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**Lagrangians and Euler morphisms on
fibered-fibered frame bundles from
projectable-projectable classical linear connections**

ABSTRACT. We classify all $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operators A transforming projectable-projectable torsion-free classical linear connections ∇ on fibered-fibered manifolds Y of dimension (m_1, m_2, n_1, n_2) into r th order Lagrangians $A(\nabla)$ on the fibered-fibered linear frame bundle $L^{\text{fib-fib}}(Y)$ on Y . Moreover, we classify all $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operators B transforming projectable-projectable torsion-free classical linear connections ∇ on fibered-fibered manifolds Y of dimension (m_1, m_2, n_1, n_2) into Euler morphism $B(\nabla)$ on $L^{\text{fib-fib}}(Y)$. These classifications can be expanded on the k th order fibered-fibered frame bundle $L^{\text{fib-fib}, k}(Y)$ instead of $L^{\text{fib-fib}}(Y)$.

1. Introduction. Lagrangians and Euler morphisms are important tools in the variational calculus. Several physical theories are using Euler–Lagrange equations, which are related with the Euler morphism of an r th order Lagrangian on a fibered manifold.

The idea of Lagrangians and Euler morphisms in the case of fibered manifolds was described in [2]. The aim of the present note is the generalization of results which were reached in [1] to the case of fibered-fibered manifolds.

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2. Fibered-fibered manifolds. A fibered-fibered manifold Y is any commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ q \downarrow & & \downarrow p \\ N & \xrightarrow{\pi_0} & M \end{array}$$

where maps π, π_0, q, p are surjective submersions and induced map $Y \rightarrow X \times_M N$, $y \mapsto (\pi(y), q(y))$ is a surjective submersion. A fibered-fibered manifold has dimension (m_1, m_2, n_1, n_2) if $\dim Y = m_1 + m_2 + n_1 + n_2$, $\dim X = m_1 + m_2$, $\dim N = m_1 + n_1$, $\dim M = m_1$. For two fibered-fibered manifolds Y_1, Y_2 of the same dimension (m_1, m_2, n_1, n_2) , a morphism $f: Y_1 \rightarrow Y_2$ is quadruple of local diffeomorphisms $f: Y_1 \rightarrow Y_2, f_1: X_1 \rightarrow X_2, f_2: N_1 \rightarrow N_2, f_0: M_1 \rightarrow M_2$ such that all squares of the cube in question are commutative [3].

All fibered-fibered manifolds of the given dimension (m_1, m_2, n_1, n_2) and their all morphisms form the category which we denote by $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$.

Every object from the category $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ is locally isomorphic to the standard fibered-fibered manifold

$$\begin{array}{ccc} \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} & \longrightarrow & \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \\ \downarrow & & \downarrow \\ \mathbb{R}^{m_1} \times \mathbb{R}^{n_1} & \longrightarrow & \mathbb{R}^{m_1} \end{array}$$

which we denote by $\mathbb{R}^{m_1, m_2, n_1, n_2}$, where arrows are obvious projections.

For fibered-fibered manifold Y we have the fibered-fibered linear frame bundle

$$L^{\text{fib-fib}}(Y) = \left\{ j_{(0,0,0,0)}^1 \psi \mid \psi: \mathbb{R}^{m_1, m_2, n_1, n_2} \rightarrow Y \text{ is an } \mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}\text{-map} \right\}$$

with the jet target projection $\pi_Y: L^{\text{fib-fib}}(Y) \rightarrow Y$,

$$\pi_Y(j_{(0,0,0,0)}^1 \psi) = \psi(0, 0, 0, 0),$$

where $(0, 0, 0, 0) \in \mathbb{R}^{m_1 + m_2 + n_1 + n_2}$. The bundle $L^{\text{fib-fib}}(Y)$ is a principal bundle over Y with a structure group $G_{m_1, m_2, n_1, n_2}^1 = L_{(0,0,0,0)}^{\text{fib-fib}}(\mathbb{R}^{m_1, m_2, n_1, n_2})$ acting on the right on $L^{\text{fib-fib}}(Y)$ by the composition of jets. Every $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -map $f: Y \rightarrow Y_1$ induces a fibered map (a principal bundle morphism) $L^{\text{fib-fib}}(f): L^{\text{fib-fib}}(Y) \rightarrow L^{\text{fib-fib}}(Y_1)$ over f by the composition of jets $L^{\text{fib-fib}}(f)(j_{(0,0,0,0)}^1 \psi) = j_{(0,0,0,0)}^1(f \circ \psi)$. The correspondence $L^{\text{fib-fib}}: \mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2} \rightsquigarrow \mathcal{FM}$ is a bundle functor [2].

3. Lagrangians and natural operators transforming connections into Lagrangians. An r th order Lagrangian on a fibered manifold $p:$

$X \rightarrow M$ is a base preserving morphism $\lambda: J^r(X) \rightarrow \wedge^m T^*M$ from the r -jet prolongation bundle

$$J^r(X) = \{j_x^r \sigma \mid \sigma: M \rightarrow X \text{ is a local section of } p: X \rightarrow M, x \in M\}$$

into the bundle $\wedge^m T^*M$ of $m = \dim M$ -forms on M [2].

A classical linear connection $\tilde{\nabla}$ on a fibered manifold $p: X \rightarrow M$ is projectable if there exists a (unique) classical linear connection $\underline{\tilde{\nabla}}$ on M such that a connection $\tilde{\nabla}$ is p -related with a connection $\underline{\tilde{\nabla}}$, that is $Tp \circ (\tilde{\nabla}_W Z) = (\underline{\tilde{\nabla}}_W \underline{Z}) \circ p$, where W and Z are projectable vector fields on X , which are p -related with vector fields \underline{W} and \underline{Z} on M .

Let Y be a fibered-fibered manifold

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ q \downarrow & & \downarrow p \\ N & \xrightarrow{\pi_0} & M \end{array}$$

We say that a projectable classical linear connection ∇ on Y is projectable-projectable if there exists a unique projectable classical linear connection $\underline{\nabla}$ on X such that a connection ∇ is π -related with a connection $\underline{\nabla}$.

In this paper we study the problem how a projectable-projectable torsion-free classical linear connection ∇ on a fibered-fibered manifold Y of dimension (m_1, m_2, n_1, n_2) can induce an r th order Lagrangian $A(\nabla)$ on $\pi_Y: L^{\text{fib-fib}}(Y) \rightarrow Y$ in the canonical way. To this aim we must determine $\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operators

$$A: Q_\tau^{\text{proj-proj}} \rightarrow (J^r L^{\text{fib-fib}}, \wedge^m T^*),$$

where $m = m_1 + m_2 + n_1 + n_2$, in the sense of [2].

We describe completely all such natural operators A in question.

An $\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operator

$$A: Q_\tau^{\text{proj-proj}} \rightarrow (J^r L^{\text{fib-fib}}, \wedge^m T^*)$$

(where $m = m_1 + m_2 + n_1 + n_2$) sending projectable-projectable torsion-free classical linear connections ∇ on fibered-fibered manifolds Y of dimension (m_1, m_2, n_1, n_2) into r th order Lagrangians $A(\nabla)$ on the fibered-fibered linear frame bundle $\pi_Y: L^{\text{fib-fib}}(Y) \rightarrow Y$ for Y is the family of $\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}$ -invariant regular operators

$$A_Y: Q_\tau^{\text{proj-proj}}(Y) \rightarrow \text{Lagr}^r(L^{\text{fib-fib}}(Y))$$

for $\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}$ -objects Y , where $Q_\tau^{\text{proj-proj}}(Y)$ is the space of all projectable-projectable torsion-free classical linear connections on Y and $\text{Lagr}^r(L^{\text{fib-fib}}(Y))$ is the space of all r th order Lagrangians on

$$\pi_Y: L^{\text{fib-fib}}(Y) \rightarrow Y.$$

The invariance means that if $\nabla \in Q_\tau^{\text{proj-proj}}(Y)$ and $\nabla_1 \in Q_\tau^{\text{proj-proj}}(Y_1)$ are f -related with respect to an $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -map $f: Y \rightarrow Y_1$, then $A_Y(\nabla)$ and $A_{Y_1}(\nabla_1)$ are also f -related. The regularity of a natural operator A_Y means that A_Y transforms smoothly parametrized families of connections in $Q_\tau^{\text{proj-proj}}(Y)$ into smoothly parametrized families of Lagrangians in $\text{Lagr}^r(L^{\text{fib-fib}}(Y))$.

To present an example of natural operator A in question we need some preparations. Let M be an m -manifold with a torsion-free classical linear connection $\tilde{\nabla}$. Given a linear frame $\underline{l} \in L(M)$, the linear isomorphism $\underline{l}: \mathbb{R}^m \rightarrow T_{\underline{x}}(M)$ defines a coordinate system in $T_{\underline{x}}(M)$. Therefore, $\tilde{\nabla}$ -exponential map $\text{Exp}_{\underline{x}}^{\tilde{\nabla}}: T_{\underline{x}}(M) \rightarrow M$ being the diffeomorphism, defines a normal coordinate system $\underline{\varphi}$ with center \underline{x} in M by the composition $\underline{\varphi} = \underline{l}^{-1} \circ (\text{Exp}_{\underline{x}}^{\tilde{\nabla}})^{-1}: M \rightarrow \mathbb{R}^m$. If $\underline{\varphi}_1: M \rightarrow \mathbb{R}^m$ is another $\tilde{\nabla}$ -normal coordinate system on M with center \underline{x} , then $\underline{\varphi}_1 = I \circ \underline{\varphi}$ for some linear isomorphism $I: \mathbb{R}^m \rightarrow \mathbb{R}^m$ [4].

Let $p: X \rightarrow M$ be a fibered manifold of dimension (m, n) and let $x \in X_{\underline{x}}$, $\underline{x} \in M$. Let $\tilde{\nabla}$ be a projectable torsion-free classical linear connection on X with the underlying torsion-free classical linear connection $\tilde{\nabla}$ on M . Since a connection $\tilde{\nabla}$ is p -related with $\tilde{\nabla}$, then p sends $\tilde{\nabla}$ -geodesics into $\tilde{\nabla}$ -geodesics. Consequently, the $\tilde{\nabla}$ -exponential map $\text{Exp}_x^{\tilde{\nabla}}: T_x(X) \rightarrow X$ at x is a local fibered diffeomorphism covering the $\tilde{\nabla}$ -exponential map $\text{Exp}_{\underline{x}}^{\tilde{\nabla}}: T_{\underline{x}}(M) \rightarrow M$ at \underline{x} , where $T_x(X)$ is treated as a fibered manifold $Tp: T_x(X) \rightarrow T_{\underline{x}}(M)$.

If we compose $(\text{Exp}_x^{\tilde{\nabla}})^{-1}$ with a fibered linear isomorphism (fibered linear frame) $l: \mathbb{R}^{m, n} \rightarrow T_x(X)$ covering a linear frame $\underline{l}: \mathbb{R}^m \rightarrow T_{\underline{x}}(M)$, then we obtain a fibered $\tilde{\nabla}$ -normal coordinate system $\varphi = l^{-1} \circ (\text{Exp}_x^{\tilde{\nabla}})^{-1}: X \rightarrow \mathbb{R}^{m, n}$ with center x covering a $\tilde{\nabla}$ -normal coordinate system

$$\underline{\varphi} = \underline{l}^{-1} \circ (\text{Exp}_{\underline{x}}^{\tilde{\nabla}})^{-1}: M \rightarrow \mathbb{R}^m$$

with center \underline{x} . If $\varphi_1: X \rightarrow \mathbb{R}^{m, n}$ is another fibered $\tilde{\nabla}$ -normal coordinate system on X with center x , then $\varphi_1 = I \circ \varphi$ for some fibered linear isomorphism $I: \mathbb{R}^{m, n} \rightarrow \mathbb{R}^{m, n}$.

Quite similarly as above, if ∇ is a projectable-projectable torsion-free classical linear connection on a fibered-fibered manifold Y of dimension (m_1, m_2, n_1, n_2) and $y \in Y$, then there exists a fibered-fibered ∇ -normal coordinate system $\varphi: Y \rightarrow \mathbb{R}^{m_1, m_2, n_1, n_2}$ with center y . If $\varphi_1: Y \rightarrow \mathbb{R}^{m_1, m_2, n_1, n_2}$ is another fibered-fibered ∇ -normal coordinate system with center y , then $\varphi_1 = I \circ \varphi$ for some fibered-fibered linear isomorphism $I: \mathbb{R}^{m_1, m_2, n_1, n_2} \rightarrow \mathbb{R}^{m_1, m_2, n_1, n_2}$.

4. The first main result. Let $Q_{\text{proj-proj}}^s$ be the vector space of all s -jets $j_{(0,0,0,0)}^s(\nabla)$ at $(0,0,0,0) \in \mathbb{R}^{m_1+m_2+n_1+n_2}$ of projectable-projectable torsion-free classical linear connections ∇ on $\mathbb{R}^{m_1,m_2,n_1,n_2}$ satisfying the condition

$$\sum_{j,k=1}^{m_1+m_2+n_1+n_2} \nabla_{jk}^i(x) x^j x^k = 0 \quad \text{for } i = 1, \dots, m_1 + m_2 + n_1 + n_2,$$

where $\nabla_{jk}^i: \mathbb{R}^{m_1,m_2,n_1,n_2} \rightarrow \mathbb{R}$, for $i, j, k = 1, \dots, m_1 + m_2 + n_1 + n_2$, are the Christoffel symbols of a connection ∇ in the usual fibered-fibered coordinate system $x^1, \dots, x^{m_1+m_2+n_1+n_2}$ on $\mathbb{R}^{m_1,m_2,n_1,n_2}$.

Equivalently, $x^1, \dots, x^{m_1+m_2+n_1+n_2}$ are fibered-fibered ∇ -normal coordinates with center $(0,0,0,0)$. The equivalence is a simple consequence of the well-known equations of ∇ -geodesics and the fact that in the ∇ -normal coordinate system ∇ -geodesics passing through the center are straight lines.

Let $\pi_s: Q_{\text{proj-proj}}^\infty \rightarrow Q_{\text{proj-proj}}^s$ for $s = 1, 2, \dots$ be the jet projections. Let $\pi_0^r: J^r(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})) \rightarrow L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})$ be the jet projection. Denote $l_0 := j_{(0,0,0,0)}^1(id_{\mathbb{R}^{m_1,m_2,n_1,n_2}}) \in L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})$.

Let $J_{l_0}^r(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2}))$ be the fibre of π_0^r at l_0 . Let $\mu: Q_{\text{proj-proj}}^\infty \times J_{l_0}^r(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})) \rightarrow \mathbb{R}$. We say that μ satisfies the finite determination property, if for any $\rho \in Q_{\text{proj-proj}}^\infty$ and $\sigma \in J_{l_0}^r(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2}))$ we can find an open neighborhood $U \subset Q_{\text{proj-proj}}^\infty$ of ρ , open neighborhood $V \subset J_{l_0}^r(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2}))$ of σ , a finite number s and a smooth map $f: \pi_s(U) \times V \rightarrow \mathbb{R}$ such that $\mu = f \circ (\pi_s \times id_V)$ on $U \times V$.

We are in a position to present the following example of the operator A in question.

Example 1. Let $\mu: Q_{\text{proj-proj}}^\infty \times J_{l_0}^r(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})) \rightarrow \mathbb{R}$ be a function satisfying the finite determination property. Given a projectable-projectable torsion-free classical linear connection ∇ on a fibered-fibered manifold Y of dimension (m_1, m_2, n_1, n_2) , we define the r th order Lagrangian

$$A_Y^{(\mu)}(\nabla): J^r(L^{\text{fib-fib}}(Y)) \rightarrow \wedge^m T^*(Y)$$

on $\pi_Y: L^{\text{fib-fib}}(Y) \rightarrow Y$ by

$$A_Y^{(\mu)}(\nabla)(\sigma) := \mu(j_{(0,0,0,0)}^\infty(\varphi_* \nabla), J^r(L^{\text{fib-fib}}(\varphi))(\sigma)) \cdot l_1^* \wedge \dots \wedge l_{m_1+m_2+n_1+n_2}^*,$$

where $m = m_1 + m_2 + n_1 + n_2$, $\sigma \in J_l^r(L^{\text{fib-fib}}(Y))$, $l = j_{(0,0,0,0)}^r(\varphi^{-1}) \in (L^{\text{fib-fib}}(Y))_y$, $y \in Y$, $l_i = T(\varphi^{-1})(\frac{\partial}{\partial x^i}|_{(0,0,0,0)})$ for $i = 1, \dots, m_1 + m_2 + n_1 + n_2$ is the basis in $T_y(Y)$ and l_i^* for $i = 1, \dots, m_1 + m_2 + n_1 + n_2$ is the dual basis in $T_y^*(Y)$ and $\varphi: Y \rightarrow \mathbb{R}^{m_1,m_2,n_1,n_2}$ is a fibered-fibered ∇ -normal coordinate system with center y such that $J^r(L^{\text{fib-fib}}(\varphi))(\sigma) \in J_{l_0}^r(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2}))$.

The definition of $A_Y^{(\mu)}(\nabla)(\sigma)$ is correct because $germ_y(\varphi)$ is uniquely determined.

Consequently, for given a projectable-projectable torsion-free classical linear connection ∇ on Y , we have an r th order Lagrangian

$$A_Y^{(\mu)} : J^r(L^{\text{fib-fib}}(Y)) \rightarrow \wedge^m T^*(Y), \quad \text{where } m = m_1 + m_2 + n_1 + n_2.$$

The family $A^{(\mu)} : Q_\tau^{\text{proj-proj}} \rightsquigarrow (J^r L^{\text{fib-fib}}, \wedge^m T^*)$ of operators

$$\begin{aligned} A_Y^{(\mu)} : Q_\tau^{\text{proj-proj}}(Y) &\longrightarrow \text{Lagr}^r(L^{\text{fib-fib}}(Y)), \\ \nabla &\longrightarrow A_Y^{(\mu)}(\nabla) \end{aligned}$$

for any $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -object Y is an $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operator.

The main result of the present note is the following:

Theorem 1. *Any $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operator*

$$A : Q_\tau^{\text{proj-proj}} \rightsquigarrow (J^r L^{\text{fib-fib}}, \wedge^m T^*),$$

where $m = m_1 + m_2 + n_1 + n_2$, is of the form $A = A^{(\mu)}$ for a uniquely determined function $\mu : Q_{\text{proj-proj}}^\infty \times J_{l_0}^r(L^{\text{fib-fib}}(\mathbb{R}^{m_1, m_2, n_1, n_2})) \rightarrow \mathbb{R}$ satisfying the finite determination property.

Proof. Let A be a $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operator in question. We must define a map $\mu : Q_{\text{proj-proj}}^\infty \times J_{l_0}^r(L^{\text{fib-fib}}(\mathbb{R}^{m_1, m_2, n_1, n_2})) \rightarrow \mathbb{R}$ by

$$\mu(j_{(0,0,0,0)}^\infty(\nabla), \sigma) := \langle A_{\mathbb{R}^{m_1, m_2, n_1, n_2}}(\nabla)(\sigma), (l_0)_1 \wedge \dots \wedge (l_0)_{m_1+m_2+n_1+n_2} \rangle,$$

where $l_0 = ((l_0)_1, \dots, (l_0)_{m_1+m_2+n_1+n_2})$ is the basis in $T_{(0,0,0,0)}(\mathbb{R}^{m_1+m_2+n_1+n_2})$. Then by the non-linear Peetre theorem [2], μ satisfies the finite determination property. By the invariance of A and $A^{(\mu)}$ with respect to fibered-fibered normal coordinates we obtain $A = A^{(\mu)}$. \square

Remark 1. Quite similarly one can describe all $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operators $A : Q_\tau^{\text{proj-proj}} \rightarrow (J^r L^{\text{fib-fib}, k}, \wedge^m T^*)$ transforming projectable-projectable torsion-free classical linear connections ∇ on $\mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}$ -objects Y into r th order Lagrangians $A_Y(\nabla)$ on $\pi_Y^k : L^{\text{fib-fib}, k}(Y) \rightarrow Y$, where

$$\begin{aligned} &L^{\text{fib-fib}, k}(Y) \\ &:= \left\{ j_{(0,0,0,0)}^k(\psi) \mid \psi : \mathbb{R}^{m_1, m_2, n_1, n_2} \rightarrow Y \text{ is a local } \mathcal{F}^2\mathcal{M}_{m_1, m_2, n_1, n_2}\text{-map} \right\} \end{aligned}$$

is the fibered-fibered k th order frame bundle for Y . All such natural operators in question are of the form $A^{(\mu)}$ for functions

$$\mu : Q_{\text{proj-proj}}^\infty \times J_{l_0}^r(L^{\text{fib-fib}, k}(\mathbb{R}^{m_1, m_2, n_1, n_2})) \rightarrow \mathbb{R}$$

satisfying the obviously modified finite determination property, where $J_{l_0}^r(L^{\text{fib-fib},k}(\mathbb{R}^{m_1,m_2,n_1,n_2}))$ is the fiber of

$$J_{l_0}^r(L^{\text{fib-fib},k}(\mathbb{R}^{m_1,m_2,n_1,n_2})) \rightarrow L^{\text{fib-fib},k}(\mathbb{R}^{m_1,m_2,n_1,n_2})$$

at the k th order frame $l_0 = j_{(0,0,0,0)}^k(id_{\mathbb{R}^{m_1,m_2,n_1,n_2}}) \in L^{\text{fib-fib},k}(\mathbb{R}^{m_1,m_2,n_1,n_2})$.

The generalized natural operators $A^{(\mu)}$ can be defined quite similarly as in Example 1.

5. Euler morphisms and natural operators transforming connections into Euler morphisms. We recall that the r th order Euler morphism on a fibered manifold $p: X \rightarrow M$ is a base preserving morphism $E: J^r(X) \rightarrow V^*(X) \otimes \wedge^m T^*(M)$, where $m = \dim M$. Here $V^*(X)$ denotes the vector bundle dual to the vertical vector bundle $V(X)$ for X . Special Euler morphisms can be obtained from Lagrangians by means of the well-known Euler operator [2], [5].

Quite similarly as for Lagrangians, we can describe completely all $\mathcal{F}^2 \mathcal{M}_{m_1,m_2,n_1,n_2}$ -natural operators

$$B: Q_{\tau}^{\text{proj-proj}} \rightarrow (J^r L^{\text{fib-fib}}, V^* L^{\text{fib-fib}} \otimes \wedge^m T^*),$$

where $m = m_1 + m_2 + n_1 + n_2$, transforming projectable-projectable torsion-free classical linear connections ∇ on fibered-fibered manifold Y of dimension (m_1, m_2, n_1, n_2) into r th order Euler morphisms $B_Y(\nabla)$ on

$$\pi_Y: L^{\text{fib-fib}}(Y) \rightarrow Y.$$

6. The second main result.

Example 2. We consider a function

$$\mu: Q_{\text{proj-proj}}^{\infty} \times J_{l_0}^r(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})) \rightarrow (\mathcal{L}(G_{m_1,m_2,n_1,n_2}^1))^*$$

satisfying the obviously modified finite determination property, where $\mathcal{L}(G_{m_1,m_2,n_1,n_2}^1)$ denotes the Lie algebra of Lie group G_{m_1,m_2,n_1,n_2}^1 . Given a projectable-projectable torsion-free classical linear connection on a fibered-fibered manifold Y of dimension (m_1, m_2, n_1, n_2) , we define an r th order Euler morphism $B_Y^{(\mu)}(\nabla): J^r(L^{\text{fib-fib}}(Y)) \rightarrow V^*(L^{\text{fib-fib}}(Y)) \otimes \wedge^m T^*(Y)$, where $m = m_1 + m_2 + n_1 + n_2$, on $\pi_Y: L^{\text{fib-fib}}(Y) \rightarrow Y$ by

$$\begin{aligned} & \langle B_Y^{(\mu)}(\nabla)(\sigma), \eta_l^* \rangle \\ &= \left\langle \mu(j_{(0,0,0,0)}^{\infty}(\varphi_* \nabla), J^r(L^{\text{fib-fib}}(\varphi))(\sigma)), \eta \right\rangle l_1^* \wedge \dots \wedge l_{m_1+m_2+n_1+n_2}^* \end{aligned}$$

for all $\sigma \in (J_l^r(L^{\text{fib-fib}}(Y)))$, $l = (l_1, \dots, l_m) \in (L^{\text{fib-fib}}(Y))_y$, $y \in Y$, where $m = m_1 + m_2 + n_1 + n_2$, $\eta \in \mathcal{L}(G_{m_1,m_2,n_1,n_2}^1)$, where η^* is the (vertical) fundamental vector field on the principal G_{m_1,m_2,n_1,n_2}^1 -bundle $L^{\text{fib-fib}}(Y)$ corresponding to η and $l_1^*, \dots, l_m^* \in T_y^* Y$ is the dual basis to $l_1, \dots, l_m \in T_y Y$

and $\varphi: Y \rightarrow \mathbb{R}^{m_1 m_2, n_1, n_2}$ is a fibered-fibered ∇ -normal coordinate system on Y with center y such that $\varphi(y) = (0, 0, 0, 0) \in \mathbb{R}^{m_1 + m_2 + n_1 + n_2}$ and $J^r(L^{\text{fib-fib}}(\varphi))(\sigma) \in (J_{l_0}^r(L^{\text{fib-fib}}(\mathbb{R}^{m_1, m_2, n_1, n_2})))$. The correspondence $B^{(\mu)}: Q_\tau^{\text{proj-proj}} \rightsquigarrow (J^r L^{\text{fib-fib}}, V^* L^{\text{fib-fib}} \otimes \wedge^m T^*)$, where $m = m_1 + m_2 + n_1 + n_2$, is $\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operator.

Similarly as Theorem 1 one can prove the following:

Theorem 2. Any $\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operator

$$B: Q_\tau^{\text{proj-proj}} \rightsquigarrow (J^r L^{\text{fib-fib}}, V^* L^{\text{fib-fib}} \otimes \wedge^m T^*),$$

where $m = m_1 + m_2 + n_1 + n_2$, is of the form $B = B^{(\mu)}$ for some uniquely determined function

$$\mu: Q_{\text{proj-proj}}^\infty \times J_{l_0}^r(L^{\text{fib-fib}}(\mathbb{R}^{m_1, m_2, n_1, n_2})) \rightarrow (\mathcal{L}(G_{m_1, m_2, n_1, n_2}^1))^*$$

satisfying the modified finite determination property.

Proof. Similarly as in the proof of Theorem 1 we define

$$\mu: Q_{\text{proj-proj}}^\infty \times J_{l_0}^r(L^{\text{fib-fib}}(\mathbb{R}^{m_1, m_2, n_1, n_2})) \rightarrow (\mathcal{L}(G_{m_1, m_2, n_1, n_2}^1))^*$$

by

$$\begin{aligned} & \langle \mu(j_{(0,0,0,0)}^\infty(\nabla), \sigma), \eta \rangle \\ &= \langle B_{\mathbb{R}^{m_1, m_2, n_1, n_2}}(\nabla)(\sigma), \eta_{l_0}^* \otimes (l_0)_1 \wedge \dots \wedge (l_0)_{m_1 + m_2 + n_1 + n_2} \rangle, \end{aligned}$$

where

$$\eta \in \mathcal{L}(G_{m_1, m_2, n_1, n_2}^1), \quad j_{(0,0,0,0)}^\infty(\nabla) \in Q_{\text{proj-proj}}^\infty, \quad \sigma \in J_{l_0}^r(L^{\text{fib-fib}}(\mathbb{R}^{m_1, m_2, n_1, n_2})),$$

η^* is the fundamental vector field on $L^{\text{fib-fib}}(\mathbb{R}^{m_1, m_2, n_1, n_2})$ corresponding to $\eta \in \mathcal{L}(G_{m_1, m_2, n_1, n_2}^1)$ and $l_0 = ((l_0)_1, \dots, (l_0)_{m_1 + m_2 + n_1 + n_2})$ is the basis in $T_{(0,0,0,0)}(\mathbb{R}^{m_1 + m_2 + n_1 + n_2})$. Then $B = B^{(\mu)}$. \square

Remark 2. Quite similarly one can describe all $\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}$ -natural operators $B: Q_\tau^{\text{proj-proj}} \rightarrow (J^r L^{\text{fib-fib}, k}, V^* L^{\text{fib-fib}, k} \otimes \wedge^m T^*)$, where $m = m_1 + m_2 + n_1 + n_2$, transforming projectable-projectable torsion-free classical linear connections ∇ on (m_1, m_2, n_1, n_2) -dimensional fibered-fibered manifolds Y into Euler morphisms $B_Y(\nabla)$ on $\pi_Y^k: L^{\text{fib-fib}, k}(Y) \rightarrow Y$ of fibered-fibered frames of order k of Y . All such natural operators are of the form $B^{(\mu)}$ for all

$$\mu: Q_{\text{proj-proj}}^\infty \times J_{l_0}^r(L^{\text{fib-fib}, k}(\mathbb{R}^{m_1, m_2, n_1, n_2})) \rightarrow \mathcal{L}(G_{m_1, m_2, n_1, n_2}^k)$$

satisfying the obviously modified finite determination property. The natural operators $B^{(\mu)}$ can be constructed similarly as in Example 2.

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