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On some definition of expectation of random element in metric space

Abstract. We are dealing with definition of expectation of random elements taking values in metric space given by I. Molchanov and P. Teran in 2006. The approach presented by the authors is quite general and has some interesting properties. We present two kinds of new results:

- conditions under which the metric space is isometric with some real Banach space;
- conditions which ensure "random identification" property for random elements and almost sure convergence of asymptotic martingales.

1. Introduction. Expectation of real random variable is basic characteristic which is used in probability theory. There is extension to random elements taking values in Banach spaces – a Bochner integral. There is the following question:

"What about metric spaces without linear structure?"

There is a lot of solutions of this problem. Probably the first (1949) who gave a concept of mathematical expectation of a random element with values in a metric space was Doss [5].

After this paper many other authors dealt with the problem of defining expectation, there were many different concepts and solutions of this problem in different kinds of metric spaces: Fréchet [6] and Pick [12] considered

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expectation defined as a minimizer of the "variance"; Herer [7], [8], [9] uplifted the definition given by Doss to random sets and gave new definition of expectation in spaces with negative curvature.

There is a lot of results concerning martingales in metric spaces and almost sure convergence of martingales (see Beneš [3], Herer [7], [9], Sturm [13]).

We are dealing with definition of expectation given in [14]. The authors presented new and quite general approach basing on properties which usual expectation possess. The results seem to be interesting especially because they in some sense unify different ways of defining expectation. Moreover, after slight modification the definition is restrictive enough to prove almost sure convergence of strongly tight asymptotic martingales, which is false in general.

In the beginning we present the definition of convex combination, convex combination space, the definition of random elements and their expectation and conditional expectation which can be found in [14]. Section 3 presents some problems which can be encountered when one is dealing with definition of expectation in non-linear metric space. The final result of this part is characterization theorem. Next part is devoted to the "random identification property". Section 5 gives some background in the theory of asymptotic martingales and contains the result concerning almost sure convergence of amarts. Finally Section 6 provides some examples illustrating the results.

2. Convex combinations, integrability and expectation. We will present a short introduction to approach presented by I. Molchanov and P. Teran in [14].

Let $(\mathbb{E}, d)$ be a separable, complete metric space, endowed with a *convex combination operation* which for all $n \geq 2$, numbers $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ satisfying $\sum_{i=1}^n \lambda_i = 1$, and all $u_1, u_2, \ldots, u_n \in \mathbb{E}$ produces an element of $\mathbb{E}$ denoted by $[\lambda_1 u_1; \lambda_2 u_2; \ldots; \lambda_n u_n]$ or $[\lambda_i u_i]_{i=1}^n$. Assume that $[1u] = u$ for every $u \in \mathbb{E}$ and the following properties hold.

(i) $[\lambda_i u_i]_{i=1}^n = [\lambda_{\sigma(i)} u_{\sigma(i)}]_{i=1}^n$ for any permutation σ of $\{1, 2, \ldots, n\}$;

(ii) $[\lambda_i u_i]_{i=1}^{n+2} = [\lambda_1 u_1; \lambda_2 u_2; \ldots; (\lambda_{n+1} + \lambda_{n+2}) \left[\frac{\lambda_{n+j}}{\lambda_{n+1}+\lambda_{n+2}} u_{n+j}\right]_{j=1}^2\Big]$;

(iii) for any sequence of numbers $\lambda^{(k)} \to \lambda \in (0,1)$; $k \to \infty$

$$\left[\lambda^{(k)} u; \left(1 - \lambda^{(k)}\right) v\right] \to [\lambda u; (1-\lambda) v];\ k \to \infty;$$

(iv) $\forall(\lambda \in (0,1))\ \forall(u_1, u_2, v_1, v_2 \in \mathbb{E})$:

$$d([\lambda u_1; (1-\lambda) v_1], [\lambda u_2; (1-\lambda) v_2]) \leq \lambda d(u_1, u_2) + (1-\lambda) d(v_1, v_2);$$

(v) for each $u \in \mathbb{E}$, there exists $\lim\limits_{n\to\infty} [n^{-1} u]_{i=1}^n$, which will be denoted by Ku.

Spaces satisfying conditions given above will be called convex combination spaces.

Let $(\Omega, \mathcal{A}, P)$ be a non-atomic probability space. We will use the following notation:

- A mapping $X\colon \Omega \to \mathbb{E}$ such that there is a measurable partition $\{\Omega_1, \dots, \Omega_m\}$ of Ω such that X takes a constant value u_i on each non-null set Ω_i, for $i = 1, 2, \dots, m$ is called a simple random element.
- For a simple random element X taking values $x_1, x_2, \dots, x_n$ with probabilities $p_1, p_2, \dots, p_n$ respectively, define the expectation of random element X by

$$EX = [p_i K x_i]_{i=1}^n.$$

- A random element X is called *integrable* if $d(u_0, X)$ is integrable real-valued random variable for some $u_0 \in \mathbb{E}$.

Remark 1. Any integrable random element may be approximated by a convergent sequence of simple, integrable random elements and therefore the definition of expectation may be extended to the set of all integrable random elements.

3. Characterization of Banach spaces. We will start this section with a simple lemma.

Lemma 1. *Let A be a measurable set such that $0 < P(A) < 1$ and X be an integrable random element, then*

$$EX = [P(A)E(X|A); P(A')E(X|A')].$$

Proof. Let us consider first the case of simple, integrable random element X. Assume that X takes values u_i with probabilities p_i respectively for $i = 1, \dots, n$. Using property (ii) and simple computations we have:

$$\begin{aligned} EX &= [p_i K u_i]_{i=1}^n \\ &= \left[P(A) \left[\frac{P(X = u_i \wedge A)}{P(A)} K u_i \right]_{i=1}^n ; P(A') \left[\frac{P(X = u_i \wedge A')}{P(A')} K u_i \right]_{i=1}^n \right] \\ &= \left[P(A) \left[P(X = u_i | A) K u_i \right]_{i=1}^n ; P(A') \left[P(X = u_i | A') K u_i \right]_{i=1}^n \right] \\ &= [P(A)E(X|A); P(A')E(X|A')]. \end{aligned}$$

If the random element X is not simple we may use the approximation and obtain the same result. $\square$

Note that this lemma is not obvious if this property holds in non-linear space. There is the following example:

Example 1 (Sturm [13]). Consider $\mathbb{E} = \{1, 2, 3\} \times [0, \infty)$ with metric:

$$d((i; x), (j; y)) = \begin{cases} |x - y|, & i = j, \\ x + y, & i \neq j. \end{cases}$$

This is an example of global non-positive curvature metric space.

Define a random element X:

$$X(\omega) = (1,1), \text{ for } \omega \in A_1, \quad X(\omega) = (2,1), \text{ for } \omega \in A_2,$$
$$X(\omega) = (3,1), \text{ for } \omega \in A_3,$$

where $P(A_1) = P(A_2) = P(A_3) = \frac{1}{3}$, $A_1 \cup A_2 \cup A_3 = \Omega$. Consider expectation in the sense of Fréchet i.e.

$$E_F X = \left\{ a \in \mathbb{E} : \; Ed^2(a, X) = \min_{u \in \mathbb{E}} Ed^2(u, X) \right\}.$$

It is quite obvious that

$$E_F(X) = (., 0).$$

Note that we have also for any $i = 1, 2, 3$,

$$E_F(X|A_i') = (., 0), \quad E_F(X|A_i) = (i, 1);$$
$$E_F(X|A_i)P(A_i) + E_F(X|A_i')P(A_i') = (i, 1/3) \neq E_F(X).$$

Remark 2. This example shows that the definition of expectation does not give natural convex combination as is written in [14]. The property (ii) may not be satisfied if we define the convex combination operation $[p_1, x_i]_{i=1}^n$ as an expectation of random element taking values x_i with probability p_i respectively.

The reason of this fact is that not for all definitions of expectation in metric spaces Lemma 1 is true. Fréchet expectation is one of the examples. Moreover I. Molchanov and P. Teran mentioned that strong law of large numbers proved by K. T. Sturm for random elements taking values in global NPC spaces (see [13]) follows from their strong law of large numbers but this is true only in case when Lemma 1 holds but this case is not really interesting because then we have the following:

Theorem 1 (see [2]). *Let E be a Fréchet expectation operator defined on externally convex, global NPC space $(\mathbb{E}, d)$. If the condition $EX = E(E(X|Y))$ is satisfied for any square integrable random element X and any random element Y taking two values then $(\mathbb{E}, d)$ is isometric with some strictly convex real Banach space.*

4. Random identification property. Let us slightly modify the definition of convex combination namely replace condition (iv) by:

(iv') $\forall(\lambda \in (0,1))\, \forall(u, v, w \in \mathbb{E};\; d(u,v) > 0)$:

$$0 < d([\lambda u; (1-\lambda)w], [\lambda v; (1-\lambda)w]) \leq \lambda d(u,v);$$

and add the following assumption:

(vi) $\forall(u, v \in \mathbb{E})$: $\quad d(u,v) > 0 \Rightarrow d(Ku, Kv) > 0.$

Spaces satisfying conditions (i)–(vi) and (iv') will be called smooth convex combination spaces.

In such spaces it is possible to prove "random identification property" which is crucial point in the proofs of theorems concerning almost sure convergence of asymptotic martingales.

To prove this property we will start with the following:

Lemma 2. *Let $\bar{B}_r(u) = \{x \in \mathbb{E};\ d(x,u) \le r\}$ denote the closed ball in $\mathbb{E}$. For any $u_1, u_2, \ldots, u_n \in \bar{B}_r(u)$ and any $\lambda_1, \lambda_2, \ldots, \lambda_n \ge 0$, $\sum_{i=1}^n \lambda_i = 1$,*

$$[\lambda_i K u_i]_{i=1}^n \in \bar{B}_r(Ku).$$

Proof. Let $x = [\lambda_i K u_i]_{i=1}^n$. We have

$$\begin{aligned} d(x, Ku) &= d([\lambda_i K u_i]_{i=1}^n, [\lambda_i K u]_{i=1}^n) \\ &\le \sum_{i=1}^n \lambda_i d(Ku_i, Ku) \le \sum_{i=1}^n \lambda_i d(u_i, u) \le r. \end{aligned}$$ □

Corollary 1. *Let X be a random element. If there is a closed ball $\bar{B}_r(u)$ such that $P(X \in \bar{B}_r(u)) = 1$, then X is integrable and $EX \in \bar{B}_r(Ku)$.*

Proof. If X is a simple random element, then this result is simple consequence of Lemma 2. If X is not simple, then there is a sequence of random elements satisfying conditions of Lemma 2 with the same ball and the result follows by approximation and closeness of the ball $\bar{B}_r(Ku)$. □

Theorem 2 (Random identification). *Let $X_1, X_2 \in L^1_{\mathbb{E}}$. If for any random variable $\tau : \Omega \to \{1, 2\}$, $EX_\tau = EX_1$, then $X_1 = X_2$ (a.s.).*

Proof. First suppose that X_1 and X_2 are simple random elements such that

$$P\{\omega; X_1(\omega) \ne X_2(\omega)\} > 0.$$

Let $A = \{\omega; X_1(\omega) \ne X_2(\omega)\}$. There is a subset $B \subset A$ such that for all $\omega \in B:\ X_1(\omega) = u,\ X_2(\omega) = v$, where $d(u,v) > 0$ and $P(B) = \delta > 0$.

Let

$$\tau(\omega) = \begin{cases} 1; & \omega \notin B; \\ 2; & \omega \in B. \end{cases}$$

We will show that $d(EX_\tau, EX_1) > 0$. Let

$$X_1(\omega) = \begin{cases} x_1; & \omega \in A_1; \\ \vdots & \\ x_n; & \omega \in A_n; \\ u; & \omega \in B; \end{cases}$$

where $\bigcup_{i=1}^{n} A_i \cup B = \Omega$. We have

$$\begin{aligned} EX_1 &= [P(A_1)K(x_1); P(A_2)Kx_2; \ldots; P(A_n)Kx_n; \delta Ku] \\ &= [(1-\delta)x; \delta Ku], \\ EX_\tau &= [(1-\delta)x; \delta Kv] \end{aligned}$$

where $x = \left[\frac{P(A_i)}{1-\delta} Kx_i\right]_{i=1}^{n}$. Assume that

$$0 = d(EX_1, EX_\tau) = d([(1-\delta)x; \delta Ku], [(1-\delta)x; \delta Ku]).$$

By (iv) it implies that $d(Ku, Kv) = 0$ but by (vi) it gives $d(u, v) = 0$ and this contradicts the construction of τ and ends the proof in this case.

If $X_1, X_2 \in L^1_{\mathbb{E}}$ are any random elements, such that $P(X_1 \neq X_2) > 0$, then by separability of $\mathbb{E}$ there are elements $u, v \in \mathbb{E}$ and a set $C \subset \Omega$ such that $d(u, v) > 0$, $P(C) > 0$,

$$X_1 \in \bar{B}_\varepsilon(u);\ X_2 \in \bar{B}_\varepsilon(v) \quad \text{for } \omega \in C \text{ and some } \varepsilon < d(Ku, Kv)/2.$$

Define

$$\tau = \begin{cases} 1; & \omega \notin C; \\ 2; & \omega \in C. \end{cases}$$

Using Lemma 1 we obtain

$$\begin{aligned} EX_1 &= [P(C)E(X_1|C); P(C')E(X_1|C')]; \\ EX_\tau &= [P(C)E(X_2|C); P(C')E(X_1|C')]. \end{aligned}$$

Note that by Lemma 2 we have

$$E(X_1|C) \in \bar{B}_\varepsilon(K(u)); \quad E(X_2|C) \in \bar{B}_\varepsilon(K(v)).$$

Assume that

$$\begin{aligned} 0 &= d(EX_1, EX_\tau) \\ &= d\left([P(C)E(X_1|C); P(C')E(X_1|C')], [P(C)E(X_2|C); P(C')E(X_1|C')]\right). \end{aligned}$$

Similarly to the proof for simple random elements case it implies that $E(X_1|C) = E(X_2|C)$ but this contradicts the fact that $\bar{B}_\varepsilon(K(u)) \cap \bar{B}_\varepsilon(K(u))$ $= \emptyset$ and ends the proof. □

5. Amarts. Now let $\mathbb{N}$ denote the set of natural numbers, i.e. $\mathbb{N} = \{1, 2, \ldots\}$. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $(\mathcal{A}_n, n \geq 1)$ be an increasing sequence of sub-σ-fields of $\mathcal{A}$ (i.e. $\mathcal{A}_n \subset \mathcal{A}_{n+1} \subset \mathcal{A}$ for every $n \in \mathbb{N}$). A mapping $\tau : \Omega \to \mathbb{N}$ will be called a stopping time with respect to $(\mathcal{A}_n)$ if and only if for every $n \in \mathbb{N}$ the event $\{\tau = n\}$ belongs to $\mathcal{A}_n$. A stopping time τ will be called bounded if and only if there exists $M \in \mathbb{N}$ such that $P(\tau \leq M) = 1$. A set of all bounded stopping times will be denoted by T.

We write $\tau \leq \sigma$ meaning a.s. inequality defining the partial ordering in T.

Definition 1. Let $\{X_n\}$ be an integrable family of random elements which is adapted to $\{\mathcal{A}_n\}$. We call $\{X_n, \mathcal{A}_n\}$ an amart if the net $\{EX_\tau; \tau \in T\}$ is convergent to some $u \in \mathbb{E}$,

$$EX_\tau \to u, \qquad \tau \in T.$$

An amart $\{X_n, \mathcal{A}_n\}$ is integrable ($X_n \in L^1_{\mathbb{E}}$) if for some $u_0 \in \mathbb{E}$

$$\sup_{n\geq 1} Ed(u_0, X_n) < \infty.$$

Definition 2 (Kruk, Zięba [11]). We say that a sequence $\{X_n\}$ of r.e. is *strongly tight* if for every $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset S$ such that

$$P\left(\bigcap_{n=1}^{\infty} [\omega : X_n(\omega) \in K_\varepsilon]\right) > 1 - \varepsilon.$$

Using Theorem 2 we are able to prove the following:

Theorem 3. *Every integrable strongly tight amart taking values in smooth convex combination space and such that*

$$\exists (u_0 \in E)\ \exists (Y \in L^1_{\mathbb{E}})\colon\ \sup_{n\geq 1} d(u_0, X_n) < d(u_0, Y). \tag{1}$$

converges a.s.

5.1. Proof. To prove this theorem we will start with the following:

Lemma 3. *Let $\{X_n\}$ be a strongly tight sequence of r.e. If the sequence is not (a.s.) convergent, then there exist two r.e. X^1 and X^2 such that $P(X^1 \neq X^2) > 0$ and X^1, X^2 are cluster points of the sequence X_n with probability* 1.

Proof. Fix $\varepsilon > 0$ and let $B_\varepsilon = \bigcap_{n=1}^{\infty} [\omega; X_n(\omega) \in K_\varepsilon]$. For every $\omega \in B_\varepsilon$ the sequence $\{X_n(\omega);\ n \geq 1\}$ has cluster points, so we may conclude that the sequence $\{X_n(\omega);\ n \geq 1\}$ has cluster points a.s.

Let $A(\omega)$ be a set of all cluster points of the sequence $\{X_n(\omega);\ n \geq 1\}$ and $\rho(A(\omega))$ denote the diameter of the set $A(\omega)$. We know that $A(\omega)$ is a measurable multifunction (see [10]).

If $\rho(A(\omega)) = 0$ a.s., then $X_n \xrightarrow[a.s.]{n\to\infty} A(\omega)$, if not, then there exist two measurable selections X^1 and X^2 of A such that $P(d(X^1, X^2) > 0) > 0$. $\square$

Lemma 4. *Let $\{X_n\}$ be a sequence of random elements adapted to an increasing sequence $\{\mathcal{A}_n\}$ of sub-σ-fields of $\mathcal{A}$. Let X be a random element such that $X(\omega)$ is a cluster point of a sequence $X_n(\omega)$ a.s. Then there is a sequence of stopping times $\{\tau_n\}$ such that $n \leq \tau_n \leq \tau_{n+1}$ and $X_{\tau_n} \to X$ (a.s.) as $n \to \infty$.*

Proof. It is enough to show that there is a sequence of stopping times τ_n satisfying $P(d(X_{\tau_n}, X) \leq \varepsilon) > 1 - \varepsilon$ because we can always choose a subsequence of such sequence which is convergent a.s. So we need to show that for any $n \in \mathbb{N}$ and any $\varepsilon > 0$ there is $\tau \in T$ such that

$$P(d(X_\tau; X) \le \varepsilon) > 1 - \varepsilon.$$

Given $\varepsilon > 0$ and n_0 we can find $n' \ge n_0$ and random element X' such that X' is $\mathcal{A}_{n'}$ measurable and

$$P\left(d(X'; X) \le \frac{\varepsilon}{3}\right) > 1 - \frac{\varepsilon}{3}$$

(we know that X is $\mathcal{A}_\infty$ measurable where $\mathcal{A}_\infty = \sigma\left(\bigcup_{n=1}^\infty \mathcal{A}_n\right)$).

Further since $X(\omega)$ is a cluster point of the sequence $\{X_n(\omega)\}$ (a.s.), it follows that

$$\left\{\omega; d\left(X'(\omega); X(\omega)\right) \le \frac{\varepsilon}{3}\right\} \subset \left\{\omega; d(X'(\omega); X_n(\omega)) \le \frac{2\varepsilon}{3} \text{ for some } n \ge n'\right\}.$$

Now we can find $n'' \ge n'$ such that $P(A) > 1 - \frac{2\varepsilon}{3}$, where

$$A = \left\{\omega; d(X'(\omega); X_n(\omega)) \le \frac{2\varepsilon}{3} \text{ for some } n' \le n \le n''\right\}.$$

Define τ by

$$\tau(\omega) = \begin{cases} \min\{n;\ n' \le n \le n'',\ d(X_n(\omega); X'(\omega)) \le \frac{2\varepsilon}{3}\}; & \omega \in A; \\ n''; & \omega \notin A. \end{cases}$$

Then τ is $\mathcal{A}_{n''}$ measurable, $\tau \in T$ and

$$P\left(d(X_\tau; X) > \varepsilon\right) \le P\left(d\left(X_\tau; X'\right) > \frac{2\varepsilon}{3}\right) + P\left(d\left(X; X'\right) > \frac{\varepsilon}{3}\right) < \varepsilon.$$

This ends the proof. □

Finally we are in position to justify Theorem 3.

Proof of Theorem 3. Assume that this is false. By Lemma 3 and condition (1) there exist two random elements $X_1', X_2' \in L^1_{\mathbb{E}}$ such that $X_1'(\omega)$ and $X_2'(\omega)$ are cluster points of the sequence $\{X_n, \mathcal{A}_n\}$ almost surely and $P(X_1' \ne X_2') > 0$.

In view of Theorem 2 there exist random elements $X_1^*,\ X_2^* \in L^1_{\mathbb{E}}$ such that for almost every $\omega \in \Omega$, $X_1^*(\omega)$ and $X_2^*(\omega)$ are cluster points of $\{X_n, \mathcal{A}_n\}$, $P(X_1^* \ne X_2^*) > 0$ and $d(EX_1^*, EX_2^*) > 0$ (if X_1' and X_2' do not satisfy the last condition one may take $X_1^* = X_1'$ and $X_2^* = X_\tau'$ for some $\tau : \Omega \to \{1, 2\}$).

Then by Theorem 4 there exist two sequences $\{\tau_n \in \Sigma\}$ and $\{\sigma_n \in \Sigma\}$ such that $X_{\tau_n} \xrightarrow{a.s.} X_1^*$ and $X_{\sigma_n} \xrightarrow{a.s.} X_2^*$, and hence by the definition of amart it follows that $EX_{\tau_n} \to u$ and $EX_{\sigma_n} \to u$, which yields $EX_1^* = EX_2^* = u$. This contradiction ends the proof. □

6. Examples.

Example 2 (Example 5, Molchanov, Teran [14]). Define the convex combination operation on a linear normed space $\mathbb{E}$ as

$$[\lambda_i u_i]_{i=1}^n = \sum \lambda_i^p u_i$$

for some $p > 1$. This operation does not satisfy condition (vi) and the random identification does not hold since for any integrable random element X taking values in this space $EX = 0$.

Remark 3. If the space $\mathbb{E}$ contains two points u, v such that $Ku = Kv$ and $d(u, v) > 0$, then we can construct two random elements $X_1 = u$ (a.s.) and $X_2 = v$ (a.s) such that $P(X_1 = X_2) = 0$ and for any measurable

$$\tau : \Omega \to \{1, 2\}, \qquad EX_\tau = EX_1 = EX_2.$$

This shows that in fact condition (vi) is a necessary condition if we want random identification to hold.

Example 3. Consider the following example. Let $\mathbb{E}$ be a space of random elements or a probabilistic distributions satisfying the following condition

$$\limsup_{t\to 0} |f'_X(t)| < \infty$$

where f_X denotes the characteristic function of the random variable X. On this set we introduce the function given by:

$$\mu(X, Y) = \sup\{|t|^{-1}|f_X(t) - f_Y(t)|\}.$$

This function is an ideal probability metric of order $s = 1$ (see [15], Example 1.4.2.) and note that $(\mathbb{E}, \mu)$ is a metric space. By the properties of ideal probability metric we know that:

- $\mu(X + Z, Y + Z) \leq \mu(X, Y)$ for any two random variables X, Y and independent variable Z.
- $\mu(cX, cY) = |c|\mu(X, Y)$ for any two random variables X, Y and any constant $c \neq 0$.

Furthermore on the set $\mathbb{E}$ introduce a convex combination operation of random variables $X_1, X_2, \ldots, X_n$ in the following way:

$[\lambda_i, X_i]_{i=1}^n$ is a random variable distributed as $\sum_{i=1}^n \lambda_i X_i$, where X_i are chosen to be independent.

It is quite easy to check that conditions (i)–(v) hold.

The strong law of large numbers implies that KX is the expectation of X if this expectation exists. If the expectation does not exist the law of large numbers does not hold, but still there may be a convergence in distribution to some stable law of order $s = 1$, i.e. to a random variable which has distribution with logarithm of characteristic function of the form

$$\ln f_X(t) = -\sigma|t| \left(1 + i\beta\frac{2}{\pi}(\operatorname{sgn} t)\ln|t|\right) + it\mu,$$

where $\mu \in \mathbb{R}$, $\beta \in [-1, 1]$, $\sigma \in \mathbb{R}_+$. For example, if X has a Cauchy distribution, KX has the same distribution.

Now consider $K(\mathbb{E}) = \mathbb{E}_1$. On this subspace also condition (vi) holds which implies that $\mathbb{E}_1$ is a smooth convex combination space.

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