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An extension of the univalence criterion for a family of integral operators

ABSTRACT. The main object of the present paper is to extend the univalence condition for a family of integral operators. Relevant connections of some of the results obtained in this paper with those in earlier works are also provided.

1. Introduction and preliminaries. Let A denote the class of functions f normalized by

(1.1)
$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and satisfy the conditions f(0) = f'(0) - 1 = 0.

Consider $S = \{ f \in A : f \text{ is a univalent function in } \mathcal{U} \}.$

A function $f(z) \in \mathcal{A}$ is said to be a member of the class $\mathcal{B}(\gamma)$ if and only if

(1.2)
$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 - \gamma, \quad 0 \le \gamma < 1.$$

Recently, Frasin and Darus (see [6]) defined and studied the class $\mathcal{B}(\gamma)$. In his paper Frasin (see [4]) obtained some results for functions belonging

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to this class and also he showed that if $f(z) \in \mathcal{B}(\gamma)$ then f(z) satisfies the following inequality

(1.3)
$$\left| \frac{zf''(z)}{f'(z)} \right| \le \frac{(1-\gamma)(2+|z|)}{1-|z|} \quad (z \in \mathcal{U}).$$

For $\gamma = 0$ the class $\mathcal{B}(0) = \mathcal{T}$ was studied by Ozaki and Nunokawa (see [8]).

We denote by \mathcal{W} the class of functions w which are analytic in \mathcal{U} satisfying the conditions |w(z)| < 1 and w(0) = w'(0) = 0 for all $z \in \mathcal{U}$.

Now, by Schwarz's lemma, it follows that

$$(1.4) |w(z)| < |z|.$$

In [7], we see that if $w(z) \in \mathcal{W}$, then w(z) satisfies

(1.5)
$$|w'(z)| \le \frac{1 - |w(z)|^2}{1 - |z|^2} (z \in \mathcal{U}).$$

In [11], N. Seenivasagan and D. Breaz considered the following family of integral operators $\mathcal{F}_{\alpha_1,\alpha_2,...,\alpha_n,\beta}(z)$ defined as follows

(1.6)
$$\mathcal{F}_{\alpha_1,\alpha_2,\dots,\alpha_n,\beta}(z) := \left(\beta \int_0^z t^{\beta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha_i}} dt\right)^{\frac{1}{\beta}},$$

where $f_i \in \mathcal{A}$, $f_i''(0) = 0$ and $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta \in \mathbb{C}$ for all $i \in \{1, 2, \ldots, n\}$. When $\alpha_i = \alpha$ for all $i \in \{1, 2, \ldots, n\}$, $\mathcal{F}_{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta}(z)$ becomes the inte-

gral operator $\mathcal{F}_{\alpha,\beta}(z)$ considered in (see [1]).

We begin by recalling each of the following theorems dealing with univalence criterion, which will be required in our present paper.

In [10], Pascu proved the following theorem.

Theorem 1 (Pascu [10]). Let β be a complex number with $\text{Re}(\beta) > 0$ and $f \in \mathcal{A}$. If

$$\frac{1 - |z|^{2\operatorname{Re}(\beta)}}{\operatorname{Re}(\beta)} \left| \frac{zf''(z)}{f'(z)} \right| \le 1,$$

for $z \in \mathcal{U}$, then the function

$$F_{\beta}(z) := \left(\beta \int_{0}^{z} t^{\beta - 1} f'(t) dt\right)^{\frac{1}{\beta}}$$

belongs to S.

In [9], Pascu and Pescar obtained the next result.

Theorem 2 (Pascu and Pescar [9]). Let β and μ be complex numbers, and $g \in \mathcal{S}$. If $\operatorname{Re}(\beta) > 0$ and $|\mu| \leq \min\left\{\frac{1}{2}\operatorname{Re}(\beta); \frac{1}{4}\right\}$, then the function

$$\mathcal{G}_{\beta,\mu}(z) \coloneqq \left(\beta \int_{0}^{z} t^{\beta-1} \left(\frac{g(t)}{t}\right)^{\mu} dt\right)^{\frac{1}{\beta}}$$

belongs to S.

Note that Theorem 2 includes the special case of Pascu and Pescar's theorem (see [9]) when $Re(\alpha) = Re(\beta)$.

In the present paper, we propose to investigate further univalence condition involving the general a family of integral operators defined by (1.6).

2. Main results. In this section, we first state an inclusion for $f(z) \in \mathcal{B}(\gamma)$, then we give the main univalence condition involving the general integral operator given by (1.6).

Theorem 3. If $f(z) \in \mathcal{B}(\gamma)$, then the inequality is satisfied

(2.1)
$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \le \frac{(1 - \gamma)(1 + |z|)}{1 - |z|}$$

for all $z \in \mathcal{U}$.

Proof. Let $f(z) \in \mathcal{B}(\gamma)$. Then we have

(2.2)
$$\frac{z^2 f'(z)}{f^2(z)} = 1 + (1 - \gamma)w(z),$$

where $w(z) \in \mathcal{W}$. By applying the logarithmic differentiation, we obtain from (2.2) that

$$\frac{zf''(z)}{f'(z)} = \frac{(1-\gamma)zw'(z)}{1+(1-\gamma)w(z)} + \frac{2zf'(z)}{f(z)} - 2$$

and

$$\frac{zf'(z)}{f(z)} - 1 = \frac{1}{2} \left(\frac{zf''(z)}{f'(z)} - \frac{(1 - \gamma)zw'(z)}{1 + (1 - \gamma)w(z)} \right),$$

thereby, it follows that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{1}{2} \left(\frac{zf''(z)}{f'(z)} - \frac{(1 - \gamma)zw'(z)}{1 + (1 - \gamma)w(z)} \right) \right|$$

$$\leq \frac{1}{2} \left(\left| \frac{zf''(z)}{f'(z)} \right| + \left| \frac{(1 - \gamma)zw'(z)}{1 + (1 - \gamma)w(z)} \right| \right)$$

$$\leq \frac{1}{2} \left(\left| \frac{zf''(z)}{f'(z)} \right| + \frac{(1 - \gamma)|z||w'(z)|}{1 - (1 - \gamma)|w(z)|} \right).$$

From (1.3) and (1.5), we have

$$(2.3) \left| \frac{zf'(z)}{f(z)} - 1 \right| \le \frac{1}{2} \left(\frac{(1-\gamma)(2+|z|)}{1-|z|} + \frac{(1-\gamma)|z|}{1-(1-\gamma)|w(z)|} \frac{1-|w(z)|^2}{1-|z|^2} \right)$$

and for $0 \le \gamma < 1$, it is easy to show that

(2.4)
$$\frac{1 - |w(z)|}{1 - (1 - \gamma)|w(z)|} \le 1 \quad (z \in \mathcal{U}).$$

From (1.4), (2.3) and (2.4), we obtain that

(2.5)
$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \le \frac{(1 - \gamma)(1 + |z|)}{1 - |z|}.$$

This evidently completes the proof of Theorem 3.

Next we prove the following main theorem.

Theorem 4. Let $f_i(z) \in \mathcal{B}(\gamma)$ for $i \in \{1, 2, ..., n\}$. Let β be a complex number with $\text{Re}(\beta) > 0$. If

(2.6)
$$\sum_{i=1}^{n} \frac{1}{|\alpha_i|} \le \min\left\{\frac{1}{2(1-\gamma)} \operatorname{Re} \beta; \frac{1}{4(1-\gamma)}\right\}$$

for all $z \in \mathcal{U}$, then the function

$$\mathcal{F}_{\alpha_1,\alpha_2,\dots,\alpha_n,\beta}(z) := \left(\beta \int_0^z t^{\beta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha_i}} dt\right)^{\frac{1}{\beta}}$$

belongs to S.

Proof. Define function

$$h(z) = \int_{0}^{z} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha_i}} dt.$$

We have h(0) = h'(0) - 1 = 0. Also, a simple computation yields

(2.7)
$$h'(z) = \prod_{i=1}^{n} \left(\frac{f_i(z)}{z}\right)^{\frac{1}{\alpha_i}}.$$

Making use of logarithmic differentiation in (2.7), we obtain

(2.8)
$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \frac{1}{\alpha_i} \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right).$$

We thus have from (2.8) that

$$\left| \frac{zh''(z)}{h'(z)} \right| \le \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right|.$$

By using the Theorem 3, we get the inequality

(2.9)
$$\left| \frac{zh''(z)}{h'(z)} \right| \le \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \frac{(1-\gamma)(1+|z|)}{1-|z|}.$$

From (2.9), we obtain

(2.10)
$$\frac{1 - |z|^{2\operatorname{Re}(\beta)}}{\operatorname{Re}(\beta)} \left| \frac{zh''(z)}{h'(z)} \right| \le \frac{1 - |z|^{2\operatorname{Re}(\beta)}}{\operatorname{Re}(\beta)} \frac{(1 - \gamma)(1 + |z|)}{1 - |z|} \sum_{i=1}^{n} \frac{1}{|\alpha_{i}|}$$

$$\le \frac{1 - |z|^{2\operatorname{Re}(\beta)}}{1 - |z|} \frac{2(1 - \gamma)}{\operatorname{Re}(\beta)} \sum_{i=1}^{n} \frac{1}{|\alpha_{i}|}$$

for all $z \in \mathcal{U}$.

Let us denote |z| = x, $x \in [0,1)$, $\text{Re}(\beta) = a > 0$ and $\psi(x) = \frac{1-x^{2a}}{1-x}$. It is easy to prove that

(2.11)
$$\psi(x) \le \begin{cases} 1, & \text{if } 0 < a < \frac{1}{2} \\ 2a, & \text{if } \frac{1}{2} < a < \infty. \end{cases}$$

From (2.6), (2.10) and (2.11), we have

$$\frac{1 - |z|^{2\operatorname{Re}(\beta)}}{\operatorname{Re}(\beta)} \left| \frac{zh''(z)}{h'(z)} \right| \le \begin{cases} \frac{2(1-\gamma)}{\operatorname{Re}(\beta)} \sum_{i=1}^{n} \frac{1}{|\alpha_{i}|}, & \text{if } 0 < \operatorname{Re}(\beta) < \frac{1}{2} \\ 4(1-\gamma) \sum_{i=1}^{n} \frac{1}{|\alpha_{i}|}, & \text{if } \frac{1}{2} < \operatorname{Re}(\beta) < \infty \end{cases}$$

for all $z \in \mathcal{U}$.

Finally, by applying Theorem 1, we conclude that the function $\mathcal{F}_{\alpha_1,\alpha_2,...,\alpha_n,\beta}(z)$ defined by (1.6) is in the function class \mathcal{S} . This evidently completes the proof of Theorem 4.

3. Some applications of Theorem 4. In this section, we give some results of Theorem 4.

First of all, upon setting $\alpha_i = \alpha$, for all $i \in \{1, 2, ..., n\}$ in Theorem 4, we immediately arrive at the following application of Theorem 4.

Corollary 1. Let $f_i(z) \in \mathcal{B}(\gamma)$ for $i \in \{1, 2, ..., n\}$. Let β be a complex number with $\text{Re}(\beta) > 0$. If

(3.1)
$$\frac{1}{|\alpha|} \le \min\left\{\frac{1}{2n(1-\gamma)}\operatorname{Re}\beta; \frac{1}{4n(1-\gamma)}\right\}$$

holds for all $z \in \mathcal{U}$, then the function

$$\mathcal{F}_{\alpha,\beta}(z) \coloneqq \left(\beta \int_{0}^{z} t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_{i}(t)}{t}\right)^{\frac{1}{\alpha}} dt\right)^{\frac{1}{\beta}}$$

belongs to S.

Next we set n = 1 in Theorem 4, we thus obtain the following interesting consequence of Theorem 4.

Corollary 2. Let the functions $f(z) \in \mathcal{B}(\gamma)$. Let β be a complex number with Re $\beta > 0$. If

(3.2)
$$\frac{1}{|\alpha|} \le \min\left\{\frac{1}{2(1-\gamma)}\operatorname{Re}\beta; \frac{1}{4(1-\gamma)}\right\}$$

holds for all $z \in \mathcal{U}$, then the function

$$\mathcal{G}_{\beta,\alpha}(z) := \left(\beta \int_{0}^{z} t^{\beta-1} \left(\frac{f(t)}{t}\right)^{\frac{1}{\alpha}} dt\right)^{\frac{1}{\beta}}$$

belongs to S.

Remark 1.

- (i) Corollary 2 provides an extension of Theorem 2 due to Pascu and Pescar (see [9]).
- (ii) If we set $\gamma = 0$, n = 1 and $\frac{1}{\alpha} = \mu$ in Theorem 4, we obtain Theorem 2 due to Pascu and Pescar (see [9]).
- (iii) If we put $\gamma = 0$, $\beta = 1$ and α instead of $\frac{1}{\alpha}$ in Corollary 2, we arrive at the result by Kim and Merkes (see [5]).

Remark 2. Some authors gave similar univalence conditions by using bounded functions $f(z) \in \mathcal{A}$ in their papers, see the works (for example Breaz et al. (see [2]), Breaz et al. (see [3])). We note that the functions $f \in \mathcal{A}$ do not have to be bounded.

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