0.055651	7100	30	-
64	0.054877	25590	204	1.05
128	0.054525	98037	1203	1.15

Table 6: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method.

I	T^n	n	CPU_t	s
16	0.056356	2009	6	-
32	0.055248	7436	30	-
64	0.054689	27070	328	0.98
128	0.054426	98873	1318	1.08

Remark 4.1. If we consider the problem (15)–(17) in the case where the initial data $\varphi(r) = \frac{2+\cos(\pi r)}{4}$ and $p = 1$, we see that the numerical quenching time of the discrete solution for the explicit scheme or the implicit scheme is slightly equal 0.053 (see Tables 1 and 2). We observe from Tables 3, 4, 5 and 6 that if the above initial data increases slightly, then the numerical quenching time also increases slightly. This result confirms the theory established in the previous section.

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Received April 11, 2008