R. BHUVANESWARI and V. KARUNAKARAN

Boehmians of type S and their Fourier transforms

ABSTRACT. Function spaces of type S are introduced and investigated in the literature. They are also applied to study the Cauchy problem. In this paper we shall extend the concept of these spaces to the context of Boehmian spaces and study the Fourier transform theory on these spaces. These spaces enable us to combine the theory of Fourier transform on these function spaces as well as their dual spaces.

1. Introduction. In [3], function spaces of type S are introduced and the properties of their Fourier transforms are investigated (see also [4], [5]). They are also applied to study the Cauchy problem. These spaces are constructed imposing conditions on the behavior of functions and on the growth of their derivatives at infinity as the order of the derivative increases (as is done in the case of generalized functions (see for example [9])). We shall first recall that the space S_α^β ($\alpha \geq 0$, $\beta \geq 0$) consists of all infinitely differentiable functions $\varphi(x)$ ($-\infty < x < \infty$) satisfying the inequalities

$$\left| x^k \varphi^{(q)}(x) \right| \leq C A^k B^q k^{k\alpha} q^{q\beta},$$

where the constants A, B, C depend on the function φ. It is also known that for sufficiently small values of α and β this space degenerates into the single function $\varphi(x) \equiv 0$. Hence in order to consider nontrivial spaces we shall be interested in choosing α and β sufficiently large. Further spaces

---

$S_\alpha, S^\beta, \mathscr{S}$ (here $\mathscr{S}$ denotes the usual space of rapidly decreasing functions) which can be described independently can be related to the space S_α^β. For an interesting relationship between these spaces and examples we refer the reader to [3], [4] and [5]. It should be noted that for $\beta \leq 1$, functions in S^β are already analytic in unbounded regions of the complex plane and hence cannot contain functions of bounded support. However for $\beta > 1$ it is known (see [3]) that the space S_0^β contains infinitely differentiable functions with compact support. Thus by virtue of the inequality $S_0^\beta \subset S_\alpha^\beta$ for $\alpha \geq 0$, we can safely assume that S_α^β ($\beta > 1$) certainly contains infinitely differentiable functions with compact support. This fact will be crucial for our construction of the Boehmian space. Incidentally the Boehmian space we construct contains also a copy of the dual of S_α^β ($\beta > 1$) (which is denoted as $\left(S_\alpha^\beta\right)'$) and hence a study of Fourier transforms of elements of our Boehmian space generalizes the theory of Fourier transform on $\left(S_\alpha^\beta\right)'$ which is undertaken in [3].

We shall assume the construction of Boehmian spaces and their properties (see [6], [7], [8]). In Section 2, we shall develop the preliminaries. In Section 2, we shall construct the Boehmian spaces. In Section 4, we shall define the Fourier transform and investigate its properties.

2. Preliminaries. In the following we shall define the required function spaces and obtain some properties. Fix $\alpha, \beta \geq 0$.

Definition 2.1. A function $f : \mathbb{R} \to \mathbb{C}$ is said to be in the family G if it satisfies the following conditions:

    (i) f is an infinitely differentiable function.

    (ii) $\left|f^{(k)}(x)\right| \leq c_{n,k} e^{\frac{1}{n}|x|^{1/\alpha}}$ for all $n \in \mathbb{N}$, $x \in \mathbb{R}$, $k = 0, 1, 2, \ldots$.

The topology on G is given by the following countable separating family of semi-norms

$$P_{n,k}(f) = \sup_{x \in \mathbb{R}, \, q \leq k} \left|f^{(q)}(x)\right| e^{\left(-\frac{1}{n}|x|^{1/\alpha}\right)} \quad n = 1, 2, 3, \ldots, \; k = 0, 1, 2, \ldots.$$

It is easy to show that G is a Fréchet space.

Definition 2.2. A function $\varphi : \mathbb{R} \to \mathbb{C}$ is said to be in S_α^β if

    (i) φ is an infinitely differentiable function.

    (ii) $\left|x^k \varphi^{(q)}(x)\right| \leq C A^k B^q k^{k\alpha} q^{q\beta}$ $(k = 0, 1, 2, \ldots, \, q = 0, 1, 2, \ldots)$ where A, B and C are constants depending on φ.

Here we assume that $k^{k\alpha} = 1$ for $k = 0$ and $q^{q\beta} = 1$ for $q = 0$.

From [3] it follows that the space S_α^β defined above is a countable union of Fréchet spaces which are denoted by $S_{\alpha,A}^{\beta,B}$ where $A, B = 1, 2, \ldots$. We also recall the following theorem from [1] and [2].

Theorem 2.3. *If $\alpha + \beta \geq 1$, then the following are equivalent:*

(1) $\varphi \in S_\alpha^\beta$.

(2) $\sup\limits_{x\in\mathbb{R}} |\varphi(x)|e^{(h|x|^{1/\alpha})} < \infty$ and $\sup\limits_{t\in\mathbb{R}} |\hat{\varphi}(t)|e^{(k|t|^{1/\beta})} < \infty$ for some $h, k > 0$.

Definition 2.4. A function $\varphi : \mathbb{R} \to \mathbb{C}$ is said to be in the space $T_{n,m}$ $(n, m \in \mathbb{N})$ if it satisfies the following conditions:

   (i) φ is infinitely differentiable.

   (ii) $||\varphi||_{n,m} = \sup\limits_{x\in\mathbb{R}} |\varphi(x)|e^{\left(\frac{1}{n}|x|^{1/\alpha}\right)} + \sup\limits_{t\in\mathbb{R}} |\hat{\varphi}(t)|e^{\left(\frac{1}{m}|t|^{1/\beta}\right)} < \infty$.

Here

$$\hat{\varphi}(t) = \int\limits_{\mathbb{R}} \varphi(x)e^{-itx}dx \quad (t \in \mathbb{R}).$$

It is also clear that if $n < n'$ and $m < m'$, then $T_{n,m} \subset T_{n',m'}$ and that if the sequence $\{\varphi_k\}$ is such that $\varphi_k \to \varphi$ as $k \to \infty$ in $T_{n,m}$, then $\varphi_k \to \varphi$ as $k \to \infty$ in $T_{n',m'}$. For this reason it is possible to define the countable union space containing all the spaces $T_{n,m}$ where n, m vary over the set of natural numbers. In view of Theorem 2.3 and Definition 2.4 above, it easily follows that this countable union space is identical with S_α^β.

Definition 2.5. (1) A sequence of functions $\{\varphi_n\}$ in S_α^β is said to converge to a function $\varphi \in S_\alpha^\beta$, if there exist n' and m' in $\mathbb{N}$ such that all φ_n and φ are in $T_{n',m'}$ and $\varphi_n \to \varphi$ in $T_{n',m'}$ as $n \to \infty$. This means that $||\varphi_n - \varphi||_{n',m'} \to 0$ as $n \to \infty$.

(2) By a continuous linear functional on the countable union space $S_\alpha^\beta = \bigcup\limits_{n,m} T_{n,m}$, we mean a linear functional on S_α^β which is continuous on every space $T_{n,m}$. The set of all continuous linear functionals on the above countable union space will be denoted by $\left(S_\alpha^\beta\right)'$.

Note 2.6. The above definitions are analogous to the definitions of a continuous linear functional and the dual $\left(S_\alpha^\beta\right)'$ of the countable union space S_α^β described in [3].

Using the equivalence of sup-norm and L^2-norm on the space $S_{\alpha,A}^{\beta,B}$ and Landau type inequalities, one can prove (this involves detailed computations which are routine and hence are omitted) the following.

(i) Given natural numbers n, m and $\varphi \in T_{n,m}$, we can find A, B such that $||\varphi||_{\delta,\rho} \leq C||\varphi||_{n,m}$ for every $\delta, \rho \in (0, 1]$ where $||\varphi||_{\delta,\rho}$ refers to the semi-norms defined on $S_{\alpha,A}^{\beta,B}$ as given in [3].

(ii) Given A, B $(A, B = 1, 2, 3, \ldots)$ and $\varphi \in S_{\alpha,A}^{\beta,B}$, we can find n and m such that $||\varphi||_{n,m} \leq C'||\varphi||_{\delta,\rho}$ for every $\delta, \rho \in (0, 1]$.

In view of the above observations the following remarks are obvious.

(i) The dual of S_α^β as described in [3] is identical with the dual of S_α^β as given in our Definition 2.5.

(ii) The countable union of the spaces $T_{n,m}$ (which is S_α^β) is also complete.
However, there is a vital difference between the countable union space described here and the one given in [3]. On the one hand, the spaces $S_{\alpha,A}^{\beta,B}$ given in [3] are Fréchet spaces whereas on the other, we have chosen the spaces $T_{n,m}$ as merely normed spaces.

Lemma 2.7. *Every function in G can be regarded as a continuous linear functional on S_α^β, i.e. each element of G can be considered as a regular element of the dual space $\left(S_\alpha^\beta\right)'$.*

Proof. Let $f \in G$. We show that f can be considered as a continuous linear functional on $T_{n,m}$ for each $n, m \in \mathbb{N}$ as follows:

$$f(\varphi) = \int_{\mathbb{R}} f(x)\varphi(x)dx \quad (\varphi \in T_{n,m}).$$

Using canonical estimates, it is an easy exercise to show that f defines a continuous linear functional on $T_{n,m}$. We leave the details. $\square$

For the construction of our Boehmian space, we need functions with compact support, inside S_α^β. But this will happen when $\beta > 1$. (Note that when $\beta > 1$, S_0^β contains functions of bounded support (see [3]) and $S_0^\beta \subset S_\alpha^\beta$, for any $\alpha > 0$). For this reason, hereafter we shall consider spaces S_α^β only for $\beta > 1$ unless explicitly specified otherwise.

Definition 2.8. A function $\varphi : \mathbb{R} \to \mathbb{C}$ is said to be in the space S if φ is in $\mathscr{D} \cap S_\alpha^\beta$.

It is easy to verify that if $\varphi \in S$, then $\varphi \in T_{n,m} \cap G$ for each $n \in \mathbb{N}$ and some $m \in \mathbb{N}$.

Definition 2.9. Let $f \in G$ and $\varphi \in S$. The convolution of f and φ is defined as

$$(f * \varphi)(x) = \int_{\mathbb{R}} f(x - t)\varphi(t)dt \quad (x \in \mathbb{R}).$$

Clearly the integral in the above definition exists and hence $f * \varphi$ is well-defined. Further the qth derivative of $f * \varphi$ exists and is defined by

$$(f * \varphi)^{(q)}(x) = \left(f^{(q)} * \varphi\right)(x).$$

Definition 2.10. Let $u \in \left(S_\alpha^\beta\right)'$ and $\varphi \in S$. We define the convolution product of u and φ in the canonical way by

$$(u * \varphi)(x) = u\left(\tau_x \check{\varphi}\right) \quad (x \in \mathbb{R})$$

where $\check{\varphi}(t) = \varphi(-t)$ and $(\tau_x \varphi)(t) = \varphi(x - t)$.

Note that if $\varphi \in T_{n,m}$, then for any $x \in \mathbb{R}$, $\tau_x \check{\varphi}$ is in $T_{n',m}$ with $n' \geq nk$ where

$$(1) \qquad k = \begin{cases} 2^{\left(\frac{1}{\alpha}-1\right)} & \text{for } \alpha < 1 \\ 1 & \text{for } \alpha \geq 1. \end{cases}$$

In particular, if $\varphi \in S$, then for any $x \in \mathbb{R}$, $\tau_x \check{\varphi} \in S$.

Lemma 2.11. *Let $f \in G$, $\varphi \in S$. Then $f * \varphi \in G$ and if further $f \in S$, then $f * \varphi \in S$ and $f * \varphi = \varphi * f$.*

Proof. Follows easily using simple computations. We leave the details. $\square$

Theorem 2.12. *Let $u \in \left(S_\alpha^\beta\right)'$, $\varphi \in S$. Then $u * \varphi \in G$.*

Proof. Since $\varphi \in S$, φ is in T_{n_1,m_1} for some $n_1, m_1 \in \mathbb{N}$. (Note that φ is in T_{n_1,m_1} for all $n_1 \in \mathbb{N}$). Choose $n' \geq n_1 k$ and $m' \geq m_1$ with k as in (1) so that $\tau_x \check{\varphi} \in T_{n',m'}$ for all $x \in \mathbb{R}$.

Indeed, using standard arguments and computations we can easily prove the following:
(i) If $x_n \to x$ as $n \to \infty$, then $\|\tau_{x_n}\check{\varphi} - \tau_x \check{\varphi}\|_{n',m'} \to 0$ as $n \to \infty$.
(ii) $u * \varphi$ is infinitely differentiable.
(iii) $\left|(u * \varphi)^{(q)}(x)\right| \leq c_{n,q} e^{\frac{1}{n}|x|^{1/\alpha}}$ $(n \in \mathbb{N}, \; q = 0, 1, 2, \dots \text{ and } x \in \mathbb{R})$.
Using the above steps, we see that $u * \varphi \in G$, completing the proof. $\square$

3. Construction of Boehmians. The general construction of Boehmians is given in detail in the literature (see [6], [7], [8]). We shall briefly recall this construction for the sake of completeness. Let G be an additive commutative semi group, $S \subset G$ and $* : G \times S \to G$ satisfy

(i) If $\varphi, \psi \in S$, then $\varphi * \psi \in S$ and $\varphi * \psi = \psi * \varphi$.
(ii) If $f \in G$ and $\varphi, \psi \in S$, then $(f * \varphi) * \psi = f * (\varphi * \psi)$.
(iii) If $f, g \in G$ and $\varphi \in S$, then $(f + g) * \varphi = f * \varphi + g * \varphi$.

Let Δ be the class of all sequences (φ_n) in S (called delta sequences) satisfying

(i) If $f, g \in G$, $(\varphi_n) \in \Delta$ and $f * \varphi_n = g * \varphi_n$ for all n, then $f = g$.
(ii) If $(\varphi_n), (\psi_n) \in \Delta$, then $(\varphi_n * \psi_n) \in \Delta$.

Let $\mathscr{A}$ denote the collection of all pairs of sequences $((f_n), (\varphi_n))$ where $f_n \in G$ $(n \in \mathbb{N})$ and $(\varphi_n) \in \Delta$ satisfying the following property

$$f_n * \varphi_m = f_m * \varphi_n \text{ for all } n, m \in \mathbb{N}.$$

Each element of $\mathscr{A}$ is called a quotient. Define a relation $\sim$ (which can be easily verified as an equivalence relation) on $\mathscr{A}$ by

$$((f_n), (\varphi_n)) \sim ((g_n), (\psi_n)) \iff f_n * \psi_m = g_m * \varphi_n \quad \forall n, m \in \mathbb{N}.$$

Let $\mathcal{B}(G, \Delta) = \mathcal{A}/\sim$, the set of all equivalence classes in $\mathcal{A}$. Elements of $\mathcal{B}(G, \Delta)$ are called Boehmians and a typical element containing $((f_n), (\varphi_n))$ is denoted by $\left[\frac{f_n}{\varphi_n}\right]$.

Now we construct the Boehmian space $\mathcal{B}(G, \Delta)$, where G and S are as in Section 2. The convolution product defines a map

$$* : G \times S \to G$$

having the required properties (easy to verify).

Let Δ be the class of all sequences (δ_n) in S satisfying the conditions:

(2)
$$\int_{\mathbb{R}} \delta_n(x)dx = 1 \quad (n \in \mathbb{N})$$

(3)
$$\int_{\mathbb{R}} |\delta_n(x)|dx \leq M \quad (n \in \mathbb{N})$$

(4) $\quad \displaystyle\lim_{n\to\infty} \int_{|x|\geq\epsilon} \left(e^{\left(\frac{1}{m}|x|^{1/\alpha}\right)} - 1\right)|\delta_n(x)|dx = 0,$ for each $m \in \mathbb{N}$ and $\epsilon > 0$.

We now show that the class Δ satisfies the following required properties of "*Delta sequences*".

Lemma 3.1. *Let $(\varphi_n) \in \Delta$. For each fixed $m \in \mathbb{N}$ and $\epsilon > 0$, the condition*

$$\lim_{n\to\infty} \int_{|x|\geq\epsilon} \left(e^{\left(\frac{1}{m}|x|^{1/\alpha}\right)} - 1\right)|\varphi_n(x)|dx = 0$$

is equivalent to

$$\lim_{n\to\infty} \int_{\mathbb{R}} \left(e^{\left(\frac{1}{m}|x|^{1/\alpha}\right)} - 1\right)|\varphi_n(x)|dx = 0.$$

Proof. Let $\eta > 0$ and fix $m \in \mathbb{N}$. It suffices to prove that

$$\lim_{n\to\infty} \int_{|x|\geq\epsilon} \left(e^{\left(\frac{1}{m}|x|^{1/\alpha}\right)} - 1\right)|\varphi_n(x)|dx = 0$$

$$\implies \lim_{n\to\infty} \int_{\mathbb{R}} \left(e^{\left(\frac{1}{m}|x|^{1/\alpha}\right)} - 1\right)|\varphi_n(x)|dx = 0.$$

Assume

$$\lim_{n\to\infty} \int_{|x|\geq\epsilon} \left(e^{\left(\frac{1}{m}|x|^{1/\alpha}\right)} - 1\right)|\varphi_n(x)|dx = 0 \quad (\forall\, \epsilon > 0).$$

Choose $\delta > 0$ such that

$$e^{\frac{1}{m}|x|^{1/\alpha}} - 1 < \frac{\eta}{2M} \quad \text{whenever} \ |x| < \delta$$

where M is as in (3). Now

$$\int_{\mathbb{R}} \left(e^{\left(\frac{1}{m}|x|^{1/\alpha}\right)} - 1 \right) |\varphi_n(x)| dx$$

$$= \int_{|x|<\delta} \left(e^{\left(\frac{1}{m}|x|^{1/\alpha}\right)} - 1 \right) |\varphi_n(x)| dx + \int_{|x|\geq\delta} \left(e^{\left(\frac{1}{m}|x|^{1/\alpha}\right)} - 1 \right) |\varphi_n(x)| dx$$

$$< \frac{\eta}{2} + \frac{\eta}{2} = \eta$$

for sufficiently large n. This completes the proof. □

Theorem 3.2. *If $f \in G$ and $(\varphi_n) \in \Delta$, then $f * \varphi_n \to f$ as $n \to \infty$ in G.*

Proof. We have to prove that $P_{m,p}(f * \varphi_n - f) \to 0$ as $n \to \infty$ for each $m \in \mathbb{N}$ and $p = 0, 1, 2, \ldots$. Fix m, p and a constant $K > 0$. Choose a compact set H such that for $x \notin H$,

$$e^{\left(\frac{1}{m'} - \frac{1}{m}\right)|x|^{1/\alpha}} < \frac{\epsilon}{4K},$$

where $m' \geq mk$ (with k as in (1)). Using the uniform continuity of $f^{(q)}$ ($q = 0, 1, \ldots$) on a compact set larger than H, say H_1, we can get a $\delta > 0$ such that whenever $|y| < \delta$ and $x \in H_1$, we have

$$\left| f^{(q)}(x - y) - f^{(q)}(x) \right| < \frac{\epsilon}{4M}.$$

Fix this $\delta > 0$. For $|x| \geq \delta$, we have

$$(5) \qquad e^{\frac{1}{m}|x|^{1/\alpha}} + 1 \leq C \left(e^{\frac{1}{m}|x|^{1/\alpha}} - 1 \right)$$

where $C = \frac{2}{1 - e^{\frac{-1}{m}|\delta|^{1/\alpha}}}$, which can be easily verified.

We now consider

$$P_{m,p}(f * \varphi_n - f) \leq \sup_{x \in \mathbb{R}, \, q \leq p} e^{\frac{-1}{m}|x|^{1/\alpha}} \int_{\mathbb{R}} \left| f^{(q)}(x - y) - f^{(q)}(x) \right| |\varphi_n(y)| dy$$

$$= \sup_{x \in H_1, \, q \leq p} e^{\frac{-1}{m}|x|^{1/\alpha}} \int_{\mathbb{R}} \left| f^{(q)}(x - y) - f^{(q)}(x) \right| |\varphi_n(y)| dy$$

$$+ \sup_{x \notin H_1, \, q \leq p} e^{\frac{-1}{m}|x|^{1/\alpha}} \int_{\mathbb{R}} \left| f^{(q)}(x - y) - f^{(q)}(x) \right| |\varphi_n(y)| dy$$

$$= I_1 + I_2,$$

say. Hence

$$(6) \qquad P_{m,p}(f * \varphi_n - f) \leq I_1 + I_2.$$

Now the integral I_1 can be split as a sum of two integrals, say J_1 and J_2, where

$$J_1 = \sup_{x \in H_1, q \leq p} e^{-\frac{1}{m}|x|^{1/\alpha}} \int_{|y| < \delta} \left| f^{(q)}(x-y) - f^{(q)}(x) \right| |\varphi_n(y)| dy$$

and

$$J_2 = \sup_{x \in H_1, q \leq p} e^{\frac{-1}{m}|x|^{1/\alpha}} \int_{|y| \geq \delta} \left| f^{(q)}(x-y) - f^{(q)}(x) \right| |\varphi_n(y)| dy.$$

Using the uniform continuity of $f^{(q)}$ on H_1, we have

$$J_1 \leq \frac{\epsilon}{4M} \int_{|y| < \delta} |\varphi_n(y)| dy \leq \frac{\epsilon}{4}.$$

Note that for any integer m',

$$\left| f^{(q)}(x-y) - f^{(q)}(x) \right| \leq \left| f^{(q)}(x-y) \right| + \left| f^{(q)}(x) \right|$$

(7)
$$\leq C_{m',q} \left(e^{\frac{1}{m'}|x-y|^{1/\alpha}} + e^{\frac{1}{m'}|x|^{1/\alpha}} \right)$$

$$\leq C_{m',q} \left(e^{\frac{k}{m'}\left(|x|^{1/\alpha}+|y|^{1/\alpha}\right)} + e^{\frac{1}{m'}|x|^{1/\alpha}} \right).$$

Take $m' \geq mk$ (with k as in (1)) in equation (7) and use this estimate and (5) in J_2 to get

$$J_2 \leq \sup_{x \in H_1, q \leq p} e^{\left(\frac{1}{m'} - \frac{1}{m}\right)|x|^{1/\alpha}} \int_{|y| \geq \delta} C_{m',q} \left(e^{\frac{1}{m'}|y|^{1/\alpha}} + 1 \right) |\varphi_n(y)| dy$$

$$\leq CC_{m',p} \int_{|y| \geq \delta} \left(e^{\frac{1}{m_1}|y|^{1/\alpha}} - 1 \right) |\varphi_n(y)| dy$$

$$< \frac{\epsilon}{4}$$

for sufficiently large n. Thus

(8) $$I_1 \to 0 \quad \text{as} \quad n \to \infty.$$

It is now easy to see that

$$I_2 \leq \sup_{x \notin H_1, q \leq p} e^{\left(\frac{1}{m'} - \frac{1}{m}\right)|x|^{1/\alpha}} \int_{\mathbb{R}} C_{m',q} \left(e^{\frac{1}{m'}|y|^{1/\alpha}} + 1 \right) |\varphi_n(y)| dy$$

with $m' \geq mk$. By a standard argument we can now show that

(9) $$I_2 \to 0 \quad \text{as} \quad n \to \infty.$$

The theorem follows from (6), (8) and (9). $\square$

Theorem 3.3. *Let $f, g \in G$ and $(\varphi_n) \in \Delta$. If $f * \varphi_n = g * \varphi_n$ for all $n \in \mathbb{N}$, then $f = g$ in G.*

Proof. Follows from Theorem 3.2. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 3.4. *Let (φ_n), (ψ_n) be any two sequences in Δ. Then $(\varphi_n * \psi_n)$ is also a sequence in Δ.*

Proof. Using Fubini's theorem (wherever necessary), the properties of the class Δ, Lemma 3.1 and some standard arguments, the result follows. We leave the details. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 3.3 and Theorem 3.4 together prove that the class Δ can be used as "Delta sequences" for the construction of our Boehmian space. The resulting Boehmian space will be denoted as $\mathcal{B}(G, \Delta)$.

Definition 3.5. Let $G' = \left(S_\beta^\alpha\right)'$ and $S' = \hat{S}$ where $\hat{S}$ denotes the set of all Fourier transforms of elements of $S = \mathscr{D} \cap S_\alpha^\beta$. The map $G' \times S'$ to G' is taken as the canonical product defined as follows:

$$(u\psi)(\phi) = u(\psi\phi) \quad (\phi \in S_\beta^\alpha, \ u \in G' \text{ and } \psi \in S').$$

Let

$$\Delta' = \hat{\Delta} = \{(\hat{\varphi}_n) : (\varphi_n) \in \Delta\}.$$

Instead of equipping $\left(S_\alpha^\beta\right)'$ with the canonical weak* convergence, we shall define a different notion of convergence as follows.

Definition 3.6. A sequence $\{u_n\}$ in $\left(S_\alpha^\beta\right)'$ converges to $u \in \left(S_\alpha^\beta\right)'$, if for every $m, k \in \mathbb{N}$ there exists a sequence $\{c_{n,m,k}\}$ with $c_{n,m,k} \to 0$ as $n \to \infty$ such that for all $\psi \in T_{m,k} \subset S_\alpha^\beta$,

$$|(u_n - u)(\psi)| \le c_{n,m,k} ||\psi||_{m,k}.$$

Hereafter convergence in $\left(S_\alpha^\beta\right)'$ means the above convergence only, unless stated otherwise.

Theorem 3.7. *$u_n \to u$ as $n \to \infty$ in $\left(S_\alpha^\beta\right)'$ if and only if $\hat{u}_n \to \hat{u}$ as $n \to \infty$ in $\left(S_\beta^\alpha\right)'$.*

Proof. The result follows from the definition of convergence and the facts that $\hat{u}(\varphi) = u(\hat{\varphi})$ $(u \in \left(S_\alpha^\beta\right)', \ \varphi \in S_\beta^\alpha)$ and $||\psi||_{p,q} = ||\hat{\psi}||_{q,p}$ $(\psi \in S_\alpha^\beta)$. $\quad\square$

Theorem 3.8. *If $f_n \to 0$ as $n \to \infty$ in G, then $f_n \to 0$ as $n \to \infty$ in $\left(S_\alpha^\beta\right)'$.*

Proof. Since $f_n \to 0$ as $n \to \infty$ in G, $P_{m,q}(f_n) \to 0$ as $n \to \infty$ for all $m \in \mathbb{N}$, $q = 0, 1, 2, \ldots$. For any $\varphi \in T_{n',m'} \subset S_\alpha^\beta$, choose $m > n'$ and consider

$$|f_n(\varphi)| \le \int_{\mathbb{R}} |f_n(x)||\varphi(x)|dx.$$

A simple computation shows that

$$|f_n(\varphi)| \leq c_n ||\varphi||_{n',m'},$$

where

$$c_n = P_{m,0}(f_n) \int_{\mathbb{R}} e^{\left(\frac{1}{m} - \frac{1}{n'}\right)|x|^{1/\alpha}} dx \to 0 \quad \text{as} \quad n \to \infty.$$

This shows that $f_n \to 0$ as $n \to \infty$ in $\left(S_\alpha^\beta\right)'$. This proves the theorem. $\square$

Theorem 3.9. *Let $u_n \to 0$ as $n \to \infty$ in $\left(S_\alpha^\beta\right)'$ and $\varphi \in S$ then $u_n * \varphi \to 0$ as $n \to \infty$ in G.*

Proof. Since $u_n \to 0$ as $n \to \infty$, for any $\psi \in T_{n',m'}$ there exists a sequence $\{c_n\}$ (depending on n' and m') with $c_n \to 0$ as $n \to \infty$ such that

$$|u_n(\psi)| \leq c_n ||\psi||_{n',m'}.$$

We fix m and q and consider the semi-norm $P_{m,q}$. $\varphi \in S$ ($\varphi \in T_{r,s}$ for all r and some s) implies that $\tau_x\left(\varphi^{(q)}\right)^{\vee} \in S$ ($q = 0, 1, 2, \ldots$ and for each fixed $x \in \mathbb{R}$) and hence $\tau_x\left(\varphi^{(q)}\right)^{\vee} \in T_{r',s'}$ where $r' > \max\{r, m\}$ and $s' \geq s$. Consider

$$\begin{aligned}
P_{m,q}(u_n * \varphi) &= \sup_{x \in \mathbb{R}, \, p \leq q} \left| u_n\left(\tau_x\left(\varphi^{(p)}\right)^{\vee}\right) \right| e^{\frac{-1}{m}|x|^{1/\alpha}} \\
&\leq \sup_{x \in \mathbb{R}, \, p \leq q} c_n \left\| \tau_x\left(\varphi^{(p)}\right)^{\vee} \right\|_{r',s'} e^{\frac{-1}{m}|x|^{1/\alpha}} \\
&\leq c_n \sup_{x \in \mathbb{R}} e^{\left(\frac{k}{r'} - \frac{1}{m}\right)|x|^{1/\alpha}} \sup_{p \leq q} \left\| \varphi^{(p)} \right\|_{r',s'} \\
&\leq C' c_n \left\| \varphi \right\|_{r,s}
\end{aligned}$$

(with k as in (1)) where $\left\|\left(\varphi^{(q)}\right)^{\vee}\right\|_{r',s'} \leq C_p \|\varphi\|_{r,s}$ and

$$C' = \sup_{x \in \mathbb{R}} e^{\left(\frac{k}{r'} - \frac{1}{m}\right)|x|^{1/\alpha}} \sup_{p \leq q} C_p < \infty.$$

Hence

$$P_{m,q}(u_n * \varphi) \leq C' c_n \|\varphi\|_{r,s}.$$

As $n \to \infty$, $c_n \to 0$ implies that $P_{m,q}(u_n * \varphi) \to 0$ as $n \to \infty$. This completes the proof of our theorem. $\square$

Theorem 3.10. *Let $u \in \left(S_\beta^\alpha\right)'$ and $(\hat{\varphi}_n) \in \Delta'$. Then $u\hat{\varphi}_n \to u$ as $n \to \infty$ in $\left(S_\beta^\alpha\right)'$.*

Proof. Fix $m, k \in \mathbb{N}$. Let $\psi \in T_{m',k'} \subset S_\beta^\alpha$. As $u \in \left(S_\beta^\alpha\right)'$, there is a positive constant C such that

$$|u(\psi)| \leq C ||\psi||_{m',k'}$$

for all $\psi \in T_{m',k'}$. A simple computation shows that

$$|(u\hat{\varphi}_n - u)(\psi)| \leq C||\hat{\varphi}_n\psi - \psi||_{p,q},$$

where $p > km'$ and $q > kk'$ with k as in (1).

Consider

$$||\hat{\varphi}_n\psi - \psi||_{p,q} = \sup_{x\in\mathbb{R}} |(\hat{\varphi}_n\psi)(x) - \psi(x)|e^{\frac{1}{p}|x|^{1/\beta}} + \sup_{t\in\mathbb{R}} |(\check{\varphi}_n * \hat{\psi})(t) - \hat{\psi}(t)|e^{\frac{1}{q}|t|^{1/\alpha}}.$$

Choosing p and q sufficiently large we have

$$||\hat{\varphi}_n\psi - \psi||_{p,q} \leq c'_n||\psi||_{m',k'} + d'_n \left\|(\hat{\psi})'\right\|_{p',q'},$$

where

$$c'_n = \sup_{x\in\mathbb{R}} |\hat{\varphi}_n(x) - 1|e^{\left(\frac{1}{p} - \frac{1}{m'}\right)|x|^{1/\beta}}$$

and

$$d'_n = K \int_{\mathbb{R}} |\check{\varphi}_n(y)||y|e^{\frac{1}{m}|y|^{1/\alpha}}dy$$

with $mk < q$, k as in (1) and K, a constant. Using simple computations we can prove that the sequences $\{c'_n\}$ and $\{d'_n\}$ tend to zero as $n \to \infty$. Using the continuity of differentiation, we can get that

$$\left\|(\hat{\psi})'\right\|_{p',q'} \leq C' \left\|\hat{\psi}\right\|_{k',m'} \leq C'||\psi||_{m',k'}.$$

Thus, we have

$$||\hat{\varphi}_n\psi - \psi||_{p,q} \leq c_n||\psi||_{m',k'},$$

where $c_n = C'd'_n + c'_n \to 0$ as $n \to \infty$. This completes the proof of our theorem. $\qquad\square$

The following theorems can be easily verified. We leave the details.

Theorem 3.11. *Let $v_1, v_2 \in G'$ and $(\sigma_n) \in \Delta'$. If $v_1\sigma_n = v_2\sigma_n$ for all n, then $v_1 = v_2$ in G'.*

Theorem 3.12. *If $(a_n), (b_n) \in \Delta'$, then $(a_nb_n) \in \Delta'$.*

Using G' and Δ' as G and Δ we can now construct a Boehmian space in a canonical way. This Boehmian space will be denoted by $\mathcal{B}(G', \Delta')$. We shall now obtain some properties of sequences in G and G'.

Theorem 3.13. *Let $f_n \to f$ as $n \to \infty$ in G and $\varphi \in S$. Then $f_n * \varphi \to f * \varphi$ as $n \to \infty$ in G.*

Proof. Since $f_n \to f$ as $n \to \infty$ in G, for any $m \in \mathbb{N}$ and $q = 0, 1, 2, \dots$

$$P_{m,q}(f_n - f) \to 0 \quad \text{as} \quad n \to \infty.$$

Fix $m \in \mathbb{N}$ and $q \in \mathbb{N} \cup \{0\}$. As $\varphi \in S$, choose r and m' so that $m' > mk$ and $m' > rk$ (with k as in (1)) and $\varphi \in T_{r,s}$.

Consider

$$P_{m,q}((f_n - f) * \varphi) \leq \sup_{x \in \mathbb{R},\, p \leq q} \int_{\mathbb{R}} \left| (f_n - f)^{(p)}(y) \right| |\varphi(x-y)| e^{-\frac{1}{m}|x|^{1/\alpha}} dy$$

$$\leq \sup_{x \in \mathbb{R},\, p \leq q} \int_{\mathbb{R}} \left| (f_n - f)^{(p)}(y) \right| |\varphi(x-y)| e^{-\frac{1}{m}|x|^{1/\alpha}} e^{\frac{1}{m'}|y|^{1/\alpha}} e^{-\frac{1}{m'}|y|^{1/\alpha}} dy$$

$$\leq \sup_{x \in \mathbb{R}} e^{\left(\frac{k}{m'} - \frac{1}{m}\right)|x|^{1/\alpha}} P_{m',q}(f_n - f) \int_{\mathbb{R}} |\varphi(x-y)| e^{\frac{k}{m'}|x-y|^{1/\alpha}} dy$$

(substituting $t = x - y$)

$$\leq P_{m',q}(f_n - f) \int_{\mathbb{R}} |\varphi(t)| e^{\frac{k}{m'}|t|^{1/\alpha}} dt$$

$$\leq P_{m',q}(f_n - f) \|\varphi\|_{r,s} \int_{\mathbb{R}} e^{\left(\frac{k}{m'} - \frac{1}{r}\right)|t|^{1/\alpha}} dt.$$

Since

$$\|\varphi\|_{r,s} \int_{\mathbb{R}} e^{\left(\frac{k}{m'} - \frac{1}{r}\right)|t|^{1/\alpha}} dt$$

is finite and $P_{m',q}(f_n - f) \to 0$ as $n \to \infty$ we have

$$P_{m,q}(f_n * \varphi - f * \varphi) \to 0 \quad \text{as} \quad n \to \infty.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 3.14. *Let $f_n \to f$ as $n \to \infty$ in G and $(\varphi_n) \in \Delta$. Then $f_n * \varphi_n \to f$ as $n \to \infty$ in G.*

Proof. Since $f_n \to f$ as $n \to \infty$ in G, for any $m \in \mathbb{N}$ and $q = 0, 1, 2, \dots$

$$P_{m,q}(f_n - f) \to 0 \quad \text{as} \quad n \to \infty.$$

Also

$$(f_n * \varphi_n) - f = ((f_n - f) * \varphi_n) + (f * \varphi_n - f).$$

By Theorem 3.2, $f * \varphi_n \to f$ as $n \to \infty$ in G. To prove the required result it is enough to prove that $(f_n - f) * \varphi_n \to 0$ as $n \to \infty$ in G.

Fix m and q and choose $m' > mk$ (k is a constant as in (1)). Consider

$$P_{m,q}((f_n - f) * \varphi_n) = \sup_{x \in \mathbb{R},\, p \leq q} \left| \left((f_n - f)^{(p)} * \varphi_n \right)(x) \right| e^{-\frac{1}{m}|x|^{1/\alpha}}$$

$$\leq \sup_{x \in \mathbb{R},\, p \leq q} e^{-\frac{1}{m}|x|^{1/\alpha}} \int_{\mathbb{R}} \left| \left(f_n^{(p)} - f^{(p)} \right)(x-y) \right| |\varphi_n(y)| dy.$$

A simple computation shows that

$$P_{m,q}((f_n - f) * \varphi_n) \leq M' P_{m',q}(f_n - f),$$

where
$$M' = \sup_{x \in \mathbb{R}} e^{\left(\frac{k}{m'} - \frac{1}{m}\right)|x|^{1/\alpha}} \int\limits_{x \in \mathbb{R}} |\varphi_n(y)| e^{\frac{1}{m}|y|^{1/\alpha}} dy < \infty$$

(since $(\varphi_n) \in \Delta$). $P_{m',q}(f_n - f) \to 0$ as $n \to \infty$ implies that

$$P_{m,q}((f_n - f) * \varphi_n) \to 0 \quad \text{as} \quad n \to \infty.$$

This completes the proof. $\qquad\qquad\square$

Theorem 3.15. *Let $v_n \to v$ as $n \to \infty$ in G' and $\psi \in S'$. Then $v_n\psi \to v\psi$ as $n \to \infty$ in G'.*

Proof. Follows easily from the Theorems 3.7, 3.8 and 3.9. $\qquad\square$

Theorem 3.16. *Let $v_n \to v$ as $n \to \infty$ in G' and $(\hat{\varphi}_n) \in \Delta'$. Then $v_n\hat{\varphi}_n \to v$ as $n \to \infty$ in G'.*

Proof. Since $v_n \to v$ as $n \to \infty$ in G', for any $\phi \in T_{m',n'} \subset S_\beta^\alpha$, there exists constants c_n depending on m' and n' with $c_n \to 0$ as $n \to \infty$ such that

(10) $$|(v_n - v)(\phi)| \leq c_n \|\phi\|_{m',n'}.$$

As $v \in G'$, there exists a constant C such that

(11) $$|v(\phi)| \leq C\|\phi\|_{m',n'} \quad (\varphi \in T_{m',n'}).$$

Now

(12) $$|(v_n\hat{\varphi}_n - v)(\phi)| \leq |(v_n - v)(\hat{\varphi}_n\phi)| + |v(\hat{\varphi}_n\phi - \phi)|.$$

As in the proof of Theorem 3.10, we have

$$\|\hat{\varphi}_n\phi - \phi\|_{q,p} \leq c'_n\|\phi\|_{m',n'} \quad \text{where } c'_n \to 0 \text{ as } n \to \infty$$

and hence

$$\|\hat{\varphi}_n\phi\|_{q,p} \leq (c'_n + 1)\|\phi\|_{m',n'}.$$

Thus from (10), (11) and (12), we have

$$|(v_n\hat{\varphi}_n - v)(\phi)| \leq d_n\|\phi\|_{m',n'},$$

where $d_n = c_n(c'_n + 1) + Cc'_n \to 0$ as $n \to \infty$. This completes the proof. $\quad\square$

Definition 3.17. A sequence of Boehmians $\{X_n\}$ in $\mathcal{B}(G, \Delta)$ is said to δ-converge to a Boehmian $X \in \mathcal{B}(G, \Delta)$ denoted by $X_n \overset{\delta}{\to} X$, if there exists a delta sequence $(\delta_k) \in \Delta$ such that $X_n * \delta_k$ and $X * \delta_k$ are in G for $k = 1, 2, \ldots$ and

$$X_n * \delta_k \to X * \delta_k$$

in G as $n \to \infty$, for each $k = 1, 2, 3, \ldots$.

Theorem 3.18. *The mapping $T_1 : \left(S_\alpha^\beta\right)' \to \mathcal{B}(G, \Delta)$ defined by $T_1(u) = \left[\frac{u*\varphi_n}{\varphi_n}\right]$ where $(\varphi_n) \in \Delta$ is any delta sequence, is a continuous imbedding of $\left(S_\alpha^\beta\right)'$ in $\mathcal{B}(G, \Delta)$.*

Proof. If $u \in \left(S_\alpha^\beta\right)'$ and $(\varphi_n) \in \Delta$, then by Theorem 2.12, $f_n = u * \varphi_n \in G$, for all n. Using the associativity of convolutions we can prove that

$$f_n * \varphi_m = f_m * \varphi_n, \quad (m, n \in \mathbb{N}).$$

Hence $\left[\frac{f_n}{\varphi_n}\right] \in \mathcal{B}(G, \Delta)$. We first show that T_1 is one-one.

If $\left[\frac{u*\varphi_n}{\varphi_n}\right] = \left[\frac{v*\varphi_n}{\varphi_n}\right]$ in $\mathcal{B}(G, \Delta)$, then

$$(u * \varphi_n) * \varphi_m = (v * \varphi_m) * \varphi_n \quad \text{in } G \quad (m, n \in \mathbb{N}).$$
$$(u * \varphi_m) * \varphi_n = (v * \varphi_m) * \varphi_n \quad \text{in } G \quad (m, n \in \mathbb{N}).$$

By Theorem 3.3, we get $u * \varphi_m = v * \varphi_m$ in G for all $m \in \mathbb{N}$. Taking Fourier transform on both sides, we get that

$$\hat{\varphi}_m \hat{u} = \hat{\varphi}_m \hat{v} \quad \text{in } \left(S_\beta^\alpha\right)' \quad (m \in \mathbb{N}).$$

Applying Theorem 3.11 we get $\hat{u} = \hat{v}$ in $\left(S_\beta^\alpha\right)'$ and hence $u = v$ in $\left(S_\alpha^\beta\right)'$. This proves that T_1 is one-one.

To prove that T_1 is continuous, let us assume that $u_m \to 0$ as $m \to \infty$ in $\left(S_\alpha^\beta\right)'$. Corresponding to each $u_m \in \left(S_\alpha^\beta\right)'$, consider $X_m = \left[\frac{u_m*\varphi_n}{\varphi_n}\right] \in \mathcal{B}(G, \Delta)$. We now prove that $X_m \xrightarrow{\delta} 0$ as $m \to \infty$ in $\mathcal{B}(G, \Delta)$. Take $(\varphi_k) \in \Delta$ and consider $X_m * \varphi_k = u_m * \varphi_k \in G$ for $k \in \mathbb{N}$. Since $u_m \to 0$ as $m \to \infty$ in $\left(S_\alpha^\beta\right)'$. Applying Theorem 3.9 we get that

$$u_m * \delta_k \to 0$$

in G as $m \to \infty$ ($k \in \mathbb{N}$). This shows that $X_m \xrightarrow{\delta} 0$ as $m \to \infty$ in $\mathcal{B}(G, \Delta)$. This completes the proof of our theorem. $\qquad\square$

Theorem 3.19. *The mapping $T_2 : G' \left(= \left(S_\beta^\alpha\right)'\right) \to \mathcal{B}(G', \Delta')$ defined by $T_2(v) = \left[\frac{v\hat{\varphi}_n}{\hat{\varphi}_n}\right]$ where $(\hat{\varphi}_n) \in \Delta'$ is any delta sequence, is a continuous imbedding of G' in $\mathcal{B}(G', \Delta')$.*

Proof. Since $v \in G'$ and $(\hat{\varphi}_n) \in \Delta'$, $v\hat{\varphi}_n \in G'$ for all $n \in \mathbb{N}$. Simple computation shows that

$$(v\hat{\varphi}_n)\hat{\varphi}_m = (v\hat{\varphi}_m)\hat{\varphi}_n \quad \text{in} \quad G' \quad (m, n \in \mathbb{N}).$$

Hence $\left[\frac{v\hat{\varphi}_n}{\hat{\varphi}_n}\right] \in \mathcal{B}(G', \Delta')$. As above in the proof of Theorem 3.18, using Theorems 3.11, 3.15 and 3.16, we can show that T_2 is a continuous one-one imbedding of G' in $\mathcal{B}(G', \Delta')$. $\qquad\square$

4. Fourier Transform. Let $X = \left[\frac{f_n}{\varphi_n}\right] \in \mathcal{B}(G, \Delta)$. Since $f_n \in G$, $\hat{f}_n \in G'$ for $n = 1, 2, 3, \ldots$. As $(\varphi_n) \in \Delta$, $(\hat{\varphi}_n) \in \Delta'$. Also $((f_n), (\varphi_n))$ is a quotient in $\mathcal{B}(G, \Delta)$ implies that

$$f_m * \varphi_n = f_n * \varphi_m, \quad (m, n \in \mathbb{N}).$$

Taking Fourier transform on both sides, we have

$$\hat{f}_m \hat{\varphi}_n = \hat{f}_n \hat{\varphi}_m \quad \text{in} \quad G' \quad (m, n \in \mathbb{N}).$$

This proves that $\left((\hat{f}_n), (\hat{\varphi}_n)\right)$ is a quotient in $\mathcal{B}(G', \Delta')$. We can therefore define the Fourier transform of X (denoted as $\hat{X}$) by $\hat{X} = \left[\frac{\hat{f}_n}{\hat{\varphi}_n}\right] \in \mathcal{B}(G', \Delta')$. It is also easy to see that this is well-defined.

Definition 4.1. Let $X = \left[\frac{f_n}{\varphi_n}\right], Y = \left[\frac{g_n}{\psi_n}\right] \in \mathcal{B}(G, \Delta)$ and $\psi \in S$. Define

(i) $X + Y = \left[\frac{f_n * \psi_n + g_n * \varphi_n}{\varphi_n * \psi_n}\right] \in \mathcal{B}(G, \Delta)$.

(ii) $\alpha X = \left[\frac{\alpha f_n}{\varphi_n}\right] \in \mathcal{B}(G, \Delta) \quad (\alpha \in \mathbb{C})$.

(iii) $X * \psi = \left[\frac{f_n * \psi}{\varphi_n}\right] \in \mathcal{B}(G, \Delta)$.

(iv) $\hat{X} + \hat{Y} = \left[\frac{\hat{\psi}_n \hat{f}_n + \hat{\varphi}_n \hat{g}_n}{\hat{\varphi}_n \hat{\psi}_n}\right] \in \mathcal{B}(G', \Delta')$.

(v) $\alpha \hat{X} = \left[\frac{\alpha \hat{f}_n}{\hat{\varphi}_n}\right] \in \mathcal{B}(G', \Delta') \quad (\alpha \in \mathbb{C})$.

(vi) $\hat{\psi} \hat{X} = \left[\frac{\hat{\psi} \hat{f}_n}{\hat{\varphi}_n}\right] \in \mathcal{B}(G', \Delta')$.

The following properties follow easily from the definitions.

Theorem 4.2. *The Fourier transform $T : \mathcal{B}(G, \Delta) \to \mathcal{B}(G', \Delta')$ defined by $T(X) = \hat{X}$ is a bijective, bi-continuous mapping with the following properties.*

(i) $(X + Y)\hat{\ } = \hat{X} + \hat{Y} \quad (X, Y \in \mathcal{B}(G, \Delta))$.

(ii) $(\alpha X)\hat{\ } = \alpha \hat{X} \quad (\alpha \in \mathbb{C})$.

(iii) $(X * \psi)\hat{\ } = \hat{\psi} \hat{X} \quad (\psi \in S)$.

Proof. Let us first prove that T is one-one. If $\left[\frac{\hat{f}_n}{\hat{\varphi}_n}\right] = \left[\frac{\hat{g}_n}{\hat{\psi}_n}\right] \in \mathcal{B}(G', \Delta')$. Then

$$\hat{f}_n \hat{\psi}_m = \hat{g}_m \hat{\varphi}_n \quad \text{in} \quad G' \quad (m, n \in \mathbb{N})$$

and taking the Fourier transform, we get

$$f_n * \psi_m = g_m * \varphi_n \quad \text{in} \quad \left(S_\alpha^\beta\right)' \quad (m, n \in \mathbb{N}).$$

Thus

$$f_n * \psi_m * \varphi_n = g_m * \varphi_n * \varphi_n \quad \text{in} \quad G \quad (m, n \in \mathbb{N}).$$

Hence

$$\left[\frac{f_n}{\varphi_n}\right] = \left[\frac{f_n * \varphi_n}{\varphi_n * \varphi_n}\right] = \left[\frac{g_n}{\psi_n}\right] \in \mathcal{B}(G, \Delta).$$

This proves that T is one-one.

T is onto: Let $Y = \left[\frac{g_n}{\hat{\varphi}_n}\right] \in \mathcal{B}(G', \Delta')$. Since $g_n \in G'$, there exists $f_n \in \left(S_\alpha^\beta\right)'$ such that $g_n = \hat{f}_n$ $(n \in \mathbb{N})$. As $(\varphi_n) \in \Delta$, $f_n * \varphi_n \in G$ for $n \in \mathbb{N}$ and hence $X = \left[\frac{f_n * \varphi_n}{\varphi_n * \varphi_n}\right] \in \mathcal{B}(G, \Delta)$ such that $\hat{X} = Y$. This proves that T is onto.

T is continuous: Let $X_n \xrightarrow{\delta} X$ as $n \to \infty$ in $\mathcal{B}(G, \Delta)$. Then there exists a delta sequence $(\delta_k) \in \Delta$ such that $X_n * \delta_k$, $X * \delta_k \in G$ $(n, k \in \mathbb{N})$ and

$$X_n * \delta_k \to X * \delta_k \quad \text{as} \quad n \to \infty \quad \text{in} \quad G \quad (k = 1, 2, 3, \dots).$$

By Theorem 3.8,

$$X_n * \delta_k \to X * \delta_k \quad \text{as} \quad n \to \infty \quad \text{in} \quad \left(S_\alpha^\beta\right)' \quad (k = 1, 2, 3, \dots).$$

By Lemma 2.7 and Theorem 3.7 and by the property (iii) of T (which can be easily verified), we have $\hat{X}_n \hat{\delta}_k = (X_n * \delta_k)\hat{}$, $\hat{X}\hat{\delta}_k = (X * \delta_k)\hat{} \in G'$ $(n, k \in \mathbb{N})$ and

$$(X_n * \delta_k)\hat{} \to (X * \delta_k)\hat{} \quad \text{as} \quad n \to \infty \quad \text{in} \quad G' \quad (k \in \mathbb{N})$$

or that

$$\hat{X}_n \hat{\delta}_k \to \hat{X}\hat{\delta}_k \quad \text{as} \quad n \to \infty \quad \text{in} \quad G' \quad (k \in \mathbb{N}).$$

Thus $\hat{X}_n \xrightarrow{\delta} \hat{X}$ as $n \to \infty$ in $\mathcal{B}(G', \Delta')$. Hence T is continuous.

T^{-1} is continuous: Let $\hat{X}_n \xrightarrow{\delta} \hat{X}$ as $n \to \infty$ in $\mathcal{B}(G', \Delta')$ where $X_n, X \in \mathcal{B}(G, \Delta)$. Then there exists a delta sequence $(\hat{\epsilon}_k) \in \Delta'$ such that $\hat{X}_n \hat{\epsilon}_k, \hat{X}\hat{\epsilon}_k \in G'$ $(n, k \in \mathbb{N})$ such that

$$\hat{X}_n \hat{\epsilon}_k \to \hat{X}\hat{\epsilon}_k \quad \text{as} \quad n \to \infty \quad \text{in} \quad G' \quad (k \in \mathbb{N}).$$

By Theorem 3.7 and by the property (iii) of T, we have

$$(X_n * \epsilon_k), (X * \epsilon_k) \in \left(S_\alpha^\beta\right)'$$

and

$$X_n * \epsilon_k \to X * \epsilon_k \quad \text{as} \quad n \to \infty \quad \text{in} \quad \left(S_\alpha^\beta\right)' \quad (k \in \mathbb{N}).$$

By Theorem 3.9, we have

$$X_n * \epsilon_k * \epsilon_k \to X * \epsilon_k * \epsilon_k \quad \text{as} \quad n \to \infty \quad \text{in} \quad G \quad (k \in \mathbb{N}).$$

Thus there exists a delta sequence $(\epsilon_k * \epsilon_k) \in \Delta$ such that

$$X_n * \epsilon_k * \epsilon_k, X * \epsilon_k * \epsilon_k \in G \quad (n, k \in \mathbb{N})$$

and

$$X_n * \epsilon_k * \epsilon_k \to X * \epsilon_k * \epsilon_k \quad \text{as} \quad n \to \infty \quad \text{in} \quad G \quad (k \in \mathbb{N}).$$

Hence $X_n \xrightarrow{\delta} X$ as $n \to \infty$ in $\mathcal{B}(G, \Delta)$.

The remaining properties follow easily from the definitions. We prefer to omit the details. This completes the proof.   □

References

[1] Chung, J., Chung, S. Y. and Kim, D., *A characterization of the Gelfand–Shilov spaces via Fourier transform*, Prod. Amer. Math. Soc. **124** (1996), 2101–2108.

[2] Chung, S. Y., Kim, D. and Lee, S., *Characterization for Beurling–Bjorck space and Schwartz space*, Prod. Amer. Math. Soc. **125** (11) (1997), 3229–3234.

[3] Gelfand, I. M., Shilov, G. E., *Generalized Functions*, Vol. I and II, Academic Press, New York, 1967.

[4] Ishihara, T., *On the structure of S space*, Osaka Math. J. **13** (1961), 251–264.

[5] Kashpirovskii, A. I., *Equality of the spaces S_α^β and $S_\alpha \cap S^\beta$*, (English. Russian original) Funct. Anal. Appl. 14, 129 (1980); translation from Funkts. Anal. Prilozh. 14, No.2, 60 (1980).

[6] Karunakaran, V., Kalpakam, N. V., *Boehmians and Fourier transform*, Integral Transform. Spec. Funct. **9** (3) (2000), 197–216.

[7] Mikusinski, P., *Convergence of Boehmians*, Japan J. Math. **9** (1983), 159–179.

[8] Mikusinski, P., *Boehmians and generalized functions*, Acta. Math. Hung. **51** (1988), 271–281.

[9] Zemanian, A. H., *Distribution Theory and Transform Analysis*, McGraw-Hill Book Co., New York, 1965.

V. Karunakaran
School of Mathematics
Madurai Kamaraj Universityy
Madurai, India - 625 021
e-mail: `vkarun_mku@yahoo.co.in`

R. Bhuvaneswari
School of Mathematics
Madurai Kamaraj University
Madurai, India - 625 021
e-mail: `buvibarathi@yahoo.co.in`