doi: 10.2478/umcsmath-2014-0003

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXVIII, NO. 1, 2014 SECTIO A 21–29

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The Turán number of the graph $3P_4$

Abstract. Let $ex(n, G)$ denote the maximum number of edges in a graph on n vertices which does not contain G as a subgraph. Let P_i denote a path consisting of i vertices and let mP_i denote m disjoint copies of P_i. In this paper we count $ex(n, 3P_4)$.

1. Introduction. Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The Turán number of the graph G, denoted by $ex(n, G)$, is the maximum number of edges in a graph on n vertices which does not contain G as a subgraph. Let P_i denote a path consisting of i vertices and let mP_i denote m disjoint copies of P_i. By C_q we denote a cycle of order q. For two vertex disjoint graphs G and F by $G \cup F$ we denote the vertex disjoint union of G and F, and by $G + F$ we denote the join of the graphs. By $\overline{G}$ we denote the complement of the graph G. For a vertex $x \in V(G)$ we define $N_G(x) = \{y \in V(G) | \{x, y\} \in E(G)\}$. Let F be a subgraph of G. Let $\deg_F(x) = N_G(x) \cap V(F)$. Moreover, for $A \subseteq V(G)$ let $G|_A$ denote the subgraph of G induced by A. The basic notions not defined in this paper can be found in [5]. First we present the following important lemma which is used to prove our main results.

Lemma 1 (Erdős, Gallai [2]). *Suppose that* $|V(G)| = n$. *If the following inequality*

$$\frac{(n-1)(l-1)}{2} + 1 \leq |E(G)|$$

2000 *Mathematics Subject Classification.* 05C35, 05C38.
Key words and phrases. Forests, trees, Turán number.

is satisfied for some $l \in \mathbf{N}$*, then there exists a cycle* C_q *in* G *for some* $q \geq l$*.*

We will use the following famous theorem.

Theorem 1 (Faudree and Schelp [3]). *If* G *is a graph with* $|V(G)| = kn + r$ $(0 \leq k, 0 \leq r < n)$ *and* G *contains no* P_{n+1}*, then* $|E(G)| \leq kn(n-1)/2 + r(r-1)/2$ *with the equality if and only if* $G = kK_n \cup K_r$ *or* $G = tK_n \cup (K_{(n-1)/2} + \overline{K}_{(n+1)/2+(k-t-1)n+r})$ *for some* $0 \leq t < k$*, where* n *is odd, and* $k > 0$*,* $r = (n \pm 1)/2$*.*

Gorgol [4] studied the Turán number for disjoint copies of graphs. She counted $ex(n, 2P_3)$ and $ex(n, 3P_3)$.

Theorem 2 (Gorgol [4]).

$$ex(n, 2P_3) = \left\lfloor \frac{n-1}{2} \right\rfloor + n - 1, \text{ for } n \geq 9.$$

$$ex(n, 3P_3) = \left\lfloor \frac{n-1}{2} \right\rfloor + 2n - 4, \text{ for } n \geq 14.$$

Moreover, she proved more general results concerning the properties of some extremal Turán graphs for disjoint copies of a given graph. Bushaw and Kettle [1] extended some of Gorgol's results as follows.

Theorem 3 (Bushaw and Kettle [1]).

$$ex(n, kP_3) = \left\lfloor \frac{n-k+1}{2} \right\rfloor + (n-k+1)(k-1) + \binom{k-1}{2}, \text{ for } n \geq 7k.$$

$$ex(n, kP_t) = \left(n - k\left\lfloor \frac{t}{2} \right\rfloor + 1\right)\left(k\left\lfloor \frac{t}{2} \right\rfloor - 1\right) + \binom{k\lfloor \frac{t}{2} \rfloor - 1}{2} + \epsilon,$$

for $n \geq 2t\left(1 + k\left(\lceil \frac{t}{2} \rceil + 1\right)\binom{t}{\lfloor \frac{t}{2} \rfloor}\right)$*, where* $\epsilon = 1$ *for odd* t *and* $\epsilon = 0$ *for even* t*.*

In particular, Bushaw and Kettle [1] counted $ex(n, 3P_4)$ for the case $n \geq 440$. We present $ex(n, 3P_4)$ for all positive integers n.

2. Results. First we prove the following result.

Theorem 4. *Let* $n \geq 15$*. Then*

$$ex(n, 3P_4) = 5n - 15. \tag{1}$$

Proof. First note that the graph $K_5 + \overline{K}_{n-5}$ does not contain $3P_4$ as a subgraph. Therefore, $ex(n, 3P_4) \geq 5n - 15$ and we would like to prove the opposite inequality. Suppose that there exists a graph G with $|V(G)| = n \geq 15$ and $|E(G)| = 5n - 14$ without $3P_4$ as a subgraph. Applying Lemma 1 to the graph G, we obtain

$$\frac{(n-1)(l-1)}{2} + 1 \leq 5n - 14,$$

$$l \leq 11 - \frac{20}{n-1}.$$

By $n \geq 15$,

$$\frac{20}{n-1} < 2$$

and we conclude that $l \leq 9$. It means that the graph G contains a cycle C_q, $q \geq 9$. Let $0, 1, 2, \ldots, q-1$ be the consecutive vertices in C_q. We should consider the following cases:

Case 1. Let $q \geq 12$. We have C_{12} in G, so $3P_4$ is a subgraph of G, a contradiction.

Case 2. Let $q = 11$.

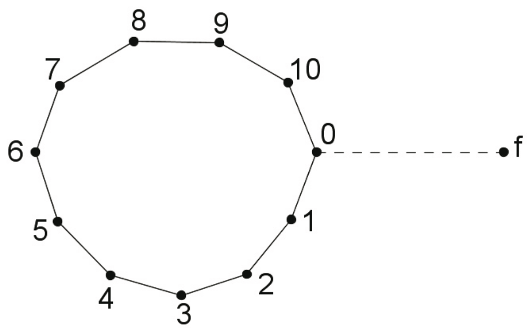

FIGURE 1. A graph G with the cycle C_{11}.

Let $F = G - V(C_{11})$. Note that C_{11} cannot be connected by an edge with F (see Figure 1 for an illustration). The minimum number of edges in F is equal to $5n - 14 - 55 = 5n - 69$. By Theorem 1 we know that

$$ex(k, P_4) = 3\left\lfloor \frac{k}{3} \right\rfloor + \binom{r}{2}, \; k \equiv r \pmod 3$$

where r is the rest from dividing k by 3. We set $k = n - 11$. If

$$3\left\lfloor \frac{n-11}{3} \right\rfloor + \binom{r}{2} < 5n - 69$$

then it means that P_4 is a subgraph of F. We check this.

(a) $r = 0$

$$5n - 3\frac{n-11}{3} > 69$$
$$n > 14.$$

(b) $r = 1$

$$5n - 3\frac{n-12}{3} > 69$$
$$n > 14.$$

(c) $r = 2$

$$5n - 3\frac{n-13}{3} > 1 + 69$$
$$n > 14.$$

So we get P_4 in F, a contradiction.

Case 3. Let $q = 10$.

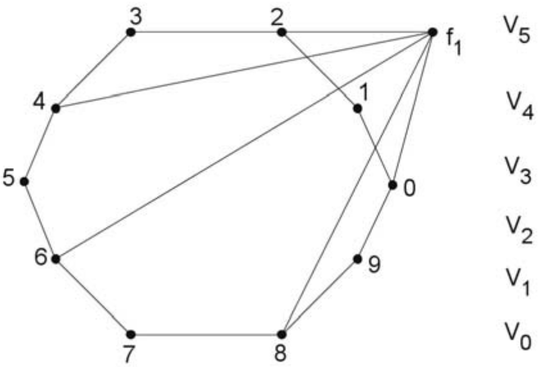

FIGURE 2. A graph G with the cycle C_{10}.

Let $F = G - V(C_{10})$. Note that $|V(F)| = n - 10$. The set of edges containing a vertex of F we can divide into:

• edges connecting C_{10} and F, i.e. the edges $\{x, f\}$ with $x \in V(C_{10})$, $f \in V(F)$,

• edges connecting both vertices inside F, i.e. the edges $\{f_i, f_j\}$ with f_i, $f_j \in V(F)$, $i \neq j$.

Notice that if the edge $\{0, f_1\}$ exists for some $f_1 \in V(F)$, then there cannot exist edges $\{1, f_1\}$ and $\{9, f_1\}$, in the opposite case we obtain a longer cycle, i.e. C_{11}. So at most 5 vertices of C_{10} can be adjacent to the vertex $f_1 \in V(F)$ (see Figure 2 for an illustration). Moreover, $\{j, f\} \notin E(G)$ for $f \in V(F) - \{f_1\}$ and $j \neq 2l+1$, $l = 0, 1, 2, 3, 4$, $j \in V(C_{10})$, in the opposite case we get $3P_4$ in G. Let

$$V(F) = V_0 \cup V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$$

be the partition of $V(F)$ such that each vertex from V_i has exactly i neighbors in C_{10}. Note that vertices from sets V_i, $i > 0$ cannot be connected between them and $\deg_F(u) = 0$ for each $u \in \bigcup_{i=1}^{5} V_i$. Vertices from the set V_0 can be connected only between them. So if $|V_0| = k$, then $|E(G|_{V_0})| \leq ex(k, P_4)$. First we show that $|E(G|_{V_0})| \leq |V_0|$. If $r \equiv 0 \pmod 3$, then

$$ex(k, P_4) = 3\frac{k}{3} = k = |V_0|.$$

If $r \equiv 1 \pmod 3$, then

$$ex(k, P_4) = 3\frac{k-1}{3} = k - 1 \leq |V_0|.$$

If $r \equiv 2 \pmod 3$, then

$$ex(k, P_4) = 3\frac{k-2}{3} + 1 = k - 1 \leq |V_0|.$$

We consider three subcases:

Case 3.1. Let $V_5 \neq \emptyset$. Then

$$|E(G)| \leq \binom{10}{2} - \binom{5}{2} + \sum_{i=1}^{5} i \cdot |V_i| + |E(V_0)| \leq 35 + \sum_{i=1}^{5} i \cdot |V_i| + |V_0|$$

$$\leq 35 + 5\sum_{i=1}^{5} |V_i| + 5|V_0| = 35 + 5(n-10) = 5n - 15.$$

Recall that $|E(G)| = 5n-14$. So we must add one more edge and we obtain $3P_4$ in G, a contradiction.

Case 3.2. Let $V_5 = \emptyset$ and $V_4 \neq \emptyset$. Then

$$|E(G)| \leq \binom{10}{2} - 5 + \sum_{i=1}^{4} i \cdot |V_i| + |V_0| \leq 45 - 5 + 4 \cdot \sum_{i=1}^{4} |V_i| + |V_0|$$
$$\leq 4n < 5n - 14$$

for $n \geq 15$. So again we must add one more edge which means that we get a $3P_4$ in G, a contradiction.

Case 3.3. Let $V_5 = \emptyset$ and $V_4 = \emptyset$. Then

$$|E(G)| \leq \binom{10}{2} + 3\sum_{i=1}^{3} |V_i| + |V_0| \leq 45 + 3(n-10) = 3n + 15 < 5n - 14$$

for $n \geq 15$. We obtain a contradiction.

Case 4. Let $q = 9$. Let $F = G - V(C_9)$. If there does not exist any edge between C_9 and F, then $|E(F)| \geq 5n - 50$. So if $ex(n-9, P_4) < 5n - 50$, then there exists a path P_4 in the graph F.

(a) $r = 0$

$$3\frac{n-9}{3} < 5n - 50,$$
$$n > 10.$$

(b) $r = 1$

$$3\frac{n-10}{3} < 5n - 50,$$
$$n > 10.$$

(c) $r = 2$

$$3\frac{n-11}{3} + 1 < 5n - 50,$$

$$n > 10.$$

In this case we obtain a contradiction.

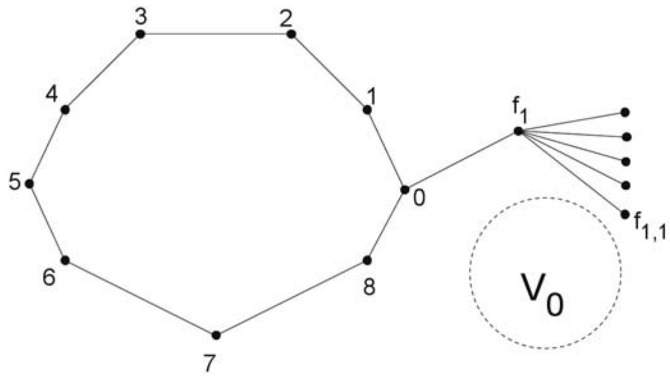

FIGURE 3. A graph G with the cycle C_9.

Suppose that there exists an edge $\{0, f_1\}$ for some $f_1 \in V(F)$. Note that the vertex f_1 can be adjacent to another vertex $f_{1,1}$ from F and we do not obtain $3P_4$ (see Figure 3 for an illustration). Now we cannot create other edges from $F - \{f_1, f_{1,1}\}$ to C_9, in the opposite case we obtain $3P_4$. Note that

$$\begin{aligned} |E(G)| &\leq \binom{9}{2} + 7 + |N_F(f_1)| + ex(n - 10 - |N_F(f_1)|, P_4) \\ &\leq 43 + |N_F(f_1)| + (n - 10 - |N_F(f_1)|) = n + 33 < 5n - 14 \end{aligned}$$

for $n \geq 12$.

So we have $3P_4$ in graph G, a contradiction. The proof is completed. □

Remark 1. Note that if $n \in \{1, \ldots, 11\}$, then $ex(n, 3P_4) = \binom{n}{2}$. It is clear because the total number of vertices does not exceed 12 and K_n does not contain $3P_4$. Moreover, $\binom{n}{2} \geq 5n - 15$ for $n \in \{1, \ldots, 11\}$.

Remark 2. For $n = 12$ we have $ex(n, 3P_4) \geq \binom{11}{2} = 55$. It is clear because $K_{11} \cup K_1$ does not contain $3P_4$. Let $|E(G)| = 56$. Applying Lemma 1, we obtain that

$$\frac{11(l-1)}{2} + 1 \leq 56,$$

$$l \leq 11.$$

So there exists a cycle C_q, $q \geq 11$. It is clear that if $q = 12$, there exists $3P_4$.

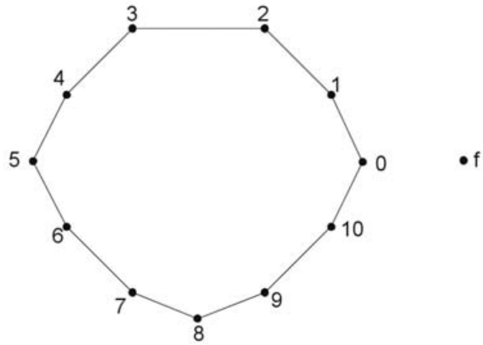

FIGURE 4. A graph with and C_{11} for $ex(12, 3P_4)$.

If $q = 11$, then we have $\deg_{C_{11}}(f) = 0$ for $f \in V(F)$ and $|E(G|_{C_{11}})| \leq \binom{11}{2} = 55$ (see Figure 4 for an illustration). We get a contradiction.

For $n = 13$ we have $ex(n, 3P_4) \geq \binom{11}{2} + \binom{2}{2} = 56$. It follows from the fact that $K_{11} \cup K_2$ does not contain $3P_4$. Let $|E(G)| = 57$. Applying Lemma 1, we obtain that

$$\frac{12(l-1)}{2} + 1 \leq 57,$$

$$l \leq 10.$$

If $q \geq 12$, then there exists $3P_4$. If $q = 11$, then we have a cycle C_{11} and a path P_2. But these two graphs cannot have edges between them and the total number of edges is equal to 56. So we must add one more edge and we obtain $3P_4$, a contradiction. If $q = 10$, then we have at most 45 edges in $G|_{V(C_{10})}$ and we need at least 12 more edges. We have 3 vertices outside C_{10}, say f_1, f_2, f_3. If $\{f_1, f_2\} \in E(G)$, then $N(f_i) \cap V(C_{10}) = \emptyset$ for $i = 1, 2$, in the opposite case we get $3P_4$. Thus $\deg_{C_{10}}(f_i) \geq 4$ for some $i = 1, 2, 3$. Note that $\deg_{C_{10}}(f_i) \leq 5$, $i = 1, 2, 3$, in the opposite case we get C_{11}. If f_1 has 4 edges with C_{10}, then we must delete at least $\binom{4}{2} = 6$ edges from K_{10}. So we need 14 more edges. So $\deg_{C_{10}}(f_i) > 5$ and we have a contradiction.

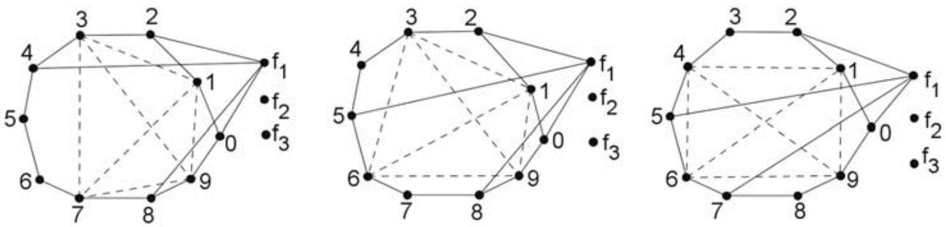

FIGURE 5. Graphs with C_{10} for $ex(13, 3P_4)$.

Figure 5 presents a subgraph of G with the cycle C_{10}. Dotted lines denote edges in $\overline{G}$, in the opposite case we get a longer cycle in G.

For $n = 14$ we have $ex(n, 3P_4) \geq \binom{11}{2} + \binom{3}{2} = 58$. It follows from the fact that $K_{11} \cup K_3$ does not contain $3P_4$. Let $|E(G)| = 59$. Applying Lemma 1, we have

$$\frac{13(l-1)}{2} + 1 \leq 59,$$

$$l \leq 9.$$

If $q \geq 12$, then we have $3P_4$ in G. If $q = 11$, then we have 55 edges in K_{11} and 3 edges in K_3 and K_{11} and K_3 must be disjoint. But we have 58 edges so we must add one more edge and we obtain $3P_4$, a contradiction. Let $q = 10$. We have 45 edges in K_{10} and we need 14 more edges. We have 4 vertices outside C_{10}. So $\deg_{C_{10}}(f_i) > 3$ for some $i = 1, 2, 3, 4$. Moreover, $\deg_{C_{10}}(f_i) \leq 5$ for $i = 1, 2, 3, 4$, in the opposite case we get a cycle C_{11}. If f_1 creates 5 edges with the vertices of C_{10}, then we must delete 10 edges from K_{10}. So we need 19 more edges. But we have only 3 vertices in $V(F) - \{f_1\}$, so $\deg_{C_{10}}(f_i) > 5$ for some $i = 2, 3, 4$ and we have a contradiction. Similarly, if $\deg_{C_{10}}(f_1) = 4$ and $\deg_{C_{10}}(f_i) \leq 4$ for $i = 2, 3, 4$, then we must delete at least 6 edges from K_{10}. So we need 20 more edges. We have three vertices in $V(F) - \{f_1\}$. Hence $\deg_{C_{10}}(f_i) > 5$ for some $i = 2, 3, 4$ and we have a contradiction.

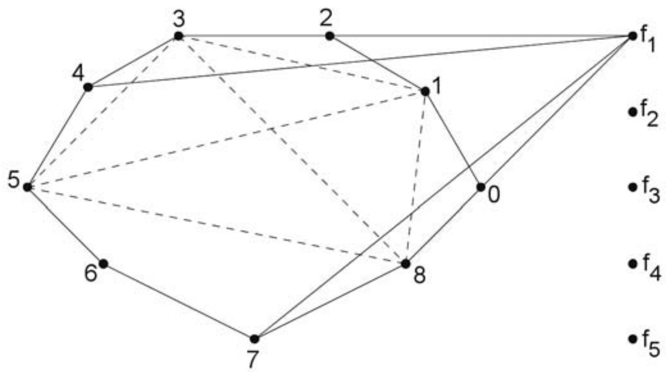

FIGURE 6. A graph with and C_9 for $ex(14, 3P_4)$.

Let $q = 9$. We have 36 edges in K_9. So we need at least 23 edges outside $G|_{C_9}$. We have 5 vertices outside the cycle C_9, i.e. in the graph F (see Figure 6 for an illustration). Recall that $ex(5, P_4) = 4$. So we have at least 19 edges between $V(C_9)$ and $V(F)$. Thus there exists a vertex $f_i \in V(F)$, such that $\deg_{C_9}(f_i) \geq 4$. Note that $\deg_{C_9}(f_i) \leq 4$ for any $f_i \in V(F)$, in the opposite case we get a cycle C_{10}. Let f_1 be the vertex adjacent to four vertices of C_9. Then $G|_{C_9}$ is not isomorphic to K_9, in the opposite case we get a longer cycle. We must delete from K_9 at least $\binom{4}{2} = 6$ edges (see dotted lines in Figure 6). So now we need at least 21 edges between $V(C_9)$

and $V(F)$. But we have only 5 vertices in F. So there exists f_i for which $\deg_{C_9}(f_i) > 4$ and we have a contradiction.

Summarizing, we collect results from above remarks in Theorem 5.

Theorem 5. *Let n be a natural number and $n \leq 14$. Then*

$$ex(n, 3P_4) = \binom{n}{2} \text{ for } n \leq 11,$$

$$ex(n, 3P_4) = \binom{11}{2} = 55 \text{ for } n = 12,$$

$$ex(n, 3P_4) = \binom{11}{2} + \binom{2}{2} = 56 \text{ for } n = 13,$$

$$ex(n, 3P_4) = \binom{11}{2} + \binom{3}{2} = 58 \text{ for } n = 14.$$

Theorems 4 and 5 present the Turán number $ex(n, 3P_4)$ for all positive integers n.

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Received June 4, 2013