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On boundary behavior of Cauchy integrals

Abstract. In this paper, we shall estimate the growth order of the n-th derivative Cauchy integrals at a point in terms of the distance between the point and the boundary of the domain. By using the estimate, we shall generalize Plemelj–Sokthoski theorem. We also consider the boundary behavior of generalized Cauchy integrals on compact bordered Riemann surfaces.

1. Introduction. Let φ be a continuous function on a smooth Jordan curve Γ in $\mathbb{C}$ and consider the Cauchy integral;

$$F(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\zeta)}{\zeta - z}d\zeta, \tag{1.1}$$

for $z \in \mathbb{C} \setminus \Gamma$. It is a holomorphic function on $\mathbb{C} \setminus \Gamma$. Let D_+ and D_- denote the bounded component of $\mathbb{C} \setminus \Gamma$ and the unbounded component, respectively. On the boundary behavior of the Cauchy integral, the following is well known and it is called Plemelj–Sokthotski formula (cf. [6]).

Theorem (Plemelj–Sokthotski). *Suppose that φ is a Hölder continuous function of order α $(0 < \alpha < 1)$ on Γ, that is, there exists a constant $A > 0$ such that*

$$|\varphi(\zeta_1) - \varphi(\zeta_2)| \leq A|\zeta_1 - \zeta_2|^\alpha. \tag{1.2}$$

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Then, the Cauchy integral (1.1) *of* φ *has a limit at each* $\zeta_0 \in \Gamma$. *Furthermore, let* $F_+(\zeta_0)$ *and* $F_-(\zeta_0)$ *denote the boundary values of* F *from* D_+ *and* D_-, *respectively, then both* F_+ *and* F_- *are Hölder continuous functions of order* α *on* Γ *and*

$$F_+(\zeta_0) = \frac{1}{2}\varphi(\zeta) + \frac{1}{2\pi i} P.V. \int_\Gamma \frac{\varphi(\zeta)}{\zeta - \zeta_0} d\zeta \tag{1.3}$$

$$F_-(\zeta_0) = -\frac{1}{2}\varphi(\zeta) + \frac{1}{2\pi i} P.V. \int_\Gamma \frac{\varphi(\zeta)}{\zeta - \zeta_0} d\zeta, \tag{1.4}$$

where P.V. means the principal value of the integral at ζ_0. *In particular,*

$$F_+(\zeta_0) - F_-(\zeta_0) = \varphi(\zeta_0). \tag{1.5}$$

From the theorem, the principal value gives a mapping from the space of Hölder continuous functions of order α to itself if $\alpha \in (0,1)$ while it does not hold when $\alpha = 1$ and it inspired interest in the theory of singular integrals (cf. [9]). Zygmund and I. E. Block ([3]) improved the theorem for functions of Zygmund class Λ^* (see also [8] for several complex variables). Bikčantaev ([2]) generalized the theorem on open Riemann surfaces. On the other hand, J. L. Walsh ([14]) showed that the equation (1.5) holds almost everywhere if Γ is the unit circle and φ is in the class L^2 with respect to the Lebesgue measure $d\theta$ on the circle.

Theorem (Walsh). *Let* φ *be in* $L^2(d\theta)$ *on the unit circle* $\{z = e^{i\theta}\}$. *Then the Cauchy integral* (1.1) *has non-tangential limits* $F_+(e^{i\theta})$ *and* $F_-(e^{i\theta})$ *almost everywhere from inside and outside the circle, respectively.*

In this paper, we have two purposes. The first purpose is to relax the condition (1.2) and show results similar to Plemelj–Sokthotski theorem hold. The second one is to extend Walsh's theorem for L^p functions on boundaries of compact bordered Riemann surfaces. In the following, we will present main results in this paper. The terminologies will be given in §2.

First, we shall show an estimate of the derivative of the Cauchy integral of φ, which gives a generalization of Hardy–Littlewood theorem (cf. [12], see also [11]).

Theorem 1. *Let* Γ *be a smooth Jordan curve in* $\mathbb{C}$ *and* φ *be a continuous function on* Γ.

Suppose that the function φ *belongs to* $\Lambda^*_\Gamma(\omega)$ *for* $\omega \in \mathcal{D}$. *Then, there exists a constant* $A > 0$ *such that the derivative of the Cauchy integral* $F(z)$ *of* φ *satisfies an inequality,*

$$|F^{(n)}(z)|\delta(z)^n \leq A\omega(\delta(z)), \tag{1.6}$$

for any $z \in \mathbb{C} \setminus \Gamma$ *near* Γ, *where* $\delta(z) = \mathrm{dist}(z, \Gamma) := \min_{\zeta\in\Gamma} |z - \zeta|$.

Using the above theorem, we prove a generalization of Plemelj–Sokthotski theorem as a corollary.

Corollary 1. *Let* Γ, φ *and* ω *be the same as in Theorem 1. Furthermore, we suppose that*

$$\int_0^1 \frac{\omega(t)}{t} dt < \infty. \tag{1.7}$$

Then, the Cauchy integral F *of* φ *has a limit at each* $\zeta_0 \in \Gamma$*. Moreover, the boundary values* $F_+(\zeta_0)$ *and* $F_-(\zeta_0)$ *are given by* (1.3) *and* (1.4).

We also give the modulus of continuity of the Cauchy integral F on Γ.

Corollary 2. *Let* Γ, φ *and* ω *be the same as in Theorem 1. Then, the boundary functions* F_+ *and* F_- *belong to* $\Lambda_\Gamma^*(Z_\omega)$*, where*

$$Z_\omega(t) = \max\left\{\int_0^t \omega(s)s^{-1}ds, \omega(t)\right\} \quad (t > 0).$$

In particular,

$$P.V. \int_\Gamma \frac{\varphi(\zeta)}{\zeta - \cdot} d\zeta \in \Lambda_\Gamma^*(Z_\omega).$$

Remark 1. Zygmund showed that if Γ is the unit circle and $\varphi \in \Lambda_\Gamma^*(\omega)$, then

$$P.V. \int_\Gamma \frac{\varphi(\zeta)}{\zeta - \cdot} d\zeta \in \Lambda_\Gamma^*(Z_\omega^0),$$

where

$$Z_\omega^0(t) = \int_0^t \frac{\omega(s)}{s} ds + t\int_t^{t_0} \frac{\omega(s)}{s^2} ds, \tag{1.8}$$

for some $t_0 > 0$ (see [7] p. 106). See also [9] for some related results. However, it is sometimes hard to calculate Z_ω^0 when Z_ω can be calculated (see §5).

Finally, we consider an analogue of Walsh's theorem on compact bordered Riemann surface. Let R be a compact bordered Riemann surface and $\hat{R}$ the double of R. Let $\hat{g}$ be the genus of $\hat{R}$. Then, there exists a canonical homology basis of $\hat{R}$ $\{A_1, B_1, \ldots, A_{\hat{g}}, B_{\hat{g}}\}$ such that

$$A_i \cap A_j = \emptyset,\ B_i \cap B_j = \emptyset \quad (i \neq j),$$

and the intersection number of A_i and B_j is δ_{ij}.

Take a point $\hat{P}_0 \in \hat{R} \setminus (R \cup \partial R)$ and fix it. Let $\omega_{\hat{P}_0,P}$ be an abelian differential of the third kind on $\hat{R}$ with simple poles at $\hat{P}_0$ and P $(\neq \hat{P}_0)$ such that the residue at $\hat{P}_0$ is -1 and 1 at P. We assume that the differential $\omega_{\hat{P}_0,P}$ is *normalized*, that is,

$$\int_{A_j} \omega_{\hat{P}_0,P} = 0 \quad (j = 1, 2, \ldots, \hat{g}).$$

For $\varphi \in L^1(\partial R)$, we define a generalized Cauchy integral F of φ by

$$F(P) = \frac{1}{2\pi i}\int_{\partial R} \varphi\omega_{\hat{P}_0,P} \quad (P \notin \partial R). \tag{1.9}$$

Then, we have the following theorem which extends theorems of Plemelj–Sokthotski and Walsh. The proof will be given in §7.

Theorem 2. *Let R be a compact Riemann surface and $\omega_{\hat{P}_0,P}$ the normalized abelian differential of the third kind with poles at $\hat{P}_0$ and P as above.*

(1) *If $\varphi \in L^1(\partial R)$, then the generalized Cauchy integral* (1.9) *is holomorphic on $\hat{R} \setminus \partial R$.*
(2) *The same statements of Theorem 1 and Corollary 1 hold for $F(P)$.*
(3) *Let φ be in $L^1(\partial R)$. Then, the generalized Cauchy integral* (1.9) *of φ has non-tangential limits almost everywhere on ∂R. Furthermore, if $\varphi \in L^p(\partial R)$ for $p > 1$, the equation* (1.5) *holds almost everywhere on ∂R. Namely, let F_+ and F_- denote the boundary functions from R and from $\hat{R} \setminus (R \cup \partial R)$, respectively. Then,*

$$F_+ - F_- = \varphi$$

almost everywhere on ∂R.

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2. Preliminaries.

2.1. Modulus of continuity. Let ω be a continuous function on $[0,\infty)$. We denote by $\mathcal{D}$ the set of all ω satisfying the following conditions.

(1) $\omega(0) = 0$, and it is an increasing function on $[0,\infty)$.
(2) $\omega(t)$ is doubling, i. e. there exists a constant $A > 0$ such that

$$\omega(s) \le \omega(t) \le A\omega(s),$$

if $0 < s < t \le 2s$.
(3) For any α $(0 < \alpha < 1)$, $t^\alpha \le \omega(t)$ if $t > 0$ is less than some $\delta > 0$.

It is easy to see that $\omega_\alpha(t) = \min\{|\log t|^{-\alpha}, A\}$ $(\alpha > 0, A > 0)$ satisfies the above conditions.

For a continuous function ω on $[0,\infty)$, we say that a function φ on Γ has the modulus of continuity ω if there exists a constant $A > 0$ such that

$$|\varphi(\zeta_1) - \varphi(\zeta_2)| \le A\omega(|\zeta_1 - \zeta_2|), \tag{2.1}$$

for every $\zeta_1, \zeta_2 \in \Gamma$. We denote by $\Lambda^*_\Gamma(\omega)$ the set of functions on Γ which have the modulus of continuity ω.

2.2. Compact bordered Riemann surfaces. Let R be an open Riemann surface. A holomorphic function f on R is said to be of class $H^p(R)$ $(1 \leq p < \infty)$ if there exists a harmonic function u on R such that $|f|^p \leq u$. For $p = \infty$, $H^\infty(R)$ is the space of bounded holomorphic functions on R. The space $H^p(R)$ is called *Hardy space* on R.

We say that the Riemann surface is *compact bordered* if there exists a closed Riemann surface R_0 such that R is a subdomain of R_0 bounded by a finite number of analytic Jordan curves. A compact bordered Riemann surface R is called of type (g, n) if the genus of R is g and the number of boundary components is n.

For each compact bordered Riemann surface R, we may consider *the double* of R. We denote it by $\hat{R}$ and by $\pi : \hat{R} \to \hat{R}$ the anticonformal involution of $\hat{R}$. If R is of type (g, n), then $\hat{R}$ is a closed Riemann surface of genus $2g + n - 1$.

Let $\{C_1, C_2, \ldots, C_n\}$ be the set of boundary curves of a compact bordered Riemann surface R of type (g, n). For each C_i $(i = 1, 2, \ldots, n)$ there exists an annular domain U_i in R such that $\partial U_i = C_i \cup \gamma_i$, where γ_i is a smooth curve parallel to C_i. Then there exists a conformal mapping $f_i : U_i \cup \partial U_i \to A_i := \{0 < r_i \leq |z| \leq 1\}$ for some r_i such that $f_i(C_i) = \{|z| = 1\}$. We say that a function F on R has a *non-tangential limit* at $p \in C_i$ if $F \circ f_i^{-1}$ has a non-tangential limit at $f_i(p)$, and that F has non-tangential limits almost everywhere on ∂R if $F \circ f_i^{-1}$ does so on $\{|z| = 1\}$ for all i $(1 = 1, 2, \ldots, n)$. It is not hard to see that those notions do not depend on the choice of the annular domain U_i and the conformal mapping f_i.

We define function spaces on ∂R by using f_i.

Definition 1. For $p \geq 1$, we define $L^p(\partial R)$ by the set of all functions φ on ∂R so that $\varphi \circ f_i^{-1}$ $(i = 1, 2, \ldots, n)$ belong to L^p space on the unit circle with respect to the Lebesgue measure on the circle.

Definition 2. A function φ on ∂R is said to be a Hölder continuous function of order α if $\varphi \circ f_i^{-1}$ $(i = 1, 2, \ldots, n)$ are Hölder continuous functions of order α on the unit circle. For $\omega \in \mathcal{D}$, we denote by $\Lambda^*_{\partial R}(\omega)$ the set of all functions φ on ∂R such that $\varphi \circ f_i^{-1} \in \Lambda^*_{\{|z|=1\}}(\omega)$ $(i = 1, 2, \ldots, n)$.

Definition 3. Let ψ_P be an abelian differential on a neighborhood ∂R with a pole at $P \in C_i$. For a continuous function φ on ∂R, we define the principal value P. V. $\int_{\partial R} \varphi \psi_P$ at P by

$$\sum_{j \neq i} \int_{C_j} \varphi \psi_P + \text{P. V.} \int_{|z|=1} (\varphi \circ f_i^{-1}) \psi_P \circ f_i^{-1}.$$

Definitions 1 and 2 do not depend on the choice of U_i and f_i while Definition 3 may depend on them.

Here, we note the following on the boundary values of H^p-functions.

Proposition 1. *Let R be a compact bordered Riemann surface. Then, every $f \in H^p(R)$ $(1 \leq p \leq \infty)$ has non-tangential limits almost everywhere on ∂R and the boundary function belongs to $L^p(\partial R)$.*

From the proposition, the set of non-tangential boundary functions of $H^p(R)$, which we denote by $H^p(R)|_{\partial R}$, is regarded as a subspace of $L^p(\partial R)$. M. Heins clarifies the relationship between $H^p(R)$ and $L^p(\partial R)$. To describe the result, we consider Green's function $g(\cdot, p_0)$ of R with pole at $p_0 \in R$. It is a positive harmonic function on $R \setminus \{p_0\}$ with logarithmic singularity at p_0 and vanishes identically on ∂R. Hence, it is extended to the double $\hat{R}$ of R by $g(p, p_0) = -g(\pi(p), p_0)$ for $p \in \hat{R} \setminus R$, and $\omega_{p_0} := -dg(\cdot, p_0) - i^*dg(\cdot, p_0)$ defines an abelian differential of the third kind on $\hat{R}$, where ${}^*\psi$ stands for the conjugate differential of a differential ψ. The abelian differential ω_{p_0} has the simple poles at p_0 and $\hat{p}_0 := \pi(p_0)$, where the residues are 1 and -1, respectively. We denote by δ the devisor of ω_{P_0} in $\hat{R}$. Then, Heins ([10]) shows the following:

Proposition 2. *For $p \in (1, \infty)$, the decomposition*

$$L^p(\partial R) = H^p(R)|_{\partial R} + \overline{H_0^p(R)}|_{\partial R} + M(\delta^{-1})|_{\partial R} \tag{2.2}$$

holds, where $H_0^p(R)$ is the space of holomorphic functions in $H^p(R)$ which vanish at P_0 and $M(\delta^{-1})$ is the space of meromorphic functions on $\hat{R}$ whose devisors are multiple of δ^{-1}.

3. Proof of Theorem 1. Let z_0 be a point in $\mathbb{C} \setminus \Gamma$ and $\zeta_0 \in \Gamma$ a point with $\delta(z_0) = |z_0 - \zeta_0|$. We take an interval $I(z_0, \zeta_0)$ whose end points are z_0 and ζ_0. Then, for any $z \in I(z_0, \zeta_0)$, $|z - \zeta_0| = \delta(z)$. Namely, ζ_0 is the nearest point of Γ from z.

Since

$$F'(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(\zeta) - \varphi(\zeta_0)}{(\zeta - z)^2} d\zeta,$$

we have

$$|F^{(n)}(z)| \leq \frac{n!}{2\pi} \int_\Gamma \frac{|\varphi(\zeta) - \varphi(\zeta_0)|}{|\zeta - z|^{n+1}} |d\zeta|.$$

Since $\zeta = \zeta(s)$ is differentiable, there exists $\delta > 0$ such that for each $t \in [-\delta, \delta]$ there exists a point $\zeta_t \in \Gamma$ such that $\arg(\zeta_t - z_0) - \arg(\zeta_0 - z_0) = t$ and the segment I_t between z_0 and ζ_t intersects only at ζ_t with Γ. It suffices to show that (1.6) is valid in a neighborhood of Γ. Hence, we may assume that there exists a constant $\varepsilon_0 > 0$ such that $\delta, \delta(z_0) > \varepsilon_0$.

Then, for $z \in I_0 \subset I(z_0, \zeta_0)$ we have

$$\begin{aligned} |F^{(n)}(z)| &\leq \frac{n!}{2\pi}\int_\Gamma \frac{|\varphi(\zeta)-\varphi(\zeta_0)|}{|\zeta-z|^{n+1}}|d\zeta| \\ &= \frac{n!}{2\pi}\int_{\Gamma_\delta} \frac{|\varphi(\zeta)-\varphi(\zeta_0)|}{|\zeta-z|^{n+1}}|d\zeta| + \frac{n!}{2\pi}\int_{\Gamma\backslash\Gamma_\delta} \frac{|\varphi(\zeta)-\varphi(\zeta_0)|}{|\zeta-z|^{n+1}}|d\zeta| \\ &=: A_1(z)+A_2(z), \end{aligned} \tag{3.1}$$

where $\Gamma_\delta \ni \zeta_0$ is the subarc of Γ between $\zeta_{-\delta}$ and ζ_δ.

Now, we consider the behavior of $|F^{(n)}(z)|$ as $I_0 \ni z \to \zeta_0$. We may assume that $|z-z_0| > \delta(z_0)/2$.

Since the point z lies in a neighborhood of Γ, by taking a sufficiently small $\delta > 0$ if necessary, we suppose that

$$|d\zeta_t| \leq 2dt,$$

on Γ_δ.

There is a point z_t on I_t with $|z_t - z_0| = \delta(z_0) \leq |\zeta_t - z_0|$ for each $t \in [-\delta, \delta]$. Thus, we have

$$\begin{aligned} A_1(z) &\leq \frac{n!}{2\pi}\int_{\Gamma_\delta} \frac{|\varphi(\zeta)-\varphi(\zeta_0)|}{|z_t-z|^{n+1}}|d\zeta| \\ &\leq \frac{1}{\pi}\int_{-\delta}^{\delta} \frac{|\varphi(\zeta_t)-\varphi(\zeta_0)|}{|z_t-z|^{n+1}}dt \\ &\leq \frac{An!}{\pi\delta(z_0)^{n+1}}\int_{-\delta}^{\delta} \frac{\omega(|\zeta_t-\zeta_0|)}{(1-2r\cos t+r^2)^{(n+1)/2}}dt \end{aligned}$$

where $r = |z-z_0|\delta(z_0)^{-1} \in (\frac{1}{2}, 1)$. Noting that

$$1-2r\cos t + r^2 \geq (1-r)^2 + \frac{4rt^2}{\pi^2},$$

we have

$$A_1(z) \leq \frac{An!}{2\pi\delta(z_0)^{n+1}}\int_{-\delta}^{\delta} \frac{\omega(|t|)}{\{(1-r)^2+4r(t/\pi)^2\}^{(n+1)/2}}dt.$$

Setting $C_r = \pi^2(1-r)^2/4r$ and $t = \sqrt{C_r}\tan\theta$, we have

$$\begin{aligned} B(z) &:= \frac{An!}{2\pi\delta(z_0)^{n+1}}\int_{-\delta}^{\delta} \frac{\omega(|t|)}{\{(1-r)^2+4r(t/\pi)^2\}^{(n+1)/2}}dt \\ &= \frac{A\pi^2 n!}{4r\delta(z_0)^{n+1}}\int_{-\delta}^{\delta} \frac{\omega(|t|)}{\{t^2+C_r\}^{(n+1)/2}}dt \end{aligned}$$

$$= \frac{A\pi^2 n!}{4r\delta(z_0)^{n+1}(\sqrt{C_r})^n}\int_{-\beta_r}^{\beta_r}\cos\theta^{n-1}\omega(|\sqrt{C_r}\tan\theta|)d\theta$$
$$\leq \frac{A\pi^2 n!}{2r\delta(z_0)^{n+1}(\sqrt{C_r})^n}\int_{0}^{\beta_r}\omega(|\sqrt{C_r}\tan\theta|)d\theta,$$

where $\beta_r = \arctan\frac{\delta}{\sqrt{C_r}} \in (0,\pi/2)$. As $z \to \zeta_0$, $r \to 1$, $C_r \to 0$ and $\beta_r \to \pi/2$. We take $z \in I_0$ sufficiently close to ζ_0 so that $C_r < 1$.

When $\theta \in (0, \frac{\pi}{4}]$, $\tan\theta \in (0,1]$. Hence, $|\sqrt{C_r}\tan\theta| \leq \sqrt{C_r}$ and $\omega(|\sqrt{C_r}\tan\theta|) \leq \omega(\sqrt{C_r})$. Thus, we have

$$\int_0^{\pi/4}\omega(|\sqrt{C_r}\tan\theta|)d\theta \leq \int_0^{\pi/4}\omega(\sqrt{C_r})d\theta = \frac{\pi}{4}\omega(\sqrt{C_r}),$$

and

$$\frac{1}{(\sqrt{C_r})^n}\int_0^{\pi/4}\omega(|\sqrt{C_r}\tan\theta|)d\theta \leq \frac{\pi}{4(\sqrt{C_r})^n}\omega(\sqrt{C_r}). \tag{3.2}$$

For any $\sigma \in (0,1)$, we put $\lambda = 1-\sigma \in (0,1)$ and $\gamma_r := \arctan\left(\frac{1}{\sqrt{C_r}}\right)^{\lambda}$. We may assume that $\gamma_r < \beta_r$. When $\theta \in (\frac{\pi}{4}, \gamma_r]$, $\tan\theta \in \left(1, C_r^{-\lambda/2}\right]$ and we have

$$\omega(\sqrt{C_r}\tan\theta) \leq \omega\left(\sqrt{C_r^{1-\lambda}}\right).$$

Therefore,

$$\int_{\pi/4}^{\gamma_r}\omega(\sqrt{C_r}\tan\theta)d\theta \leq \frac{\pi}{4}\omega\left(\sqrt{C_r^{1-\lambda}}\right),$$

and

$$\frac{1}{(\sqrt{C_r})^n}\int_{\pi/4}^{\gamma_r}\omega(|\sqrt{C_r}\tan\theta|)d\theta \leq \frac{\pi}{4(\sqrt{C_r})^n}\omega\left(\sqrt{C_r^{1-\lambda}}\right). \tag{3.3}$$

Finally, we consider the case where $\theta \in (\gamma_r, \beta_r]$. Since $\arctan x = \int_0^x \frac{1}{1+t^2}dt$, we have

$$\beta_r - \gamma_r = \int_{(\sqrt{C_r})^{-\lambda}}^{\delta/\sqrt{C_r}}\frac{dx}{x^2+1} \leq \int_{(\sqrt{C_r})^{-\lambda}}^{\delta/\sqrt{C_r}}\frac{dx}{x^2} \leq A_1\sqrt{C_r^{\lambda}},$$

where the constant $A_1 > 0$ depends only on r, and $A_1 \to 1$ as $r \to 1$. On the other hand,

$$\left(\frac{1}{\sqrt{C_r}}\right)^{-\lambda} \leq \tan\theta \leq \left(\frac{\delta}{\sqrt{C_r}}\right),$$

and

$$\omega(\sqrt{C_r}\tan\theta) \leq \omega(\delta).$$

Therefore, we conclude

$$\text{(3.4)} \qquad \frac{1}{(\sqrt{C_r})^n}\int_{\gamma_r}^{\beta_r} \omega\big(\big|\sqrt{C_r}\tan\theta\big|\big)d\theta \le \frac{A_1\sqrt{C_r^{\lambda}}}{(\sqrt{C_r})^n}\omega(\delta).$$

Combining (3.2), (3.3) and (3.4), we have

$$B(z) \le \frac{A_2}{(\sqrt{C_r})^n}\left\{\omega(\sqrt{C_r}) + \omega\left(\sqrt{C_r^{1-\lambda}}\right) + \sqrt{C_r^{\lambda}}\omega(\delta)\right\},$$

for some constant A_2 depending only on r. Since $0 < 1-\lambda < 1$, we have $\omega\big(\sqrt{C_r^{1-\lambda}}\big) \ge \omega(\sqrt{C_r})$ as $C_r \to 0$. By the definition of $\mathcal{D}$, $\sqrt{C_r^{\lambda}} \le \omega(\sqrt{C_r})$. Hence,

$$B(z) \le \frac{A_3}{(\sqrt{C_r})^n}\omega\left(\sqrt{C_r^{1-\lambda}}\right),$$

where the constant $A_3 > 0$ depends only on $r < 1$ and $\delta > 0$, and it is bounded if r is sufficiently close to 1 and δ is sufficiently small.

Noting that $\delta(z_0) > \varepsilon$, $1-r = (\delta(z_0) - |z-z_0|)\delta(z_0)^{-1} = \delta(z)\delta(z_0)^{-1}$ and

$$\frac{\pi^2}{4}(1-r)^2 \le C_r \le \frac{\pi^2}{2}(1-r)^2,$$

we obtain, for $\sigma = 1-\lambda \in (0,1)$,

$$A_1(z) \le B(z) \le \frac{A_3}{\delta(z)^n}\omega((\varepsilon^{-1}\pi\delta(z))^{\sigma}) \le \frac{A_3A^{\log\varepsilon^{-1}\pi/\log 2}}{\delta(z)^n}\omega(\delta(z)^{\sigma}),$$

by the doubling property of ω. Here, A is the constant of the doubling property of ω.

Since $\delta > \varepsilon$, it is not hard to see that $A_2(z)$ is bounded by some constant $M > 0$ as $z \to \zeta_0$. Hence, we have

$$|A_2(z)| \le M = \frac{M}{\delta(z)^n}\delta(z)^n \le \frac{M}{\delta(z)^n}\omega(\delta(z)^{\sigma}).$$

if $\delta(z) > 0$ is sufficiently small. Therefore, we conclude that there exists a constant $A_5 > 0$ such that

$$|F^{(n)}(z)|\delta(z)^n \le A_5\omega(\delta(z)^{\sigma}).$$

Since the constant A_5 is independent of σ, we have

$$\text{(3.5)} \qquad |F^{(n)}(z)|\delta(z)^n \le A_5\omega(\delta(z)).$$

By using the differentiability, we see that we can take $\varepsilon > 0$ sufficiently small so that there exists a neighborhood U of Γ such that for any $z \in U\setminus\Gamma$, there exist points z_0, ζ_0 satisfying the above conditions. Thus, we verify that there (3.5) holds for any $z \in \mathbb{C}\setminus\Gamma$ near Γ.

Remark 2. By using a similar argument, we extend the Hardy–Littlewood theorem on the unit disk which gives an estimate of the derivatives of holomorphic functions on the disk and continuous on the closed disk ([12]). Recently, H. Aikawa ([1]) estimates the norms of gradient vectors of harmonic functions on certain domains in $\mathbb{R}^n$.

4. Proof of Corollary 1. At first, we shall show that the principal value of the integral exists. Let $\zeta = \zeta(s)$ be the arc-length parametrization of Γ with $\zeta(0) = \zeta_0$. Since $\zeta(s)$ is differentiable, there exists a constant $A > 0$ such that

$$A^{-1}|s| \leq |\zeta(s) - \zeta_0| \leq A|s|$$

holds near $t = 0$. From the doubling condition of ω and (2.1), we have

$$|\varphi(\zeta(s)) - \varphi(\zeta_0)| \leq A\omega(|s|),$$

near $s = 0$. Hence, it follows from (1.7) that the principal value $P.V. \int_\Gamma \varphi(\zeta)d\zeta/(\zeta - \zeta_0)$ exists.

Next, we consider the existence of the boundary value $F_+(\zeta_0)$ of $F(z)$. For each $\zeta_0 \in \Gamma$, we may take a point $z_0 \in D_+$ so that $\delta(z_0) = |z_0 - \zeta_0|$. Let $I(z_0, \zeta_0)$ denote the interval between z_0 and ζ_0 and take two points z, z' on $I(z_0, \zeta_0)$ so that $\delta(z) \geq \delta(z') > 0$. It follows from Theorem 1 that for any $\sigma \in (0,1)$

$$\begin{aligned}
|F(z) - F(z')| &= \left| \int_{z'}^{z} F'(z)dz \right| \\
&\leq \int_{z'}^{z} |F'(z)||dz| \\
&\leq A \int_{\delta(z')}^{\delta(z)} \frac{\omega(t^\sigma)}{t} dt \qquad (s = t^\sigma) \\
&\leq \frac{1}{\sigma} \int_{\delta(z')^{1/\sigma}}^{\delta(z)^{1/\sigma}} \frac{\omega(s)}{s} ds.
\end{aligned}$$

From (1.7), we verify that $\lim_{z \to \zeta_0} F(z)$ exists along $I(z_0, \zeta_0)$ and the convergence is uniform. Hence, the boundary function is continuous and it guarantees the existence of $F_+(\zeta_0)$. By the same proof, we can show the existence of $F_-(\zeta_0)$.

Finally, we show (1.3) and (1.4). In fact, a standard argument gives the proof from the existence of F_+ and F_- (cf. [6]). We shall give the proof for convenience of the reader.

For $\delta > 0$, we put $\Gamma(\delta) = \{|z - \zeta_0| < \delta\} \cap \Gamma$ and

$$\Phi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(\zeta) - \varphi(\zeta_0)}{\zeta - z} d\zeta,$$

for $z \in D_+$. Then,

$$\begin{aligned}
\Phi(z) &- \frac{1}{2\pi i} P.V. \int_\Gamma \frac{\varphi(\zeta)-\varphi(\zeta_0)}{\zeta-\zeta_0} d\zeta \\
&= \frac{z-\zeta_0}{2\pi i} P.V. \int_\Gamma \frac{\varphi(\zeta)-\varphi(\zeta_0)}{(\zeta-\zeta_0)(\zeta-z)} d\zeta \\
&= \frac{z-\zeta_0}{2\pi i} P.V. \int_{\Gamma(\delta)} \frac{\varphi(\zeta)-\varphi(\zeta_0)}{(\zeta-\zeta_0)(\zeta-z)} d\zeta \\
&+ \frac{z-\zeta_0}{2\pi i} \int_{\Gamma\setminus\Gamma(\delta)} \frac{\varphi(\zeta)-\varphi(\zeta_0)}{(\zeta-\zeta_0)(\zeta-z)} d\zeta \\
&:= I_1(z) + I_2(z).
\end{aligned}$$

We have already seen that the boundary value $F_+(\zeta_0)$ exists. Therefore, we may assume that $z \in D_+$ approach to ζ_0 along $I(z_0, \zeta_0)$. It is not hard to see that there exists a constant $A > 0$ such that for and $z \in I(z_0, \zeta_0)$ for any $\zeta \in \Gamma(\delta)$, an inequality

$$|z-\zeta| \geq A|z-\zeta_0|$$

holds. Hence, we have

$$\begin{aligned}
|I_1(z)| &\leq \frac{|z-\zeta_0|}{2\pi} P.V. \int_{\Gamma(\delta)} \frac{|\varphi(\zeta)-\varphi(\zeta_0)|}{|(\zeta-\zeta_0)(\zeta-z)|} |d\zeta| \\
&\leq \frac{A}{2\pi} P.V. \int_{\Gamma(\delta)} \frac{|\varphi(\zeta)-\varphi(\zeta_0)|}{|(\zeta-\zeta_0)|} |d\zeta| \\
&\leq \frac{A}{2\pi} P.V. \int_{\Gamma(\delta)} \frac{\omega(|\zeta-\zeta_0|)}{|(\zeta-\zeta_0)|} |d\zeta|.
\end{aligned}$$

Thus, for arbitrary small $\varepsilon > 0$ it follows from (1.7) that $|I_1(z)| < \varepsilon$ if $\delta > 0$ is sufficiently small.

On the other hand, $\lim_{z\to\zeta_0} I_2(z) = 0$ and we have

$$\lim_{z\to\zeta_0} \Phi(z) = \frac{1}{2\pi i} P.V. \int_\Gamma \frac{\varphi(\zeta)-\varphi(\zeta_0)}{\zeta-\zeta_0} d\zeta.$$

Noting that

$$\frac{1}{2\pi i} P.V. \int_\Gamma \frac{1}{\zeta-\zeta_0} d\zeta = \frac{1}{2},$$

we obtain

$$F_+(\zeta_0) - \varphi(\zeta_0) = \frac{1}{2\pi i} P.V. \int_\Gamma \frac{\varphi(\zeta)}{\zeta-\zeta_0} d\zeta - \frac{1}{2}\varphi(\zeta_0)$$

and it shows (1.3) as desired. The proof of (1.4) is the same.

5. Proof of Corollary 2. Let ζ_1, ζ_2 be on Γ. Then, it follows from Theorem 1 that

$$\begin{aligned}|F_+(\zeta_1) - F_+(\zeta_2)| &= \left| \int_{I(z_1,\zeta_1)+\gamma+I(z_2,\zeta_2)} F'(z)dz \right| \\ &\le A \int_{I(z_1,\zeta_1)+\gamma+I(z_2,\zeta_2)} \frac{\omega(\delta(z))}{\delta(z)} |dz|,\end{aligned}$$

where $z_j \in D_+$ $(j = 1, 2)$ are in a neighborhood fo ζ_j and $\gamma \subset D_+$ is a smooth arc connecting z_1 and z_2. We may take ζ_1 and ζ_2 sufficiently close to each other so that the length of γ is less than $A|\zeta_1 - \zeta_2|$ and

$$A^{-1}\delta(z_1) \le \delta(z) \le A\delta(z_1) \tag{5.1}$$

for any $z \in \gamma$. We may also take z_j so that the length of $I(z_j, \zeta_j)$ is $\delta(z_j)$ $(j = 1, 2)$. Then, we have

$$\int_{I(z_j,\zeta_j)} \frac{\omega(\delta(z))}{\delta(z)} |dz| = \int_0^{\delta(z_j)} \frac{\omega(t)}{t} dt \quad (j = 1, 2).$$

As for the integral along γ, from (5.1) and the doubling property of ω, we obtain

$$\int_\gamma \frac{\omega(\delta(z))}{\delta(z)} |dz| \le A \int_\gamma \frac{\omega(\delta(z_1))}{\delta(z_1)} |dz| \le A' \frac{\omega(\delta(z_1))}{\delta(z_1)} |\zeta_1 - \zeta_2|.$$

Therefore, by taking $\delta(z_1) = |\zeta_1 - \zeta_2|$, we obtain

$$|F_+(\zeta_1) - F_+(\zeta_2)| \le AZ_\omega(|\zeta_1 - \zeta_2|),$$

and the proof is completed.

6. Examples. In this section, we shall give examples for our theorems.

Example 1. We have seen that $\omega_\alpha(t) = \min\{|\log t|^{-\alpha}, A\}$ $(\alpha > 0)$ belongs to $\mathcal{D}$. Therefore, from Theorem 1 we have

$$|F^{(n)}(z)\delta(z)^n| \le A|\log \delta(z)|^{-\alpha},$$

for $\varphi \in \Lambda^*_\Gamma(\omega_\alpha)$.

We also see that

$$\int_0^\delta \frac{\omega_\alpha(t)}{t} dt < \infty$$

for small $\delta > 0$ if and only if $\alpha > 1$. Hence, from Corollary 1, if $\alpha > 1$, then

$$P.V. \int_\Gamma \frac{\varphi(\zeta)}{\zeta - \cdot} d\zeta \in \Lambda^*_\Gamma(\omega_{\alpha-1}).$$

Now, we consider Zygmund's estimate $Z^0_{\omega_\alpha}$ given by (1.8). For $\omega = \omega_\alpha$ $(\alpha > 1)$, it is not hard to see that

$$t \int_t^{t_0} \frac{\omega_\alpha(s)}{s^2} ds \le \omega_\alpha(t).$$

Example 2. Put $\hat{\omega}_\alpha(t) = \min\{|\log t|^{-1}|\log|\log t||^{-\alpha}, A\}$ $(\alpha > 0)$. Then, we see that $\hat{\omega}_\alpha \in \mathcal{D}$. Hence, for $\varphi \in \Lambda^*_\Gamma(\hat{\omega}_\alpha)$, Theorem 1 gives

$$|F^{(n)}(z)\delta(z)^n| \leq A|\log\delta(z)|^{-1}|\log|\log\delta(z)||^{-\alpha}.$$

It is easily seen that

$$\int_0^\delta \frac{\hat{\omega}_\alpha(t)}{t}dt < \infty$$

if and only if $\alpha > 1$. Therefore, we see that if $\alpha > 1$, then

$$P.V.\int_\Gamma \frac{\varphi(\zeta)}{\zeta - \cdot}d\zeta \in \Lambda^*_\Gamma\left(|\log|\log t||^{1-\alpha}\right).$$

More generally, we may consider $\hat{\omega}_{n,\alpha}(t) = \prod_{k=1}^{n-1} f_k(t)^{-1}f_n(t)^{-\alpha}(t)$ $(\alpha > 0)$, where $f_1(t) = |\log t|$ and $f_{n+1}(t) = f_n(f_1(t))$. Then, $\hat{\omega}_{n,\alpha} \in \mathcal{D}$.

By putting $x = f_n(s)$, we get

$$\int_0^t \frac{\hat{\omega}_{n,\alpha}(s)}{s}ds = \int_{f_n(t)}^\infty x^{-\alpha}dx,$$

for sufficiently small $t > 0$. Therefore, we obtain that if $\varphi \in \Lambda^*_\Gamma(\hat{\omega}_{n,\alpha})$ for $\alpha > 1$, then

$$P.V.\int_\Gamma \frac{\varphi(\zeta)}{\zeta - \cdot}d\zeta \in \Lambda^*_\Gamma\left(f_n(t)^{1-\alpha}\right).$$

On the other hand, if we consider Zygmund's estimate (1.8), we have to estimate the integral

$$t\int_t^{t_0} \frac{\hat{\omega}_{n,\alpha}(s)}{s^2}ds = t\int_{f_n(t_0)}^{f_n(t)} \frac{1}{f_n^{-1}(x)}dx.$$

as $t \to 0$. At least for the author, it is much harder than the above one.

7. Proof of Theorem 2. First, we consider the behavior of sequences of abelian differentials on closed Riemann surfaces.

Let X be a closed Riemann surface of genus g and $\{P_{1,n}\}_{n=1}^\infty$, $\{P_{2,n}\}_{n=1}^\infty$ be distinct sequences on X converging to a point P_0. We assume that both $P_{1,n}$ and $P_{2,n}$ are contained in a parametric disk U of P_0. Let $\zeta : U \to \{|\zeta| < 1\}$ be a local coordinate on U with $\zeta(P_0) = 0$. Put $b_{1,n} \coloneqq \zeta(P_{1,n})$ and $b_{2,n} \coloneqq \zeta(P_{2,n})$.

We consider abelian differentials ω_n $(n = 0, 1, 2, \ldots)$ on X satisfying the following conditions.

(1) The differentials ω_n $(n = 1, 2, \ldots)$ are holomorphic on $X\backslash\{P_{1,n}, P_{2,n}\}$ and ω_0 is holomorphic on X;

(2) For each j $(j = 1, 2, \ldots, g)$,

$$\alpha_j(n) \coloneqq \int_{A_i} \omega_n \to \int_{A_j} \omega_0 \coloneqq \alpha_j \quad (n \to \infty), \tag{7.1}$$

where $\{A_j, B_j\}_{j=1}^{g}$ is a canonical homology basis of X defined as in §2;

(3) Let

$$\left\{\sum_{k=1}^{l_{1,n}} c_{1,-k}(n)(\zeta - b_{1,n})^{-k} + \sum_{k=0}^{\infty} c_{1,k}(n)(\zeta - b_{1,n})^{k}\right\} d\zeta$$

and

$$\left\{\sum_{k=1}^{l_{2,n}} c_{2,-k}(n)(\zeta - b_{2,n})^{-k} + \sum_{k=0}^{\infty} c_{2,k}(n)(\zeta - b_{2,n})^{k}\right\} d\zeta$$

be Laurent expansions of ω_n at $P_{1,n}$ and $P_{2,n}$, respectively. Then, $l_{1,n} = l_{2,n} = l$ for some $l \in \mathbb{N}$ and $\lim_{n\to\infty} c_{j,-k}(n)$ $(j = 1, 2)$ exist for each k $(k = 1, 2, \ldots, l_{1,n})$, which satisfy

$$\lim_{n\to\infty} c_{1,-k}(n) = -\lim_{n\to\infty} c_{2,-k}(n). \tag{7.2}$$

Under those conditions, we have the following.

Lemma 1. *Let Q be a point in $X \setminus \bigcup_{n=1}^{\infty}\{P_{1,n}, P_{2,n}\} \cup \{P_0\}$ and z a local coordinate at Q with $z(Q) = 0$. If ω_0 and ω_n, which are holomorphic at Q, have expansions:*

$$\omega_0 = \sum_{m=0}^{\infty} a_m(Q) z^m dz, \quad \omega_n = \sum_{m=0}^{\infty} a_{m,n}(Q) z^m dz$$

with respect to z at Q, then $\lim_{n\to\infty} a_{m,n}(Q) = a_m(Q)$ $(m = 0, 1, 2, \ldots)$. Furthermore, the convergence is locally uniform in $X \setminus \bigcup_{n=1}^{\infty}\{P_{1,n}, P_{2,n}\} \cup \{P_0\}$.

Proof. By considering $\omega_n - \omega_0$, we may assume that $\alpha_j = 0$ for $j \in \{1, 2, \ldots g\}$ and $a_m = 0$ for any $m \in \mathbb{N} \cup \{0\}$.

Let ψ_m be an abelian differential holomorphic on $X \setminus \{Q\}$ with pole of order $m + 2$ at Q which has the expansion:

$$\psi_m = \left\{\frac{1}{z^{m+2}} + (\text{holomorphic})\right\} dz \tag{7.3}$$

with respect to the local coordinate z. For the abelian integral $\Psi_m(P) = \int_P \psi_m$ on $X \setminus \bigcup_{j=1}^{g} A_j \cup B_j$, we have

$$\begin{aligned} 2\pi i \sum_{P\in X} \mathrm{res}_P \Psi_m \omega_n &= \sum_{j=1}^{g}\left[\int_{A_j}\psi_m \int_{B_j}\omega_n - \int_{B_j}\psi_m\int_{A_j}\omega_n\right] \\ &= -\sum_{j=1}^{g}\alpha_j(n)\int_{B_j}\psi_m \end{aligned}$$

from the bilinear relation (cf. [5]).

The poles of $\Psi_m\omega_n$ are $P_{1,n}, P_{2,n}$ and Q. As for Q, from (7.3) we obtain the residue at Q:

$$\mathrm{res}_Q \Psi_m\omega_n = \frac{-1}{m+1} a_{m,n}(Q).$$

Since Ψ_m is holomorphic at $P_{1,n}$ and $P_{2,n}$, it has expansions:

$$\Psi_m(\zeta) = \sum_{k=0}^{\infty} \beta_{j,k}(n)(\zeta - b_{j,n})^k$$

at $b_{j,n}$ $(j = 1, 2)$. Thus,

$$\mathrm{res}_{P_{j,n}} \Psi_m\omega_n = \sum_{k=1}^{l} c_{j,-k}(n)\beta_{j,k-1}(n) \quad (j = 1, 2),$$

and we get

$$\begin{aligned}
&\frac{-1}{m+1} a_{m,n}(Q) + \sum_{k=1}^{l} c_{1,-k}(n)\beta_{1,k-1}(n) + \sum_{k=1}^{l} c_{2,-k}(n)\beta_{2,k-1}(n) \\
&= -\frac{1}{2\pi i} \sum_{j=1}^{g} \alpha_j(n) \int_{B_j} \psi_m.
\end{aligned}$$

We have

$$\lim_{n\to\infty} \beta_{1,k}(n) = \lim_{n\to\infty} \beta_{2,k}(n) \quad (j = 0, 1, \dots, l-1).$$

Therefore, from (7.2) we obtain

$$\lim_{n\to\infty} a_{m,n}(Q) = 0.$$

The uniform convergence is also easily shown. □

Now, we proceed to prove Theorem 2.

Proof of (1). Let φ be in $L^1(\partial R)$. It is known that the normalized abelian differential $\omega_{\hat{P}_0,P}$ is holomorphic on $X \setminus \{\hat{P}_0\}$ for P (cf. [5] III. 3). Thus, $F(P) = \frac{1}{2\pi i}\int_{\partial R} \varphi\omega_{\hat{P}_0,P}$ is holomorphic on $R \setminus \partial R \cup \{\hat{P}_0\}$. Furthermore, it follows from Lemma 1 that $\omega_{\hat{P}_0,P}$ uniformly converges to zero on ∂R as $P \to \hat{P}_0$. We conclude that $F(P)$ is holomorphic on $\hat{R} \setminus \partial R$.

Proof of (2). The proof is done by localization. Let $Q \in C_i$ and U_i an annular domain in R as in §2. Then, $\hat{U}_i := U_i \cup C_i \cup \pi(U_i)$ is an annular neighborhood of C_i in $\hat{R}$, where $\pi : \hat{R} \to \hat{R}$ is the anti-conformal involution of $\hat{R}$. We may assume that $\hat{P}_0 \notin \hat{U}_i$. There exists a conformal mapping $f : \hat{U}_i \to A_{r_i} := \{0 < r_i < |z| < r_i^{-1}\}$ such that $f(C_i) = \{|z| = 1\}$, $f(Q) = 1$ and $f(\partial\hat{U}_i \cap R) = \{|z| = r_i\}$. Then, a differential

$$\theta_P = \omega_{\hat{P}_0,P} \circ f^{-1}(z) - \frac{1}{z - f(P)} dz$$

is holomorphic in A_{r_i}. Hence, there exists a holomorphic function h_P on A_{r_i} such that $\theta_P = h_P(z)dz$. Because of the uniform convergence of $\omega_{\hat{P}_0,P} \circ f^{-1}$ on ∂A_{r_i} as $P \to Q$, holomorphic functions h_P converges to h_Q uniformly on ∂A_{r_i}. Therefore, from the maximum principle, h_P converges to h_Q uniformly on $\{|z|=1\} = f(C_i)$ and we have

$$\lim_{P\to Q} \int_{|z|=1} \varphi \circ f^{-1}\theta_P = \int_{|z|=1} \varphi \circ f^{-1}\theta_Q.$$

Noting that

$$\int_{C_i} \varphi\omega_{\hat{P}_0,P} = \int_{|z|=1} \frac{\varphi \circ f^{-1}(z)}{z - f(P)}dz + \int_{|z|=1} \varphi \circ f^{-1}\theta_P,$$

we verify that the behavior of $F(P)$ as $P \to Q \in C_i$ is determined by that of the Cauchy integral of $\varphi \circ f^{-1}$ on the unit circle. This implies the conclusion of (2).

Proof of (3). The proof of the non-tangential limits is done by the localization as in (2) since the statement is true when R is the unit disk (cf. [4]).

Now, we show the second statement of (3) holds. Since $\varphi \in L^p(\partial R)$ $(p > 1)$, from Proposition 2 we have

$$\varphi = f_1 + \overline{f_2} + m \tag{7.4}$$

on ∂R, where $f_1 \in H^p(R)$, $f_2 \in H_0^p(R)$ and $m \in M(\delta^{-1})$. (If $p = \infty$, we may take any finite number greater than 1 as p in (7.4).) Then, from Cauchy's integral formula for $H^p(R)$, we have

$$\frac{1}{2\pi i}\int_{\partial R} f_1\omega_{\hat{P}_0,P} = f_1(P),$$

for $P \in R$. Since $F_2 := \overline{f_2 \circ \pi}$ is in $H^p(\hat{R} \setminus \bar{R})$ and $F_2 = \bar{f}_2$ on ∂R, we have

$$\frac{1}{2\pi i}\int_{\partial R} \overline{f_2}\omega_{\hat{P}_0,P} = -F_2(\hat{P}_0) = -\overline{f_2(P_0)} = 0.$$

Let $Q_1^{n_1} \ldots Q_k^{n_k}$ be the polar divisor of m in R and

$$m(z_j) = \sum_{l=1}^{n_j} \frac{d_{j,l}}{z^l} + \sum_{l=0}^{\infty} d_{j,l}^+ z^l \quad (j = 1, 2, \ldots, k)$$

the Laurent expansion of m at Q_j with respect to a local coordinate z_j of Q_j with $z_j(Q_j) = 0$. For $P \neq Q_j$, we put

$$\omega_{\hat{P}_0,P} = \sum_{l=0}^{\infty} a_{j,l}(P)z_j^l dz_j$$

near Q_j. Then, we obtain

$$\frac{1}{2\pi i}\int_{\partial R} m\omega_{\hat{P}_0,P} = m(P) + \sum_{j=1}^{k}\sum_{l=1}^{n_j} d_{j,l}a_{j-1,l}(P), \tag{7.5}$$

if $P \neq Q_j$ $(j = 1, 2, \ldots, k)$. Therefore, we obtain

$$F(P) = f_1(P) + m(P) + \sum_{j=1}^{k}\sum_{l=1}^{n_j} d_{j,l}a_{j-1,l}(P),$$

if $P \in R\setminus\bigcup_{j=1}^{k} Q_j$. Hence, it follows from Lemma 1 that the non-tangential limit $F_+(Q)$ of F from R exists for almost all Q on ∂R. By the same argument, F_- exists almost everywhere on ∂R.

On the other hand, for $P \in R$ and $P' \in \pi(R)$,

$$F(P) - F(P') = f_1(P) + \overline{f_2(\pi(P'))} + \frac{1}{2\pi i}\int_{\partial R} m\omega_{P,P'}, \tag{7.6}$$

because $\omega_{P',P} = \omega_{\hat{P}_0,P} - \omega_{\hat{P}_0,P}$. By the same argument as in (7.5), we obtain

$$\frac{1}{2\pi i}\int_{\partial R} m\omega_{P',P} = m(P) + \sum_{j=1}^{k}\sum_{l=1}^{n_j} d_{j,l}b_{j-1,l}(P,P'),$$

where $\omega_{P',P} = \sum_{l=0}^{\infty} b_{j,l}(P,P')z_j^l dz_j$ at Q_j $(j = 1, 2, \ldots, k)$. Since $b_{j,l}(P,P') \to 0$ as $P, P' \to Q$, we have

$$\lim_{P,P'\to Q}\frac{1}{2\pi i}\int_{\partial R} m\omega_{P',P} = m(Q).$$

Therefore, from (7.6), we verify

$$F_+ - F_- = \varphi,$$

almost everywhere on ∂R and the proof is completed.

References

[1] Aikawa, H., *Modulus of continuity of the Dirichlet solutions*, Bull. London Math. Soc. **42** (2010), 857–867.

[2] Bikčantaev, I. A., *Analogues of a Cauchy kernel on a Riemann surface and some applications of them*, Mat. Sb. (N.S.) **112 (154)**, no. 2 (6) (1980), 256–282 (Russian); translation in Math. USSR Sb. **40**, no. 2 (1981), 241–265.

[3] Block, I. E., *The Plemelj theory for the class Λ^* of functions*, Duke Math. J. **19** (1952), 367–378.

[4] Duren, P. L., *Theory of H^p Spaces*, Academic Press, New York–San Francisco–London, 1970.

[5] Farkas, H. M., Kra, I., *Riemann Surfaces*, Springer-Verlag, New York–Heidelberg–Berlin, 1980.

[6] Gakhov, F. D., *Boundary Value Problems*, Pergamon Press, Oxford–New York–Paris, 1966.

[7] Garnett, J. B., *Bounded Analytic Functions*, Academic Press, New York–London, 1981.
[8] Gong, S., *Integrals of Cauchy type on the ball*, International Press, Cambridge, 1993.
[9] Guseinov, E. G., *The Plemelj-Privalov theorem for generalized Hölder classes*, Mat. Sb. **183**, no. 2 (1992), 21–37 (Russian); translation in Russian Acad. Sci. Sb. Math. **75** (1993), 165–182.
[10] Heins, M., *Hardy Classes on Riemann Surfaces*, Springer-Verlag, Berlin–New York, 1969.
[11] Shiga, H., *Riemann mappings of invariant components of Kleinian groups*, J. London Math. Soc. **80** (2009), 716–728.
[12] Shiga, H., *Modulus of continuity, a Hardy–Littlewood theorem and its application*, RIMS Kokyuroku Bessatsu, 2010, 127–133.
[13] Pommerenke, C., *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin, 1992.
[14] Walsh, J. L., *Polynomial expansions of functions defined by Cauchy's integral*, J. Math. Pures Appl. **31** (1952), 221–244.

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